Fundamental Errors in Papers

Proof of the Polignac Prime Conjecture and other Conjectures
Stephen Marshall, 1702.0150

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ABSTRACT

The paper *Proof of the Polignac Prime Conjecture and other Conjectures*, (although listed under the title "Elementary Proof of the Goldbach Conjecture") first published in 2017 claimed to have proven Polignac's conjecture, and in doing so also the twin prime conjecture. The said paper had several problems, not least of which was a catastrophic basic error that completely invalidated the proof. Polignac's conjecture remains unproven, as does the twin primes conjecture. In this paper we outline the fundamental mistake that was made.

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1 Analysis

Author attempts proof by contradiction. Polignac's conjecture states that for any positive integer $k > 0$, the set of primes that lie consecutively and that differ by exactly $2k$ is infinite. Author begins by supposing that such a set is finite for any given $k > 0$:

$$\mathcal{A} = \{n_1, n_2, \cdots, n_p\} \tag{1}$$

(following the original author's notation, which we will do throughout this paper), where $n_1 > n_2 > \cdots > n_p$. This set is finite; let's assume it has p elements. This is important.

Author then constructs a sum:

$$S = \frac{1}{2n_1} + \frac{1}{3n_2} + \cdots + \frac{1}{m\,n_p}. \tag{2}$$

In the paper the author states that m "is the denominator factor for the smallest Polignac prime number that exists in our finite set." This is a clumsy way of simply saying that $m = p + 1$. Remember that the set $\mathcal{A}$ is finite and has p elements. If we multiply every consecutive member of the set $\mathcal{A}$ by an integer multiplier that likewise increases, (beginning with 2), then clearly the last such multiplier will be $p + 1$. We will always write $m = p + 1$ throughout this paper:

$$S = \frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{(p+1)\,n_p}.$$

Author then rewrites this finite sum as follows:

$$\begin{aligned} S = \frac{1}{2n_1} + \frac{1}{6n_2} + \frac{1}{12n_3} + \cdots + \frac{1}{p(p+1)\,n_p} \\ + \frac{1}{6n_2} + \frac{1}{12n_3} + \cdots + \frac{1}{p(p+1)\,n_p} \\ + \frac{1}{12n_3} + \cdots + \frac{1}{p(p+1)\,n_p} \\ \vdots \\ + \frac{1}{p(p+1)\,n_p} \end{aligned} \tag{3}$$

Here the author displays the error that will invalidate his proof. The author keeps writing the sums like this, *without* the final term in the *finite* sum:

$$\frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots \tag{4}$$

(although he leaves out the addition symbol for some reason). This is the notation for an *infinite* sum. Our sums are not infinite. They have a terminating term (i.e., the final term). By leaving this final term out he has confused himself into making a basic mistake. We will do the sums properly by *never* forgetting the final term. For example, instead of the incorrect statement (4) above, we will write the correct statement:

$$\frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} + \cdots + \frac{1}{(p+1)\,n_p}. \tag{5}$$

Let us continue. In equation (3), the author identifies each line with a distinct sum:

$$A = \frac{1}{2n_1} + \frac{1}{6n_2} + \frac{1}{12n_3} + \cdots + \frac{1}{p(p+1)\,n_p}, \tag{6}$$

$$B = \frac{1}{6n_2} + \frac{1}{12n_3} + \cdots + \frac{1}{p(p+1)\,n_p}, \tag{7}$$

$$C = \frac{1}{12n_3} + \cdots + \frac{1}{p(p+1)\,n_p}, \tag{8}$$

etc.

The author then rewrites each term in these sums into partial fractions. For example, for A we will have:

$$A = \left(\frac{1}{n_1} - \frac{1}{2n_1}\right) + \left(\frac{1}{2n_2} - \frac{1}{3n_2}\right) + \cdots + \left(\frac{1}{p\,n_p} - \frac{1}{(p+1)n_p}\right), \tag{9}$$

where, again, we have written the final term explicitly whereas the author neglected it. Author continues:

$$A = \frac{1}{n_1} + \left(\frac{1}{2n_2} - \frac{1}{2n_1}\right) + \left(\frac{1}{3n_3} - \frac{1}{3n_2}\right) + \cdots + \left(\frac{1}{p\,n_p} - \frac{1}{p\,n_{p-1}}\right) - \frac{1}{(p+1)n_p}. \tag{10}$$

Author then states that since $n_1 > n_2$,

$$\frac{1}{2n_2} - \frac{1}{2n_1} > 0, \tag{11}$$

and likewise for all the terms in brackets in equation (10). The author then makes the fatal error. Author states on page 4, that therefore

$$A > \frac{1}{n_1}. \tag{12}$$

This is incorrect. Clearly, the correct statement is:

$$A > \frac{1}{n_1} - \frac{1}{(p+1)n_p}. \tag{13}$$

The author has made this mistake because he has consistently neglected the final term in the *finite* sum.

