Extending an Irrationality Proof of Sondow: From e to $\zeta(n)$

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Introduction

Jonathan Sondow's geometric proof that e is irrational [1] uses nested closed intervals and the Bolzano-Weierstrass theorem. It's a trap: the endpoints of the intervals are systematically excluded as possible values for e. They are collectively all possible rational values, so e is proven to be irrational.

Here we re-frame Sondow's idea, replacing his intervals with concentric circles with classes from natural number moduli on them. We call such sets of points a circular moduli lattice (CML). This idea leads to a general criterion for irrationality of a series.

We explore some applications of the CML idea by giving Sondow's original proof for the irrationality of e using the CML associated with it. Next, we see how Sondow's proof doesn't generalize to show

$$\zeta(n) - 1 = z_n = \sum_{k=2}^{\infty} \frac{1}{k^n}$$

is irrational. Finally, we give proofs for the irrationality of e and z_2 using the criterion given earlier in the article.

Circular moduli lattice

Let's suppose the circle in Figure 1 has a radius of $1/\sqrt{\pi}$. Then its area is 1. We've placed equally spaced moduli classes for modulus 5 around the circle. Now sector areas correspond to fractions with numerators given by classes and denominators with the value of the modulus. The area associated with the radial in the figure is 3/5. Clearly, for any rational number m/n, $0 < m < n$, this procedure can be done.

Definition 1. *We will designate the set of such points in this arrangement with CK_n, where n is the modulus used and refer to such sets as clocks.*

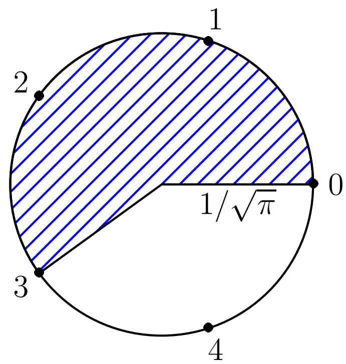

Figure 1: The shaded area is given by a modulo class.

Additional clocks can be added. In order to make them all sweep the same areas we use radii of $\sqrt{n/\pi}$. For example, in Figure 2(a) there are a 3-clock and a 5-clock. The radial given in this figure sweeps the same area in the inner circle and the annulus formed from the two circles. In this way the clocks can be used as a crude measurement device. We can infer from Figure 2(b) that the area associated with the sector given by the radial shown in Figure 2(a) measures neither thirds or fifths of the inner circle's area. It is in this sense that it is a very crude measuring device for sums of fractions: it doesn't say what the sum is equal to, but only what it is not equal to.

The circles can also be used to construct areas corresponding to the addition of fractions. In Figure 2(b) an addition method is given. It is similar to the head to tail method of vector addition. The 5-clock is rotated so as to place its 0 (head) position at the 1 position (tail) of the 3-clock. The new 1 position of the 5-clock corresponds, gives the area $1/3 + 1/5$. The radial generated is the same as that in Figure 2(a). Thus we can infer that $1/3+1/5$ is not in the set $\{1/3, 2/3, 1/5, 2/5, 3/5, 4/5\}$ or any un-reduced form of these

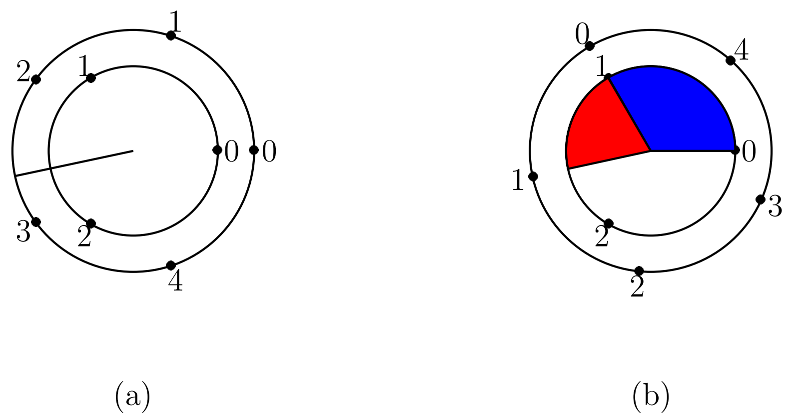

Figure 2: Circles as measuring device (a) and construction device (b).

fractions. The clocks give both a way to construct addition of fractions and measure the result.

