# Maxwell Field Equation Tutorial: Its Causal Lorentz-Covariant Solution in Integral Form

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**Abstract**

This tutorial sets out the detailed steps which produce the causal solution in integral form of the Maxwell field equation set, the causal solution being the one which vanishes in the absence of the source charge-current density, including during the time prior to that source being switched on. After warming up by obtaining the causal solution in integral form for the static source and field case, the Lorentz-covariant antisymmetric electromagnetic field tensor is defined, and the Maxwell field equation set is converted to its Lorentz-covariant form, whose formal causal solution is obtained in nine lines, but whose Green's function initially is ill-defined, a subtle issue whose ultimate causal resolution prompts a substantial effusion of ink. The integral solution which emerges can be exhibited either as manifestly Lorentz covariant or as very closely related to the static-case solution, except for systematic causal time retardation. Although direct solution for the electromagnetic field tensor is emphasized, the algebraically less involved four-vector intermediary potential approach is outlined in parallel.

## Introduction: static electromagnetism's causal solution in integral form

For *time-independent* charge-current densities and electromagnetic fields, the Maxwell field equations are,

$$\nabla\cdot\mathbf{E}=4\pi\rho,\qquad \nabla\times\mathbf{E}=\mathbf{0},\qquad \nabla\cdot\mathbf{B}=0,\qquad \nabla\times\mathbf{B}=(4\pi/c)\,\mathbf{j}. \tag{1a}$$

The last one of these four equations imposes on the time-independent current density $\mathbf{j}$ the requirement,

$$\nabla\cdot\mathbf{j}=0. \tag{1b}$$

Taking the curl of the second Eq. (1a) static Maxwell field equation produces,

$$\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}=\mathbf{0}, \tag{1c}$$

into which we substitute the right side of the first Eq. (1a) Maxwell field equation to obtain,

$$-\nabla^2\mathbf{E}=4\pi(-\nabla\rho). \tag{1d}$$

Likewise, taking the curl of the fourth Eq. (1a) static Maxwell field equation produces,

$$\nabla(\nabla\cdot\mathbf{B})-\nabla^2\mathbf{B}=(4\pi/c)(\nabla\times\mathbf{j}), \tag{1e}$$

into which we substitute the right side of the third Eq. (1a) Maxwell field equation to obtain,

$$-\nabla^2\mathbf{B}=(4\pi/c)(\nabla\times\mathbf{j}). \tag{1f}$$

The *solutions* for the *static* electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ of Eqs. (1d) and (1f) *which are causal*, i.e., *which vanish when their respective sources* $4\pi(-\nabla\rho)$ *and* $4\pi(\nabla\times\mathbf{j}/c)$ *vanish*, are *formally* given by,

$$\mathbf{E}=4\pi\big(-\nabla^2\big)^{-1}(-\nabla\rho)\quad\text{and}\quad\mathbf{B}=(4\pi/c)\big(-\nabla^2\big)^{-1}(\nabla\times\mathbf{j}). \tag{1g}$$

The *mathematical meaning of* $\big(-\nabla^2\big)^{-1}(f(\mathbf{r}))$ is obtained by using Fourier-transform methodology. Since,

$$f(\mathbf{r})=\int\Big[(1/(2\pi))^3\int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,d^3\mathbf{k}\Big]f(\mathbf{r}')\,d^3\mathbf{r}', \tag{2a}$$

it follows that,

$$-\nabla^2 f(\mathbf{r})=\int\Big[(1/(2\pi))^3\int|\mathbf{k}|^2\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,d^3\mathbf{k}\Big]f(\mathbf{r}')\,d^3\mathbf{r}', \tag{2b}$$

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and since $\left(-\nabla^2\right)^{-1}\left(-\nabla^2 f(\mathbf{r})\right) = f(\mathbf{r}) = -\nabla^2\left(\left(-\nabla^2\right)^{-1}(f(\mathbf{r}))\right)$, it *also* follows that,

$$\left(-\nabla^2\right)^{-1}(f(\mathbf{r})) = \int\left[(1/(2\pi))^3 \int |\mathbf{k}|^{-2}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\, d^3\mathbf{k}\right] f(\mathbf{r}')\, d^3\mathbf{r}'. \tag{2c}$$