Author then continues with the mistake and incorrectly states:

$$\begin{aligned} A &> \frac{1}{n_1}, && \text{[wrong!]} \\ B &> \frac{1}{2n_2}, && \text{[wrong!]} \\ C &> \frac{1}{3n_3}, && \text{[wrong!]} \\ &\vdots \end{aligned} \tag{14}$$

and since $S = A + B + C + \dots$, then

$$S > \frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{3n_3} + \cdots . \qquad \text{[wrong!]} \tag{15}$$

But, by definition,

$$S = \frac{1}{2n_1} + \frac{1}{3n_2} + \cdots ,$$

and it cannot be that

$$\frac{1}{2n_1} + \frac{1}{3n_2} + \frac{1}{4n_3} \cdots > \frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{3n_3} + \cdots . \qquad \text{[wrong!]} \tag{16}$$

Hence there is a contradiction and Polignac's conjecture is proven. This concusion is, however, invalid. Let's do this properly. Taking off from equation (13), we actually have:

$$\begin{aligned} S_1 = A &> \frac{1}{n_1} - \frac{1}{(p+1)n_p}, \\ S_2 = B &> \frac{1}{2n_2} - \frac{1}{(p+1)n_p}, \\ S_3 = C &> \frac{1}{3n_3} - \frac{1}{(p+1)n_p}, \\ &\vdots \\ S_{p-1} &> \frac{1}{(p-1)n_{p-1}} - \frac{1}{(p+1)n_p}, \\ S_p &> \frac{1}{p\,n_p} - \frac{1}{(p+1)n_p}, \end{aligned} \tag{17}$$

noting that in equation (3) there are p lines, and therefore there will be p distinct sums, which we have labelled $S_1, S_2, \cdots, S_p$, where $S_1 = A$, $S_2 = B$ etc.

Since S was the sum of the lines in equation (3), we have $S = S_1 + S_2 + \cdots + S_p$ by construction. Thus,

$$S > \left(\frac{1}{n_1} + \frac{1}{2n_2} + \cdots + \frac{1}{p\,n_p}\right) - \frac{p}{(p+1)n_p}. \tag{18}$$

Hence,

$$\frac{1}{2n_1} + \frac{1}{3n_2} + \cdots + \frac{1}{(p+1)\,n_p} > \left(\frac{1}{n_1} + \frac{1}{2n_2} + \cdots + \frac{1}{p\,n_p}\right) - \frac{p}{(p+1)n_p}. \tag{19}$$

This is the correct result. Note that it differs from the author's result (equation (16)) by the term

$$\frac{p}{(p+1)n_p}.$$

This term is just the accumulation of the final terms in the *finite* sums in equations (17). It should be present precisely because the sum is a finite sum.

2 Observations

2.1 Point 1

Equation (19) is the *actual* correct result. There is also no contradiction. If you rearrange terms, you get

$$\frac{p}{(p+1)n_p} > \left(\frac{1}{n_1} - \frac{1}{2n_1}\right) + \left(\frac{1}{2n_2} - \frac{1}{3n_2}\right) + \cdots + \left(\frac{1}{p\,n_p} - \frac{1}{(p+1)\,n_p}\right), \tag{20}$$

and so

$$\frac{p}{(p+1)n_p} > \frac{1}{2n_1} + \frac{1}{6n_2} + \cdots + \frac{1}{p(p+1)\,n_p}. \tag{21}$$

This inequality is trivial to prove, apart from the derivation that was already done above. Beginning with the right-hand side:

$$\begin{aligned} T &:= \frac{1}{2n_1} + \frac{1}{6n_2} + \cdots + \frac{1}{p(p+1)\,n_p} \\ &< \frac{1}{2n_p} + \frac{1}{6n_p} + \cdots + \frac{1}{p(p+1)\,n_p} \end{aligned} \tag{22}$$

because $n_k > n_p$ for all $k < p$. Now,

$$\frac{1}{2n_p} + \frac{1}{6n_p} + \cdots + \frac{1}{p(p+1)\,n_p} = \left(\frac{p}{p+1}\right)\frac{1}{n_p} \tag{23}$$

because

$$\frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{p(p+1)} = \frac{p}{p+1}. \tag{24}$$

You can prove this last statement by induction, for example. Hence,

$$T < \frac{p}{(p+1)\,n_p}. \tag{25}$$

Thus, equation (19) is correct, in contrast to the formula originally presented by the author, which was completely wrong.

2.2 Point 2

At no point in the author's "proof" was the fact that n_k were prime numbers ever used. Therefore his conclusion would have held for *any* set of integers $n_k > 0$ that strictly decrease in size. Consider this: imagine the exact same procedure, but instead begin by assuming n_k was *any* integer. Would anything in the derivation have changed? The answer is no. Therefore the "proof" applies to *any* such set of integers.

Let $\mathcal{B}$ be a finite set of size p of *any* integers $n_k > 0$ such that $n_1 > n_2 > \cdots > n_p$. For example,

$$\mathcal{B} = \{p, p-1, p-2, \cdots, 3, 2, 1\}, \tag{26}$$

which is just the set of the first p positive integers in descending order. (Explicitly, we have $n_k = p - k + 1$ for $k = 1, 2, \cdots, p$.) Every step in the author's "proof" would then follow exactly the same as before, leading to a supposed "contradiction". Therefore if his "proof" were valid, finite sets of integers would not be mathematically allowed. Clearly this is wrong.

2.3 Point 3

It's unlikely that a proof for a long-standing problem in mathematics could be finally solved by a trivial 5-page argument, which the author himself admitted was adapted from Johann Bernoulli's work. Did it not occur to the author that perhaps Johann Bernoulli would already have arrived at this "proof" if it was indeed valid?

3 Conclusion

The paper *Proof of the Polignac Prime Conjecture and other Conjectures*, is one of the most downloaded papers on vixra. It is also completely wrong. It is unfortunate that such a prominent paper contains such a fundamental mistake. We hope that this paper will clarify the matter for interested readers.