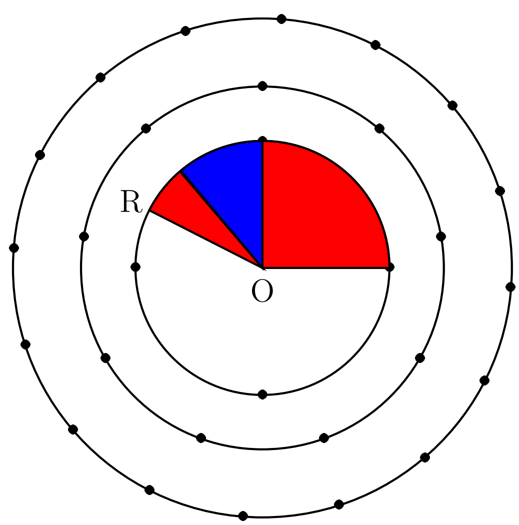

Figure 3: A partial sum for z_2 is constructed using $CML\{k^2\}$.

In Figure 3, the first few terms, $1/4, 1/9, 1/16$, for z_2 are added. Clearly, one can continue with this method for as many terms or all terms, as one likes. We formalize the idea with a definition.

Definition 2. *Given an infinite series with positive, strictly decreasing terms of the form $1/a_j$, $a_j \in \mathbb{N}$, let the set of all points on CK_{a_j} be called the circular moduli lattice for the series. Designate this set with $CML\{a_j\}$.*

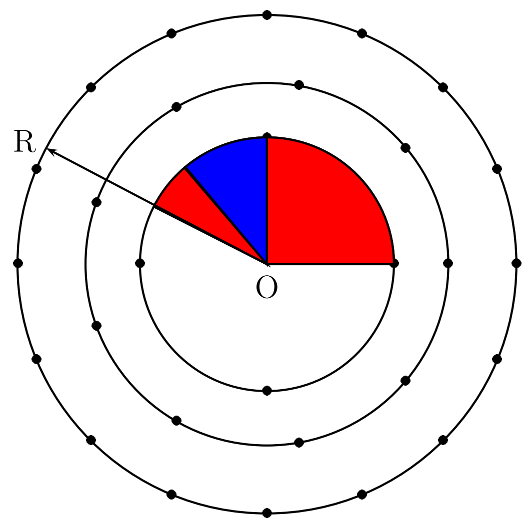

Figure 4: The radial for the partial doesn't intersect lattice points.

Using sets of clocks associated with an infinite series, we can frame the question of convergence to an irrational point. In Figure 4 the partial sum $1/4+1/9+1/16$ for z_2 is depicted using the original, un-rotated clocks. The radial OR generates a sector of this sums area and it doesn't intersect any of the points on the three circles. This means $1/4+1/9+1/16$ doesn't have a reduced form associated with CK_4, CK_9, or CK_{16}. If this is always true, i.e., if the radial for z_2, the infinite series, doesn't go through a lattice point and all the lattice points give all the possible rational areas, then z_2 is irrational. We formalize the notion of all possible rational areas with a definition.

Definition 3. *For a given series with terms $1/a_j$, if there exists for every m/n, with $0 < m < n$, CK_r and modulus class s such that $s/r = m/n$ then the CML associated with the series, $CML\{a_j\}$ is said to cover the rational numbers.*

We can give a necessary and sufficient condition for a series to converge to an irrational number.