The integral inside the square brackets in Eq. (2c) is evaluated using spherical polar $\mathbf{k}$ coordinates,

$$(1/(2\pi))^3 \int |\mathbf{k}|^{-2}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\, d^3\mathbf{k} = (1/(2\pi))^2 \int_0^\infty \left(\int_{-1}^1 e^{ik|\mathbf{r}-\mathbf{r}'|\alpha}\, d\alpha\right) dk =$$
$$(1/(2\pi^2)) \int_0^\infty (\sin(k|\mathbf{r}-\mathbf{r}'|)/(k|\mathbf{r}-\mathbf{r}'|))\, dk = (1/(4\pi))|\mathbf{r}-\mathbf{r}'|^{-1}, \tag{2d}$$

which upon insertion into Eq. (2c), yields,

$$\left(-\nabla^2\right)^{-1}(f(\mathbf{r})) = (1/(4\pi)) \int |\mathbf{r}-\mathbf{r}'|^{-1} f(\mathbf{r}')\, d^3\mathbf{r}'. \tag{2e}$$

Eq. (2e) converts the Eq. (1g) *formal* causal solution of static electromagnetism into well-defined *integrals*,

$$\mathbf{E}(\mathbf{r}) = \int |\mathbf{r}-\mathbf{r}'|^{-1}(-\nabla\rho(\mathbf{r}'))\, d^3\mathbf{r}' \quad \text{and} \quad \mathbf{B}(\mathbf{r}) = (1/c)\int |\mathbf{r}-\mathbf{r}'|^{-1}(\nabla\times\mathbf{j}(\mathbf{r}'))\, d^3\mathbf{r}'. \tag{2f}$$

It is often easier *to first calculate intermediary potentials* $A^0$ *and* $\mathbf{A}$, and then *to differentiate them to obtain* $\mathbf{E}$ *and* $\mathbf{B}$ than it is *to calculate* $\mathbf{E}$ *and* $\mathbf{B}$ *directly from* Eq. (2f). Via integration by parts, Eq. (2f) yields,

$$\mathbf{E}(\mathbf{r}) = -\nabla A^0(\mathbf{r}),\ \text{where } A^0(\mathbf{r}) = \int |\mathbf{r}-\mathbf{r}'|^{-1}\, \rho(\mathbf{r}')\, d^3\mathbf{r}' \ \text{and}$$
$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r}),\ \text{where } \mathbf{A}(\mathbf{r}) = (1/c)\int |\mathbf{r}-\mathbf{r}'|^{-1}\, \mathbf{j}(\mathbf{r}')\, d^3\mathbf{r}'. \tag{3}$$

It is to be borne in mind that the specific $A^0$ and $\mathbf{A}$ of Eq. (3) *aren't the only intermediary potentials which yield* $\mathbf{E}$ *and* $\mathbf{B}$; to that specific $A^0$ we can clearly add an arbitrary constant, and to that specific $\mathbf{A}$ we can add the gradient of an arbitrary scalar function of $\mathbf{r}$. The specific Eq. (3) $A^0(\mathbf{r})$ tends toward zero as $|\mathbf{r}| \to \infty$, and, because of the Eq. (1b) requirement that $\nabla\cdot\mathbf{j} = 0$, the specific Eq. (3) $\mathbf{A}$ satisfies $\nabla\cdot\mathbf{A} = 0$.

