Bi-metric description of two particles gravitational interactioon

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Abstract

A bi-metric formalism is introduced to describe two bodies gravitational interaction using Whitehead's version of Schwarzschhild's original solution

1 Schwarzschild solution

It is well-known that Schwarzschild's solution of Einstein's field equations to describe a spherical static gravitational field can be be written using a great variety of coordinate systems including that proposed by Whitehead who did it unconscious of the fact that instead of a new theory he was proposing a very drastic simplification of Schwarzschild's original solution:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2\hat{m}}{r^3} L_\mu L_\nu \tag{1}$$

where $\eta_{\mu\nu}$ is Minkowsky's metric with $\eta_{44} = -1$ and:

$$L_4 = r, \ L_i = x_i \ i, j = 1, 2, 3 \tag{2}$$

with:

$$r = (x_1^2 + x_2^2 + x_3^2)^{1/2} \tag{3}$$

$\hat{x}$ being the space location of the point source and $\hat{m}$ is its passive mass. Therefore:

$$\eta_{\mu\nu} L^\mu L^\nu = 0 \tag{4}$$

It has been assumed also that the point mass $\hat{m}$, source of the gravitational field, is at rest at the origin of coordinates $\hat{x}^i = 0$. Besides the simplicity of Whitehead potentials two notorious facts follow follow from them:

1) The single value of r where every potential becomes infinite is $r = 0$ i.e at the location of the passive mass $\hat{m}$: there are no "blackholes".

2) The time-component of $L, L^4/c = r$ can very well be interpreted as the time that gravitation takes to go from the the location of the passive mass $\hat{m}$ to the point where test mass point is located at distance r, emphasizing the fact that gravitation is an interaction propagating at the speed of light.

If two masses are interacting and we want to describe the motion resulting from this interaction the question is: who is observing them. Is there only one observer or many? If there is one, who is it. And if there are many as it should , how are related to one another?. The observers that I have in mind are those observers that Special relativity considers. Namely observers with a common unit of length and who are able a synchronize their clocks. Then while we deal with the dynamics of two interacting objects is a problem of general relativity to make sense out of it we need embody it in a frame description of Special relativity.

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2\hat{m}}{(L_\sigma u^\sigma)^3} L_\mu L_\nu \tag{5}$$

where u^ρ is a time-like vector of Minkowski space-time identifying the class of objects using the same time . The usual Whitehead solution corresponds $u^4 = 1, u_i = 0$.

$$g_{44} = -1 + \frac{2\hat{m}}{r}, \quad g_{14} = \frac{2\hat{m}x_1}{r^2} \tag{6}$$

$$g_{24} = \frac{2\hat{m}x_2}{r^2}, \quad g_{34} = \frac{2\hat{m}x_3}{r^2} \tag{7}$$

$$g_{11} = \frac{2\hat{m}x_1^2}{r^3}, \quad g_{22} = \frac{2\hat{m}x_2^2}{r^3} \tag{8}$$

$$g_{33} = \frac{2\hat{m}x_3^2}{r^3}, \quad g_{12} = \frac{2\hat{m}x_1 x_2}{r^3} \tag{9}$$

$$g_{23} = \frac{2\hat{m}x_3}{r^3}, \quad g_{31} = \frac{2\hat{m}x_3 x_1}{r^3} \tag{10}$$

Let us start from the simplest case where a test mass is orbiting an object of mass $\hat{m}$ that is located at a point with coordinates $\hat{x}^i = 0$. Using t as parameter the geodesic principle tells us that the orbit of the test particle will be a solution of the system of differential equations:

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{44} + 2\Gamma^i_{4j}v^j + \Gamma^i_{jk}v^jv^k = bv^i \tag{11}$$

$$b = \Gamma^4_{44} + 2\Gamma^4_{4j}v^j + \Gamma^4_{jk}v^jv^k \tag{12}$$

where the Christoffel symbols corresponding to Whitehead's metric are:

$$\Gamma^i_{ii} = \frac{2\hat{m}^2x_i^3 - 2\hat{m}r^3x_i + 3\hat{m}rx_i^3}{r^6} \tag{13}$$

$$\Gamma^i_{jj} = \frac{-2\hat{m}^2x_ix_j^2 + 2\hat{m}r^3x_i - 3\hat{m}rx_ix_j^2}{r^6} \tag{14}$$

$$\Gamma^i_{jk} = \frac{(2\hat{m}+3r)\hat{m}x_ix_jx_k}{r^6}, \quad \Gamma^i_{jk} = \frac{(2\hat{m}+3r)\hat{m}x_ix_jx_k}{r^6} \tag{15}$$