Theorem 1. *If $CML\{a_k\}$ covers the rational numbers and partial sums for the series are such that*

$$\sum_{k=2}^{n} \frac{1}{a_k} \in \bigcup_{k=2}^{\infty} CK_{a_k} \setminus \bigcup_{k=2}^{\varphi(n)} CK_{a_k}, \tag{1}$$

where $\varphi(n)$ is a natural number, strictly increasing function, then the series converges to an irrational number.

Proof. Using (1),

$$\lim_{n\to\infty} \bigcup_{k=2}^{\infty} CK_{a_k} \setminus \bigcup_{k=2}^{\varphi(n)} CK_{a_k} = \emptyset,$$

so the the limit of the partials is not in $CML\{a_k\}$ and must be irrational. □

Sondow's proof

Here's Sondow's proof that e is irrational, using the CML idea as a visual aid. The series we use drops the first term:

$$e - 1 = \sum_{k=2}^{\infty} \frac{1}{k!}.$$

Figure 5 has a final radial that sweeps an arc giving a sector of area $e-1$. To see this note that the inner most circle has two sectors each of one half area: the first term in the series for $e-1$ is $1/2! = 1/2$. So we sweep one half and then repeat the procedure to sweep another $1/3! = 1/6$ using CK_6; the annulus's blue band gives the next location of the series final radius. This procedure is repeated for $4! = 24$ in the Figure. This procedure continues to infinity via adding $CK_{k!}$ clocks. As the terms of the series are fractional multiplies of each other, factorial value denominators, the sectors perpetually nest. The $CK\{a_k\}$ covers the rationals: $p(q-1)!/q! = p/q$ with $p < q$. This implies that all possible rational convergence points are excluded.

Sondow, in his article, uses a series of lines representing intervals that give boundaries for possible convergence points. He doesn't drop the first $1/1!$ term. Dropping the first term, as we do, makes the argument clearer; and, of course, if $e-1$ is irrational, so is e.

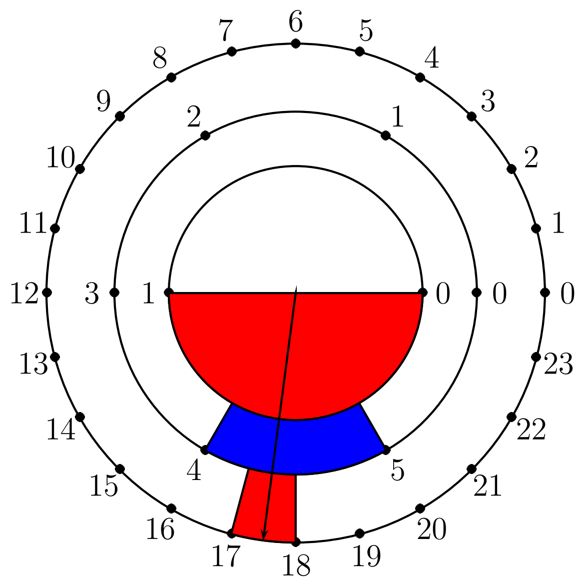

Figure 5: Sondow's proof that e is irrational using $CML\{k!\}$. The number of circles goes to infinity.

Here's z_2

In Figure 6, we attempt Sondow's strategy. It doesn't work. They don't nest. It's a mess.

Examples

The technique given in section 2 replaces interval endpoint exclusions with general, in effect, denominator exclusions. It's easy. We need to move away from the $\epsilon - \delta$ world of point set topology and analysis and use just sets without a defined metric. Here are two examples that motivate the idea.