## Dynamic electromagnetism's Lorentz-covariant causal solution in integral form

The familiar *dynamically general* Maxwell field equations are,

$$\nabla\cdot\mathbf{E} = 4\pi\rho, \qquad \nabla\times\mathbf{E} + (1/c)\dot{\mathbf{B}} = \mathbf{0}, \qquad \nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{B} - (1/c)\dot{\mathbf{E}} = (4\pi/c)\mathbf{j}. \tag{4a}$$

They become *manifestly Lorentz-covariant* when the six components of $\mathbf{E}$ and $\mathbf{B}$ are made *the six independent components of the second-rank antisymmetric electromagnetic field tensor* $F^{\mu\nu} = -F^{\nu\mu}$ *as follows*,

$$F^{00} = 0;\ \ F^{ii} = 0,\ F^{i0} = -F^{0i} = (\mathbf{E})^i \text{ for } i = 1,2,3;\ \ F^{ij} = -F^{ji} = -(\mathbf{B})^k \text{ for } ijk = 123, 231, 312. \tag{4b}$$

Noting that $j^\mu \overset{\text{def}}{=} (c\rho, \mathbf{j})$, the *first* Eq. (4a) Maxwell field equation $\nabla\cdot\mathbf{E} = 4\pi\rho$ is, in terms of $F^{\mu\nu}$ and $j^\mu$,

$$\nabla\cdot\mathbf{E} = \textstyle\sum_{i=1}^3 \partial_i(\mathbf{E})^i = \partial_0 F^{00} + \sum_{i=1}^3 \partial_i F^{i0} = \partial_\mu F^{\mu 0} = 4\pi\rho = (4\pi/c)j^0, \tag{4c}$$

while the *fourth* Maxwell field equation $\nabla\times\mathbf{B} - (1/c)\dot{\mathbf{E}} = 4\pi(\mathbf{j}/c)$ of Eq. (4a) is, for $ijk = 123, 231, 312$,

$$(\nabla\times\mathbf{B})^i - (1/c)(\dot{\mathbf{E}})^i = \partial_j(\mathbf{B})^k - \partial_k(\mathbf{B})^j - \partial_0(\mathbf{E})^i = -\partial_0 F^{i0} + \partial_i F^{ii} - \partial_j F^{ij} + \partial_k F^{ki} =$$
$$\partial_0 F^{0i} + \partial_i F^{ii} + \partial_j F^{ji} + \partial_k F^{ki} = \partial_\mu F^{\mu i} = (4\pi/c)(\mathbf{j})^i = (4\pi/c)j^i. \tag{4d}$$

Thus the *first* and *fourth* Maxwell field equations of Eq. (4a) *together are the Lorentz-covariant equation*,

$$\partial_\mu F^{\mu\nu} = (4\pi/c)j^\nu. \tag{4e}$$

The *third* Maxwell field equation $\nabla\cdot\mathbf{B} = 0$ of Eq. (4a) is,

$$\nabla\cdot\mathbf{B} = \partial_1(\mathbf{B})^1 + \partial_2(\mathbf{B})^2 + \partial_3(\mathbf{B})^3 = \partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} = 0, \tag{4f}$$

while the *second* Maxwell field equation $\nabla \times \mathbf{E} + (1/c)\dot{\mathbf{B}} = \mathbf{0}$ of Eq. (4a) is, for $ijk = 123, 231, 312$,

$$\begin{gathered}(\nabla \times \mathbf{E})^i + (1/c)(\dot{\mathbf{B}})^i = \partial_j(\mathbf{E})^k - \partial_k(\mathbf{E})^j + \partial_0(\mathbf{B})^i = \\ \partial^j F^{0k} + \partial^k F^{j0} + \partial^0 F^{kj} = \partial^0 F^{kj} + \partial^k F^{j0} + \partial^j F^{0k} = 0.\end{gathered} \tag{4g}$$

Thus the *third* and *second* Maxwell field equations of Eq. (4a) *together are the four equations*,

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0, \tag{4h}$$

in the cases where $\lambda \neq \mu$, $\mu \neq \nu$ and $\nu \neq \lambda$. However the antisymmetry $F^{\mu\nu} = -F^{\nu\mu}$ of $F^{\mu\nu}$ *ensures the validity of* Eq. (4h) *even when* $\lambda = \mu$, $\mu = \nu$ *or* $\nu = \lambda$, *as is easily verified*. In summary, the Eq. (4a) Maxwell field equations for $\mathbf{E}$ and $\mathbf{B}$ *are equivalent to the following Lorentz-covariant equations for* $F^{\mu\nu}$,

$$F^{\mu\nu} = -F^{\nu\mu}, \qquad \partial_\mu F^{\mu\nu} = (4\pi/c)j^\nu, \qquad \partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0. \tag{4i}$$