$$\Gamma^i_{ij} = \frac{(2\hat{m}+3r)\hat{m}x_i^2x_j}{r^6}, \quad \Gamma^i_{4i} = -\frac{2cx_i^2\hat{m}^2}{r^5} \tag{16}$$

$$\Gamma^i_{4j} = -\frac{2cx_ix_j\hat{m}^2}{r^5}, \quad \Gamma^i_{44} = \frac{(-2\hat{m}+r)\hat{m}x_ic^2}{r^4} \tag{17}$$

$$\Gamma^4_{ii} = \frac{2\hat{m}(\hat{m}x_i^2 - r^3 + 2rx_i^2)}{r^5c}, \quad \Gamma^4_{ij} = \frac{2(\hat{m}+2r)\hat{m}x_ix_j}{r^5c} \tag{18}$$

$$\Gamma^4_{i4} = \frac{(2\hat{m}+r)\hat{m}x_i}{r^4}, \quad \Gamma^4_{4,4} = \frac{2\hat{m}^2c}{r^3} \tag{19}$$

In these formulas i, j, k run from 1 to 3 but $i \neq j$, $i \neq k$ and $j \neq k$

2 Two bodies geodesic equations

In this paper I consider an important generalization of ([1]) problem, considering two point objects with non negligible masses m and $\hat{m}$ describing orbits $x_i(t)$ and $\hat{x}_i(t)$ as functions of a common time parameter t. This idea can be very simply implemented rewriting the Whitehead model as a two body bi-metric model introducing both:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2\hat{m}}{r^3}L_\mu L_\nu \tag{20}$$

where x_i has been replaced by $x_i - \hat{x}_i$, in (**??**)) and:

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \frac{2m}{r^3}\hat{L}_\mu \hat{L}_\nu \tag{21}$$

where x_i has been replaced by $\hat{x}_i - x_i$ and $\hat{m}$ by m. as well as in the formulae corresponding to the Christoffel symbols of the preceeding section.

The resulting system of differential equations to be considered is then the following:

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{44} + 2\Gamma^i_{4j}v^j + \Gamma^i_{jk}v^j v^k = bv^i \tag{22}$$

$$\frac{d^2\hat{x}^i}{dt^2} + \hat{\Gamma}^i_{44} + 2\hat{\Gamma}^i_{4j}\hat{v}^j + \hat{\Gamma}^i_{jk}\hat{v}^j \hat{v}^k = \hat{b}\hat{v}^i \tag{23}$$

$$\tag{24}$$

with:

$$b = \Gamma^4_{44} + 2\Gamma^4_{4j}v^j + \Gamma^4_{jk}v^j v^k \tag{25}$$

$$\hat{b} = \hat{\Gamma}^4_{44} + 2\hat{\Gamma}^4_{4j}\hat{v}^j + \hat{\Gamma}^4_{jk}\hat{v}^j \hat{v}^k \tag{26}$$

Notice that the x substitutions made can be understood as linear coordinate transformations and therefore they are indeed tensor transformations.

3 A head-on collision

I consider here the case of a head-on collision along a radial trajectory $x_2 = x_3 = \hat{x}_2 = \hat{x}_3 = 0$ of two point particles with masses m and $\hat{m}$. The relevant non zero Christoffel symbols to take into account now, dropping the suffix 1 of x_1 are the following:

$$\tag{27}$$

$$\Gamma^1_{11} = \frac{2\hat{m}^2(x-\hat{x})^3 - 2\hat{m}r^3(x\hat{-}x) + 3\hat{m}r(x-\hat{x})^3}{r^6} \tag{28}$$

$$\Gamma^1_{44} = \frac{(-2\hat{m}+r)\hat{m}(x-\hat{m})}{r^4} \tag{29}$$

$$\Gamma^1_{14} = \frac{2\hat{m}^2(x-\hat{x})^2}{r^5} \tag{30}$$

$$\Gamma^4_{44} = \frac{2\hat{m}^2}{r^3} \tag{31}$$

$$\Gamma^4_{14} = \frac{(2\hat{m} + r)\hat{m}(x - \hat{x})}{r^4} \tag{32}$$

$$\Gamma^4_{11} = \frac{(2\hat{m} + r)\hat{m}(x - \hat{x})^2 - r^3 + 2r(x - \hat{x})^2)\hat{m}}{r^5} \tag{33}$$

$$\Gamma^4_{44} = \frac{2\hat{m}^2}{r^3} \tag{34}$$

as well as the result of exchanging $\Gamma's$ by $\hat{\Gamma}'s$, $\hat{m}$ by m and $x - \hat{x}$ by $\hat{x} - x$

The system of equations to solve is the following:

$$\frac{d^2x}{dt^2} = \frac{(2v^3 + 4v^2 + 6v + 2)}{(x - \hat{x})^3}\hat{m}^2 + \frac{(v^3 + 2v^2 - 1)}{(x - \hat{x})^2}\hat{m} \tag{35}$$

$$\frac{d^2\hat{x}}{dt^2} = \frac{(2\hat{v}^3 + 4\hat{v}^2 + 6\hat{v} + 2)}{(\hat{x} - x)^3}m^2 + \frac{(\hat{v}^3 + 2\hat{v}^2 - 1)}{(\hat{x} - x)^2}m \tag{36}$$

where $v = dx/dt$, and $\hat{v} = d\hat{x}/dt$.