Consider the task of proving the limit of $1 - 1/n$ is not of the form m/n, $0 < m < n$ with $n > 4$. That is we want to show the limit is an integer and not a fraction, of one class and not another. Now

$$1 - 1/n = \frac{n-1}{n} \in \bigcup_{k=1}^{\infty} CK_k \setminus \bigcup_{k=2}^{n-1} CK_k$$

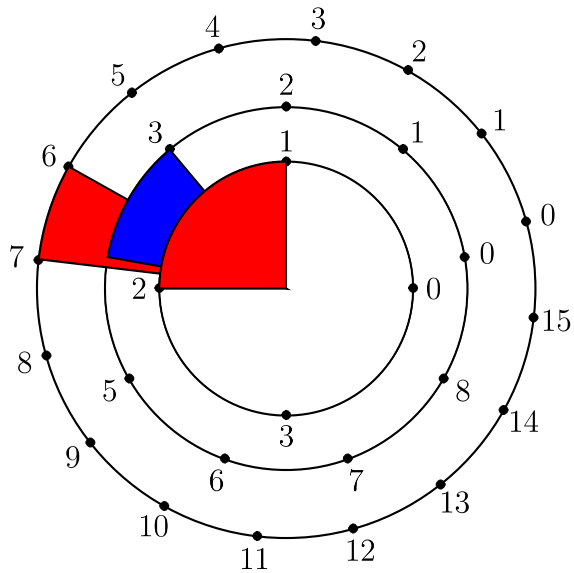

Figure 6: Sondow's interval technique fails for proving z_2 is irrational.

and, as $(n-1, n) = 1$, $(n-1)/n$ is a reduced fraction,

$$\lim_{n\to\infty} \bigcup_{k=1}^{\infty} CK_k \setminus \bigcup_{k=2}^{n-1} CK_k = CK_1.$$

This implies that the limit is in CK_1; CK_1 only has 1 as a non-zero element.

Consider next a less trivial example. Suppose we want to show that $.\overline{1}$ base 4 converges to a denominator that does not have a denominator of a power of 4. Using clocks, the set

$$\bigcup_{k=1}^{n-1} CK_{4^k}$$

gives all finite decimals of length $n-1$ or less. So $.1, .11, \ldots, .\overline{1}_{n-1}$ are in this set, where the last decimal represents $n-1$ repeated 1 decimal digits. The following set

$$\bigcup_{k=2}^{\infty} CK_k$$

gives all finite decimals in the unit interval, $[0,1]$, in all bases. Observing

$$.\overline{1}_n \in \bigcup_{k=2}^{\infty} CK_k \setminus \bigcup_{k=1}^{n-1} CK_{4^k},$$

we can infer

$$\lim_{n\to\infty} \bigcup_{k=2}^{\infty} CK_k \setminus \bigcup_{k=1}^{n-1} CK_{4^k} = F \neq \emptyset. \tag{2}$$

As, in the limit, all finite decimals in base 4 are exhausted, the convergence point must not be a finite decimal in base 4. Note: $.\overline{1}$ in base 4 converges to $1/3$ and (2) is consistent with this: $1/3 \in F$.

If, instead of $.\overline{1}$ base 4, we used $\sqrt{2}$ given as an infinite, non-repeating decimal, we would still get a non-empty set, so the test is not sufficient to show rationality. We could not infer (2) as the decimals are non-repeating and not all ones, violating the antecedents of Theorem 1. The criterion demands that the terms of the series be complete or cover the rationals; that is that the denominators of the terms, taken as bases, allow all rationals to be expressed as finite decimals within a base given by such denominators.

Irrationality proofs

There are two essential steps necessary to use Theorem 1. First, the denominators of the series must cover the rationals; second, the partials must reside in a set given by a set difference; and third, taking the limit of this set difference results in an empty set; this last result should be automatic. We illustrate these steps to show e and z_2 are irrational in this section.