The first two of these equations *impose the equation-of-continuity requirement on the source* $j^\nu$, *i.e.*,

$$\partial_\nu j^\nu = 0, \tag{4j}$$

which is the dynamical extension of the static Eq. (1b) requirement that $\mathbf{j}$ must adhere to $\nabla \cdot \mathbf{j} = 0$.

To work out the causal solution of the Eq. (4i) electromagnetic field equations, we emulate the differentiation and subsequent substitution of a source term carried out in the static case in Eqs. (1c) and (1d). We take the *divergence* of the *third* equation $\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0$ of Eq. (4i), and thus obtain,

$$(\partial_\lambda \partial^\lambda) F^{\mu\nu} + \partial^\mu \partial_\lambda F^{\nu\lambda} + \partial^\nu \partial_\lambda F^{\lambda\mu} = 0, \tag{5a}$$

into which *we substitute the source* $j^\nu$ via the *second* equation $\partial_\mu F^{\mu\nu} = (4\pi/c)j^\nu$ of Eq. (4i), aided by the antisymmetry $F^{\mu\nu} = -F^{\nu\mu}$, with the result,

$$(\partial_\lambda \partial^\lambda) F^{\mu\nu} - (4\pi/c)(\partial^\mu j^\nu - \partial^\nu j^\mu) = 0. \tag{5b}$$

Eq. (5b) immediately yields the *formal* causal solution $F^{\mu\nu}$ of the three field equations given by Eq. (4i),

$$F^{\mu\nu} = (4\pi/c)(\partial_\lambda \partial^\lambda)^{-1}(\partial^\mu j^\nu - \partial^\nu j^\mu), \tag{5c}$$

which clearly satisfies $F^{\mu\nu} = -F^{\nu\mu}$. Since the space-time gradient operator $\partial^\mu$ commutes with $(\partial_\lambda \partial^\lambda)$, it must commute with $(\partial_\lambda \partial^\lambda)^{-1}$ as well. Therefore since,

$$\begin{gathered}\partial^\lambda(\partial^\mu j^\nu - \partial^\nu j^\mu) + \partial^\mu(\partial^\nu j^\lambda - \partial^\lambda j^\nu) + \partial^\nu(\partial^\lambda j^\mu - \partial^\mu j^\lambda) = \\ (\partial^\mu \partial^\nu j^\lambda + \partial^\nu \partial^\lambda j^\mu + \partial^\lambda \partial^\mu j^\nu) - (\partial^\nu \partial^\mu j^\lambda + \partial^\lambda \partial^\nu j^\mu + \partial^\mu \partial^\lambda j^\nu) = 0,\end{gathered} \tag{5d}$$

the formal causal solution $F^{\mu\nu} = (4\pi/c)(\partial_\lambda \partial^\lambda)^{-1}(\partial^\mu j^\nu - \partial^\nu j^\mu)$ also satisfies the equation $\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0$. It as well satisfies the equation $\partial_\mu F^{\mu\nu} = 4\pi(j^\nu/c)$ because $\partial_\mu j^\mu = 0$, as noted in Eq. (4j).