4 radiation reaction terms

In the model described up to now the retarded interaction is described by a null vector L_α whose space components $L_i = x^i - \hat{x}^i$ have its origin at the mass $\hat{m}$ of the source and its end point at the point where its gravitational field is felt, and its time component is its length r, or since I have assumed $c = 1$, the travel time duration of the. Since this means the two events at x^i and $\hat{x}^i$ have been considered simultaneous This suggests that to describe the radiation reaction terms of the interaction the model can be improved by considering a null vector with space components:

$$\hat{L}_i = x^i - \hat{x}^i - r\hat{v}^i \tag{37}$$

in first approximation, or:

$$\hat{L}_i = x^i - \hat{x}^i - r(\hat{v}^i - r\hat{a}^i) \tag{38}$$

as well as that obtained exchanging un-headed x, v by headed ones $\hat{x}, \hat{v}$.

The graphs at the end of the paper show that the improvement of the model is real. Initial conditions are $x(0) = 2m$, $\hat{x}(0) = -\hat{m}$, with $m = \hat{m}$ = 1 solar mass. Without radiation reaction the numerical integration stops when $v(t) \approx 1$ without reaching $x = \hat{x} = 0$. Including first order radiation reaction term extends somewhat the approach to r=0. But this limit is only reached when second order reaction terms are included and correspond to a value of $|v|$ that is very approximately 1.

References

[1] Whitehead. A. N. Whitehead, *The principle of relativity with applications to physical science*. Cambridge University Press, (1922)

[2] A. S. Eddington, *A comparison of Whitehead,s and Einstein's formulae*, Nature,113, 192 (1922)

[3] arXiv:Physics/0505027 v2 (2005)

[4] G. Gibbons and C. Will arXiv:gr-qc/0611006 v1, (2006)

[5] L. Acedo, and Ll. Bel *On a correlation among azimuthal velocities and the flyby anomaly sign*, Astronomishe Nachrichten, (2018)

5 Appendix

Coulomb electrostatics can also be cast as a whitehead-like model introducing a potential vector such that:

$$A_\alpha = \frac{e}{(u_\rho L^\rho)^2} L_\alpha, \quad \textit{or } A_4 = \frac{e}{r},\ A_i = \frac{ex_i}{r^2} \tag{39}$$

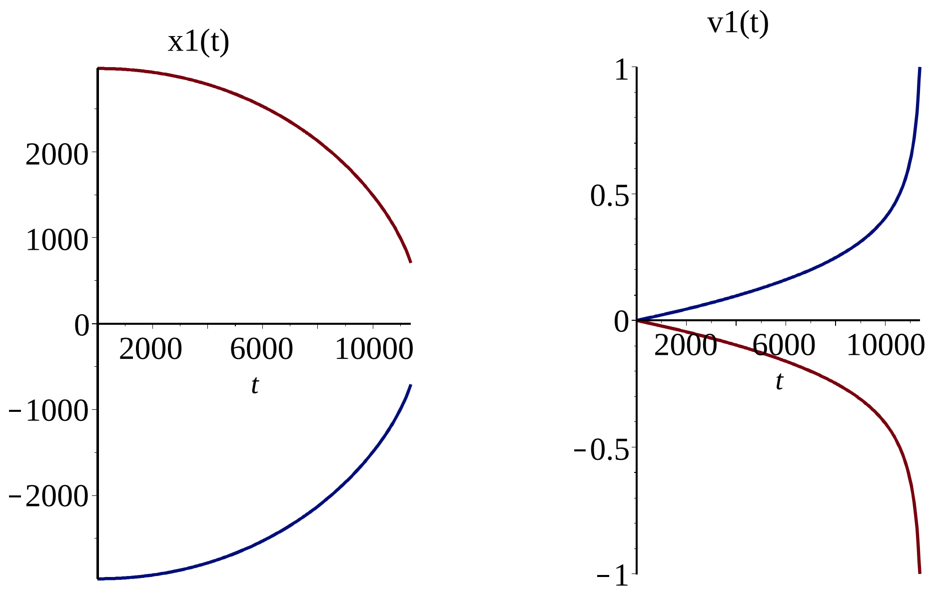
The corresponding field components being:

$$E_i = F_{i4} = -\frac{ex_i}{r^3}, B_{ij} = \partial_i A_j - \partial_j A_i = 0 \tag{40}$$

and the corresponding force-like equation becomes :

$$m\frac{d^2x^i}{dt^2} + F^i_4 + F^i_j v^j = 0,\ \text{or } m\frac{d^2x^i}{dt^2} = \frac{ex_i}{r^3} \tag{41}$$