Irrationality of $e - 1$

The denominators cover the rationals: given reduced p/q with $p < q$, $p(q-1)!/q! \in CK_{q!}$. That's step one. Step two: we must find a strictly increasing function $\phi(n)$, per Theorem 1, such that

$$\sum_{k=2}^{n} \frac{1}{k!} \in \bigcup_{k=2}^{\infty} CK_{k!} \setminus \bigcup_{k=2}^{\phi(n)} CK_{k!}.$$

We observe that

$$(n-1)!\sum_{k=2}^{n}\frac{1}{k!}=K+\frac{1}{n},$$

where K is a positive integer. This implies that

$$\sum_{k=2}^{n}\frac{1}{k!}\in\bigcup_{k=2}^{\infty}CK_{k!}\setminus\bigcup_{k=2}^{n-1}CK_{k!}. \tag{3}$$

That is $\phi(n)=n-1$ with $n>3$. Taking the limit in (3),

$$\sum_{k=2}^{n}\frac{1}{k!}\in\emptyset,$$

implying that $e-1$ is not rational; it must be irrational.

Irrationality of z_2

The denominators cover the rationals: all candidate rational numbers, p/q, can be written as $pq/q^2\in CK_{q^2}$. That's step one. Step two: we must find a strictly increasing function $\phi(n)$, per Theorem 1, such that

$$\sum_{k=2}^{n}\frac{1}{k^2}\in\bigcup_{k=2}^{\infty}CK_{k^2}\setminus\bigcup_{k=2}^{\phi(n)}CK_{k^2}.$$

We need to show that the partials for z_2, $\sum_2^n 1/k^2$, require greater than n^2 denominators, for example. At prime p partials will be given by

$$\frac{a}{b}+\frac{1}{p^2}=\frac{p^2a+b}{p^2b},$$

where $p\nmid b$. This implies that partials for upper limit a prime require more than the upper limit squared in their denominators. We can use $\phi(n)=n$. This gives

$$\sum_{k=2}^{n}\frac{1}{k^2}\in\bigcup_{k=2}^{\infty}CK_{k^2}\setminus\bigcup_{k=2}^{n}CK_{k^2}$$

and taking the limit, we get the required empty set, implying that z_2 is irrational.

Counter-example

Consider the telescoping series:

$$\frac{1}{2} = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \ldots.$$

The terms of this series are of the form $1/n(n-1)$, $n > 1$. They cover the rationals: $p/q = p(q-1)/q(q-1) \in CK_{q(q-1)}$. Is there a $\phi(n)$ such that

$$\sum_{k=3}^{n} \frac{1}{k(k-1)} \in \bigcup_{k=3}^{\infty} CK_{k(k-1)} \setminus \bigcup_{k=3}^{\phi(n)} CK_{k(k-1)}?$$

If there was this series would give a counter-example. But the partials don't force an increasing function. Using upper limits of 3, 4, 5, and 6, the partials sum to 1/6, 1/4, 2/7, 1/3.

Conclusion

The set theory used in this article seems to be of not the standard type. What does

$$\lim_{n\to\infty} \bigcup_{k=2}^{\infty} CK_{a_k} \setminus \bigcup_{k=2}^{\varphi(n)} CK_{a_k} = \emptyset, \tag{4}$$

mean, if one insists on a epsilon/delta type idea? There doesn't seem to be any metric involved. But, isn't it totally obvious: if one has a glass full of fluid and drains it to nothing, nothing is left. The oddness of the mathematics is that at any moment the number of fractions is countable infinite. How can it go from countable infinite to the empty set in a gradual way? But set theory does address this. There are orders of infinity: $\aleph_0$ and $\aleph_1$, for example. The metric of number in a set gives the idea: in the finite domain, removing marbles of a certain number from a glass full of marbles, one can at any moment say how many remain to go. But this finite world is not the world of fluids or abstract numbers that remain infinitely divisible: an interval, no matter how small, will have a uncountable cardinality, for example. This seems to be the better understanding of what (4) means.

Perhaps (4) requires an axiom in set theory: its true.

References

[1] J. Sondow, A geometric proof that e is irrational and a new measure of its irrationality, *Amer. Math. Mon.* 113 (2006), 637-641.