We next use Fourier-transform methodology *to obtain the mathematical meaning of* $(\partial_\lambda \partial^\lambda)^{-1}(f(x))$. In strict analogy with the Fourier-transform logic of Eqs. (2a)–(2c) for $(-\nabla^2)^{-1}(f(\mathbf{r}))$, we have that,

$$(\partial_\lambda \partial^\lambda)^{-1}(f(x)) = \int \big[(1/(2\pi))^4 \int (-(k^\lambda k_\lambda))^{-1} e^{ik_\mu (x-x')^\mu}\, d^4k\big] f(x')\, d^4x'. \tag{6a}$$

The entity in square brackets in Eq. (6a) is the Green's function (or kernel) of the operator $(\partial_\lambda \partial^\lambda)^{-1}$, which we denote as $K(x-x')$; by inspection it is a Lorentz-invariant function of its four-vector space-time argument $(x - x')$. It is clear from Eq. (6a) that,

$$(\partial_\lambda \partial^\lambda)^{-1}(\delta^{(4)}(x)) = K(x). \tag{6b}$$

Thus $K(x)$ describes *how* the operator $(\partial_\lambda \partial^\lambda)^{-1}$ propagates in space-time *the effect of a point source* $\delta^{(4)}(x)$ *located at the origin that acts at time zero only*. The Eq. (5c) formal electromagnetic field solution $F^{\mu\nu} =$

$(4\pi/c)(\partial_\lambda\partial^\lambda)^{-1}(\partial^\mu j^\nu - \partial^\nu j^\mu)$ therefore will be causal *only if* $K(x)$ *vanishes when* $t < 0$*, i.e., when* $x^0 < 0$. However upon writing $K(x)$ out in greater detail as,

$$K(x^0,\, \mathbf{r}) = (1/(2\pi))^3 \int \big[(1/(2\pi)) \int (-(k^\lambda k_\lambda))^{-1}\, e^{ik_0 x^0}\, dk_0\big] e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{k}, \tag{6c}$$

we note that,

$$(-(k^\lambda k_\lambda))^{-1} = \big(-\big((k_0)^2 - |\mathbf{k}|^2\big)\big)^{-1} = (2|\mathbf{k}|)^{-1}\big((k_0 + |\mathbf{k}|)^{-1} - (k_0 - |\mathbf{k}|)^{-1}\big), \tag{6d}$$

so the integral over $k_0$ inside the square brackets of Eq. (6c) *is ill-defined*. That issue *is routinely resolved* by infinitesimal displacement of the offending poles into the complex plane, *but the result for a given pole depends on whether the upper or lower half of the complex plane is selected*—in our case *it is of course crucial for* $K(x^0, \mathbf{r})$ *to vanish when* $x^0 < 0$. When $x^0 < 0$, the factor $e^{ik_0 x^0}$ in the *integrand* of the Eq. (6c) integral over $k_0$ *makes the only viable closure of the real-axis* $k_0$*-contour an arbitrarily-large-radius semicircle in the lower half of the complex* $k_0$ *plane*. The $k_0$-integral *over that closed contour* will vanish *only* if we infinitesimally displace *both* offending poles *into the upper half of the complex* $k_0$ *plane*. Doing that by using the positive-imaginary infinitesimal $i\epsilon$ changes Eq. (6c) to,

$$K(x^0,\, \mathbf{r}) = (1/(2\pi))^3 \int \big[(4\pi|\mathbf{k}|)^{-1} \int \big((k_0 + |\mathbf{k}| - i\epsilon)^{-1} - (k_0 - |\mathbf{k}| - i\epsilon)^{-1}\big)\, e^{ik_0 x^0}\, dk_0\big] e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{k}, \tag{6e}$$

When $x^0 < 0$, the $k_0$ integral in Eq. (6e) now of course vanishes, and when $x^0 > 0$, it now equals $(2\pi i)$ times the sum of the residues at the two $k_0$-poles, which works out to $4\pi \sin(|\mathbf{k}|x^0)$. Eq. (6e) thereby becomes,

$$K(x^0,\, \mathbf{r}) = (1/(2\pi))^3\, \theta(x^0) \int |\mathbf{k}|^{-1} \sin(|\mathbf{k}|x^0)\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{k}, \tag{6f}$$

where $\theta(x^0)$ is the Heaviside step function: $\theta(x^0)$ vanishes when $x^0 < 0$ and equals unity when $x^0 > 0$. We tackle the remaining Eq. (6f) integral using spherical polar $\mathbf{k}$ coordinates, as was done in Eq. (2d),

