Generalized Deng Entropy

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Abstract

Dempster-Shafer evidence theory as an extension of Probability has wide applications in many fields. Recently, A new entropy called Deng entropy was proposed in evidence theory. Deng Entropy as an uncertain measure in evidence theory. Recently, some scholars have pointed out that Deng Entropy does not satisfy the additivity in uncertain measurements. However, this irreducibility can have a huge effect. In more complex systems, the derived entropy is often unusable. Inspired by this, a generalized entropy is proposed, and the entropy implies the relationship between Deng entropy, Rényi entropy, Tsallis entropy.

Keywords: Deng entropy, Rényi entropy, Tsallis entropy. ,Uncertaintity, Dempster shafer evidence theory.

1. Introduction

Dempster-shafer evidence theory [1, 2] was proposed by Dempster [1] and developed by Shafer [2]. Evidence theory as a framework of uncertain reasoning is closely related to probability theory. It can be considered as a generalization of probability, assigning belief to power set of the propositions rather than single elements. This theory allows for the combination of evidence from different sources and draws a certain degree of conclusion, taking

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into account all available evidence. There are a lot of applications about it [3, 4, 5, 6, 7, 8].

How to measure uncertainty has always received widespread attention in evidence theory. Most of the measurements on uncertainty are related to Shannon entropy [9]. Yager [10] generalizes the entropy in probability theory to evidence theory. This entropy is based on the belief structure, which provides an indicator of the quality of the evidence. Maeda and Ichihashi [11] propose a uncertain measurement method. This uncertainty consists of two types, one containing the uncertainty of Shannon entropy determination and the other related to the cardinality of the set.

Recently, A new entropy named Deng entropy [12] has been proposed to solve uncertain measurement. This entropy is directly related to basic probability assignment. When the belief is assigned to the element of frame of discernment instead of the power set, this entropy is degenerated into the Shannon entropy. Deng entropy quickly attracts the attention of many scholars. Abellán [13] discusses the property of Deng entropy and points out that this entropy could quantify two types of uncertainty in evidence theory. Tang *et al*. [14] extended Deng entropy to an open world and applied it to information fusion. There are other discussions and applications about Deng entropy [15, 16].

Entropy is diverse [17]. After Clausius [18] proposed the concept of entropy, various entropies were raised. Rényi [19] proposed an entropy called Rényi entropy. Rényi entropy [19] has many applications in quantum information [20], information theory [21], and fractal theory [22]. Tsallis entropy is a generalization of the standard Boltzmann Gibbs entropy [23]. Tsallis entropy has been controversial since it was proposed [24]. After many complex systems are derived from Tsallis entropy [25], Tsallis entropy has received a lot of attention.

It can be proved that Shannon entropy [9] is a special case of Tsallis entropy [23], Rényi entropy [19]. So a natural question is what is the relationship between Deng entropy [12] and Rényi entropy [19] and Tsallis entropy [23]? Therefore, in order to explore the relationship between Deng entropy [12] and these two entropies. In this paper, we propose two generalized Deng entropies, which reveal the relationship with these entropies.

The structure of this article is as follows. Section 2 introduces some basic knowledge. Section 3 proposes the generalized Deng entropy. In section 4, some examples are discussed. Finally, conclusion is given.

2. Basic Knowledge

In this section, Deng entropy [12], Rényi entropy [19], Tsallis entropy [23] will be briefly introduced.

2.1. Deng Entropy

Compared to probability theory, Dempster shafer evidence theory [1, 2] has a greater advantage to deal with uncertainty. First, Dempster shafer evidence theory [1, 2] can deal with more uncertainty in the real world. In Dempster shafer evidence theory [1, 2], belief is not only assigned to a single element but also to a multi-element set [26]. In addition, it does not require prior information before combining each individual evidence [27]. Some basic knowledge about evidence theory is introduced.

Suppose the power set of the frame of discernment $X = \{\theta_1, \theta_2, \ldots, \theta_N\}$ is $P(X)$. Where the elements of X are mutually exclusive and exhaustive. For a frame of discernment X, the mass function is defined as follows [2].

$$m : P(X) \mapsto [0, 1] \tag{1}$$

where $m(\phi) = 0$ and $\sum_{F_i \in P(X)} m(F_i) = 1$.

In evidence theory, mass function is also called basic probability assignment (BPA), indicating the degree of belief in $A_i \in P(X)$.

Deng entropy in evidence theory is difined as follows [12].

$$E_d = -\sum_i m(A_i) \ln \frac{m(A_i)}{2^{|A_i|} - 1} \tag{2}$$

where $A_i \in P(X)$ and $|A_i|$ is the cardinality A_i.

Note that the base of all log functions is taken as a natural number e. $i.e.$ $\log_e \triangleq \ln$.

2.2. entropy

For a discrete random Y, its probability distribution is $P_Y = \{p_i | i = 1, 2, \ldots, N\}$. Rényi entropy is defined as follows [19].

$$H_\alpha = \frac{1}{1-\alpha} \log(\sum_i p_i^\alpha) \tag{3}$$

where $\alpha \geq 0$.