From this point of view magnetism does not appears as a proper field but as a modification of Coulomb's law when considering moving charges..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−2000
2000
6000
10000
t
v1(t)
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0.5
0
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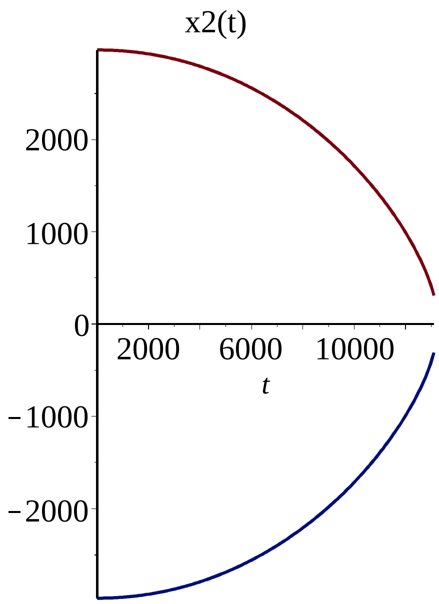
2000
6000
10000
t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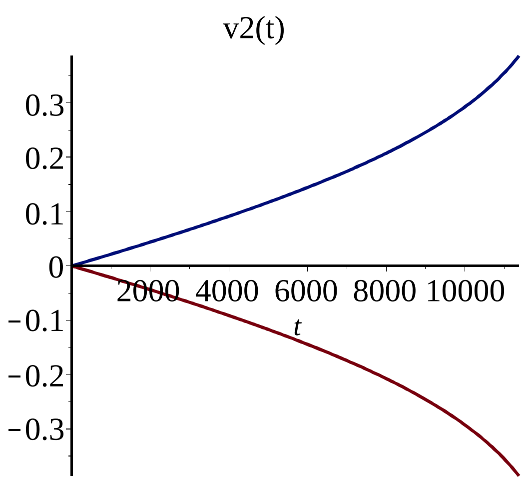
6000
10000
t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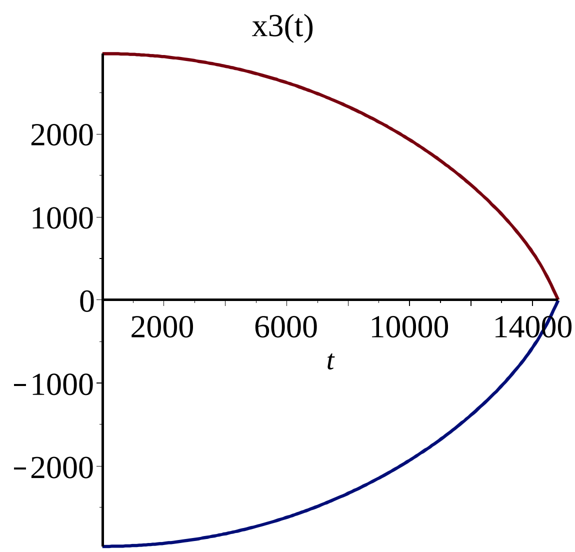
2000
4000
6000
8000
10000
t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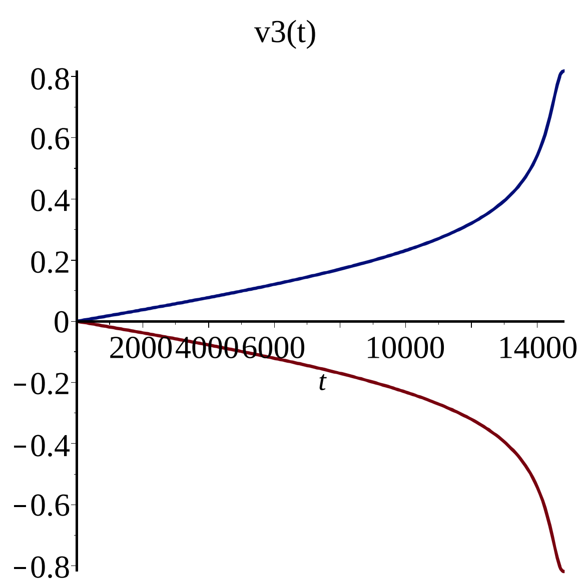
−1000
−2000
2000
6000
10000
14000
t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0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
2000
4000
6000
10000
14000
t