$$\begin{gathered} K(x^0,\, \mathbf{r}) = (1/(2\pi))^2\, \theta(x^0) \int_0^\infty \sin(kx^0)\big(\int_{-1}^{1} e^{-ik|\mathbf{r}|\alpha}\, d\alpha\big)\, k\, dk = \\ (1/(2\pi^2))\, \theta(x^0)\, |\mathbf{r}|^{-1} \int_0^\infty \sin(kx^0) \sin(k|\mathbf{r}|)\, dk = \\ (1/(8\pi^2))\, \theta(x^0)\, |\mathbf{r}|^{-1} \int_{-\infty}^{\infty} \big[\cos(k(x^0 - |\mathbf{r}|)) - \cos(k(x^0 + |\mathbf{r}|))\big]\, dk = \\ (1/(4\pi))\, \theta(x^0)\, |\mathbf{r}|^{-1}\, \delta(x^0 - |\mathbf{r}|) = (1/(2\pi))\, \theta(x^0)\, \delta\big((x^0)^2 - |\mathbf{r}|^2\big) = (1/(2\pi))\, \theta(x^0)\, \delta(x_\lambda x^\lambda), \end{gathered} \tag{6g}$$

whose last expression exhibits $K(x)$ in Lorentz-invariant form as per the discussion below Eq. (6a). We see that $K(x)$ is nonzero only on the causal (i.e., retarded) half of the light cone, which is a spherical shell centered on the origin whose radius $|ct|$ grows at speed c when $t > 0$; it is the locus of space-time points which satisfy $|\mathbf{r}|^2 = (ct)^2$ when $t > 0$, or equivalently, the space-time locus $x_\lambda x^\lambda = 0$ for $x^0 > 0$. We now insert this causal (i.e., retarded) K(x) of Eq. (6g) into Eq. (6a) to obtain the corresponding causal (i.e., retarded) version of $(\partial_\lambda\partial^\lambda)^{-1}(f(x))$,

$$(\partial_\lambda\partial^\lambda)^{-1}(f(x)) = \int K(x - x')f(x')\, d^4x' = (1/(2\pi)) \int \theta\big((x - x')^0\big)\ \delta\big((x - x')_\lambda (x - x')^\lambda\big)\, f(x')\, d^4x', \tag{6h}$$

*which reflects the Lorentz invariance of* $(\partial_\lambda\partial^\lambda)^{-1}$. An *equivalent* form of $K(x^0, \mathbf{r})$ is the third-from-the-last expression given by Eq. (6g), *which cleanly eliminates the integration over* $(x')^0$ *via its delta function*,

$$\begin{gathered} (\partial_\lambda\partial^\lambda)^{-1}(f(x^0,\, \mathbf{r})) = (1/(4\pi)) \int \theta((x - x')^0)\, |\mathbf{r} - \mathbf{r}'|^{-1}\, \delta((x - x')^0 - |\mathbf{r} - \mathbf{r}'|)\, f((x')^0,\, \mathbf{r}')\, d(x')^0\, d^3\mathbf{r}' = \\ (1/(4\pi)) \int |\mathbf{r} - \mathbf{r}'|^{-1}\, f(x^0 - |\mathbf{r} - \mathbf{r}'|,\, \mathbf{r}')\, d^3\mathbf{r}'. \end{gathered} \tag{6i}$$

This result is formally *very similar indeed to the* Eq. (2e) *result for* $(-\nabla^2)^{-1}(f(\mathbf{r}))$; the only difference is that $(\partial_\lambda\partial^\lambda)^{-1}$ in addition *causally retards* $(x)^0$ *by the distance to the source*, replacing $(x)^0$ by $((x)^0 - |\mathbf{r} - \mathbf{r}'|)$. Applying either Eq. (6h) or (6i) to the Eq. (5c) formal solution of the electromagnetic field equations for $F^{\mu\nu}$, namely to $F^{\mu\nu} = (4\pi/c)(\partial_\lambda\partial^\lambda)^{-1}(\partial^\mu j^\nu - \partial^\nu j^\mu)$, makes that formal solution *a well-defined integral which is causal*. Applying Eq. (6h) *in addition presents that integral in Lorentz-covariant form*,

$$F^{\mu\nu}(x) = (2/c) \int \theta\big((x - x')^0\big)\ \delta\big((x - x')_\lambda (x - x')^\lambda\big)\ \big((\partial j^\nu(x')/\partial(x')_\mu) - (\partial j^\mu(x')/\partial(x')_\nu)\big)\, d^4x'. \tag{6j}$$