2.3. Tsallis entropy

Given a discrete Z, its probability distribution is $P_Z = \{p_i | i = 1, 2, \ldots, N\}$. Tsallis entropy is defined as follows [23].

$$H_q = \frac{k}{q-1}(1 - \sum_i p_i^q) \tag{4}$$

where q and k are parameters. For analysis, k is set to 1, which means that Tsallis entropy can be expressed as follows.

$$H_q = \frac{1}{q-1}(1 - \sum_i p_i^q) \tag{5}$$

3. Generalized Deng Entropy

In evidence theory, Klir and Wierman define five types of uncertainty requirements: probability consistency, set consistency, range, subadditive, additivity [28]. Are all the uncertain measurements satisfying these five requirements? Abellán [13] points out that Deng entropy [12] does not satisfy additivity and sub-additiveness. In fact, Tsallis entropy [23] does not satisfy additivity [29]. Rényi pointed out that if the additivity of Rényi entropy [19] is strictly satisfied, then there are only two possible Kolmogorov-Nagumo functions [29]. For example, in some systems involving long range forces [29], this kind of nonlinear system has come to receive widespread attention [30].

Deng entropy has been proposed as an entropy in the field of information [12], although there is currently no physical explanation. However, this non-additive nature seems to imply a connection to more complex systems.

3.1. R-D entropy

In order to bridge the relationship between Deng entropy [12] and Rényi entropy [19], a generalized D-R entropy is proposed as follows.

$$E_\alpha(m(A_i)) = \frac{1}{1-\alpha} \ln \left[\sum_i (\frac{m(A_i)}{2^{|A_i|}-1})^\alpha (2^{|A_i|} - 1) \right] \tag{6}$$

Theorem 1. *When $\alpha \to 1$, D-S entropy degenerates into Deng entropy.*

Proof 1. $\lim_{\alpha\to 1} E_\alpha(m(A_i))$

$= \lim_{\alpha\to 1} \frac{\frac{\partial}{\partial\alpha}\left[\ln(\sum_i (\frac{m(A_i)}{2^{|A_i|}-1})^\alpha (2^{|A_i|}-1)\right]}{\frac{\partial}{\partial\alpha}(1-\alpha)}$

$= \frac{\sum_i e^{\alpha \ln(\frac{m(A_i)}{2^{|A_i|}-1})} (2^{|A_i|}-1) \ln(\frac{m(A_i)}{2^{|A_i|}-1})}{-\sum_i (\frac{m(A_i)}{2^{|A_i|}-1})^\alpha (2^{|A_i|}-1)}$

$= -\sum_i m(A_i) \ln \frac{m(A_i)}{2^{|A_i|}-1}$

It can be easily proved that the R-D entropy degenerates into Rényi entropy when the belief is assigned to single elements. Naturally, when $\alpha \to 1$ and belief is assigned to single elements, the R-D entropy degenerates into Shannon entropy.

3.2. T-D Entropy

T-D entropy is proposed as follows, which may expose the relationship between Deng entropy [12] and Tsallis entropy [23].

$$E_q(m(A_i)) = \frac{1}{1-q}\left[1 - \sum_i \left(\frac{m(A_i)}{2^{|A_i|}-1}\right)^q (2^{|A_i|}-1)\right] \quad (7)$$

Theorem 2. *When $q \to 1$, T-D entropy degenerates into Deng entropy.*

Proof 2. $\lim_{q\to 1} E_q(m(A_i))$

$= \lim_{q\to 1} \frac{\frac{\partial}{\partial q}\left[1-\sum_i (\frac{m(A_i)}{2^{|A_i|}-1})^q (2^{|A_i|}-1)\right]}{\frac{\partial}{\partial q}(q-1)}$

$= -\sum_i e^{q \ln(\frac{m(A_i)}{2^{|A_i|}-1})} (2^{|A_i|}-1) \ln(\frac{m(A_i)}{2^{|A_i|}-1})$

$= -\sum_i m(A_i) \ln \frac{m(A_i)}{2^{|A_i|}-1}$

Similarly, it can be proved that the T-D entropy degenerates into Tsallis entropy when the belief is assigned to single elements. Naturally, when $q \to 1$ and belief is assigned to single elements, the T-D entropy degenerates into Shannon entropy.

3.3. R-T-D Entropy

Masi [29] proposes a unified entropy that links Rényi entropy [19] and Tsallis entropy [23]. Inspired by him, a unified form of entropy is proposed which could linkRényi entropy [19], Tsallis entropy [23] and Deng entropy [12].

$$E_{t,r}(m(A_i)) = \frac{1}{1-r}\left[\left[\sum_i (\frac{m(A_i)}{2^{|A_i|}-1})^t (2^{|A_i|}-1)\right]^{\frac{1-r}{1-t}} - 1\right] \tag{8}$$

It can be proved that when r tends to t, the R-T-D entropy degenerates into T-D entropy. when r tends to 1, the R-T-D entropy degenerates into R-D entropy.

4. Conclusion

We propose a generalized entropy that links Deng entropy [12], Rényi entropy [19], Tsallis entropy [23].

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