Once again noting from Eq. (6g) that,

$$\theta\big((x-x')^0\big)\ \delta\big((x-x')_\lambda(x-x')^\lambda\big) = (1/2)\theta((x-x')^0)\,|\mathbf{r}-\mathbf{r}'|^{-1}\,\delta((x-x')^0-|\mathbf{r}-\mathbf{r}'|), \tag{6k}$$

and defining the second-rank antisymmetric "source tensor" $s^{\mu\nu}(x)$ as,

$$s^{\mu\nu}(x) \overset{\text{def}}{=} \big((\partial j^\nu(x)/\partial x_\mu) - (\partial j^\mu(x)/\partial x_\nu)\big), \tag{6l}$$

we substitute the right side of Eq. (6k) and the left side of Eq. (6l) into Eq. (6j) to produce,

$$F^{\mu\nu}(x^0,\,\mathbf{r}) = (1/c)\int \theta((x-x')^0)\,|\mathbf{r}-\mathbf{r}'|^{-1}\,\delta((x-x')^0-|\mathbf{r}-\mathbf{r}'|)\,s^{\mu\nu}((x')^0,\,\mathbf{r}')\,d(x')^0\,d^3\mathbf{r}' = \\ (1/c)\int |\mathbf{r}-\mathbf{r}'|^{-1}\,s^{\mu\nu}(x^0-|\mathbf{r}-\mathbf{r}'|,\,\mathbf{r}')\,d^3\mathbf{r}'. \tag{6m}$$

Eqs. (6l) and (6m) together *also make the causal solution of the electromagnetic field equations a well-defined integral*, just as Eq. (6j) does. Indeed Eqs. (6l) and (6m) *have been developed by using* Eq. (6j) *and are equivalent to it*, but they have a form *which is very similar to that of* Eq. (2f) *for static electromagnetism* (with *the single salient exception*, of course, of their causal retarded $x^0$ feature).

*Instead* of calculating $F^{\mu\nu}$ *directly* from Eq. (6j), or from Eqs. (6l) and (6m), it is often *easier* to *first* calculate the four-vector *causal intermediary potential*,

$$A^\nu = (4\pi/c)(\partial_\lambda\partial^\lambda)^{-1}(j^\nu), \tag{7a}$$

*by specifically using* Eq. (6h) *or* (6i) *to obtain* $(\partial_\lambda\partial^\lambda)^{-1}(j^\nu)$*, followed by the calculation of* $F^{\mu\nu}$ *by using*,

$$F^{\mu\nu} = \big(\partial^\mu A^\nu - \partial^\nu A^\mu\big), \tag{7b}$$

which is legitimate in light of Eqs. (7a) and (5c). It is to be borne in mind that the specific $A^\nu$ of Eq. (7a) *isn't the only intermediary potential which yields* $F^{\mu\nu}$ *by the application of* Eq. (7b); to that specific $A^\nu$ *we can add the four-gradient* $\partial^\nu$ *of an arbitrary scalar function* $\chi(x)$ *of the space-time variable* $x$ *because*,

$$\big(\partial^\mu(\partial^\nu\chi(x)) - \partial^\nu(\partial^\mu\chi(x))\big) = 0. \tag{7c}$$

The specific Eq. (7a) causal intermediary potential $A^\nu$ satisfies the Einstein condition $\partial_\nu A^\nu = 0$ because the source $j^\nu$ is obliged to satisfy the equation of continuity $\partial_\nu j^\nu = 0$, as is pointed out in Eq. (4j).