Extending an Irrationality Proof of Sondow: From e to $\zeta(n)$

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April 15, 2019

Abstract

We modify Sondow's geometric proof of the irrationality of e. The modification uses sector areas on circles, rather than closed intervals. Using this circular version of Sondow's proof, we see a way to understand the irrationality of a series. We evolve the idea of proving all possible rational value convergence points of a series are excluded because all partials are not expressible as fractions with the denominators of their terms. If such fractions cover the rationals, then the series should be irrational.

Introduction

Jonathan Sondow's geometric proof that e is irrational [6] uses nested closed intervals and the Bolzano-Weierstrass theorem [1]. It's a trap: the endpoints of the intervals are systematically excluded as possible values for e. They are collectively all possible rational values, so e is proven to be irrational. The intervals he uses seem a little unwieldy, so we replace them with concentric circles giving values as sector areas. The sector areas are determined at points around the circle; these points correspond to classes from natural number moduli. We call such sets of points a circular moduli lattice (CML).

Using this CML idea, we give Sondow's proof. The CML seems to help with making the proof completely transparent. Giddy, we attempt to use the same CML technique to show $\zeta(2)$ is irrational, but it doesn't work: no nesting. But from our consideration of these two series a general criterion for the irrationality of a series emerges.

We prove both e and $\zeta(n)$ are irrational using this criterion. The latter is an unsolved number theory problem [2, 3]. The criterion does involve a limit construction that seems to be novel. Proofs for the $n = 2$ and $n = 3$ cases using $\epsilon - \delta$ reasoning are given in [3].

Sondow's e proof

Here's Sondow's proof verbatim. The irrationality of e is a consequence of the following construction of a nested sequence of closed intervals I_n. Let $I_1 = [2, 3]$. Proceeding inductively, divide the interval I_{n-1} into n (≥ 2) equal intervals, and let the second one by I_n (see Figure 2. For example, $I_2 = [\frac{5}{2!}, \frac{6}{2!}]$, $I_3 = [\frac{16}{3!}, \frac{17}{3!}]$, and $I_4 = [\frac{65}{4!}, \frac{66}{4!}]$.

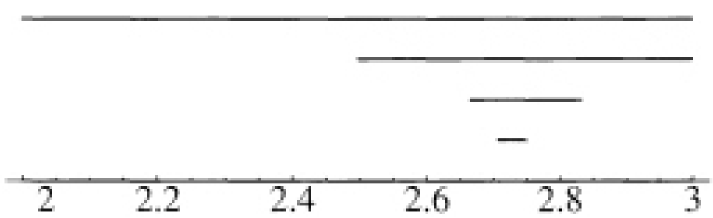


Figure 1: The intervals I_1, I_2, I_3, and I_4.

The intersection

$$\bigcap_{n=1}^{\infty} I_n = \{e\} \tag{1}$$

is then the geometric equivalent of the summation

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

When $n > 1$ the interval I_{n+1} lies strictly between the endpoints I_n, which are $a/n!$ and $(a+1)/n!$ for some integer $a = a(n)$. It follows that the point of intersection (1) is not a fraction with denominator $n!$ for any $n \geq 1$. Since a rational number p/q with $q > 0$ can be written

$$\frac{p}{q} = \frac{p \cdot (q-1)!}{q!},$$

we conclude that e is irrational.

Circular moduli lattice

Let's suppose the circle in Figure 2 has a radius of $1/\sqrt{\pi}$. Then its area is 1. We've placed equally spaced moduli classes for modulus 5 around the circle. Now sector areas correspond to fractions with numerators given by classes and denominators with the value of the modulus. The area associated with the radial in the figure is 3/5. Clearly, for any rational number m/n, $0 < m < n$, this procedure can be done.

Definition 1. *We will designate the set of such points in this arrangement with CK_n, where n is the modulus used and refer to such sets as clocks.*

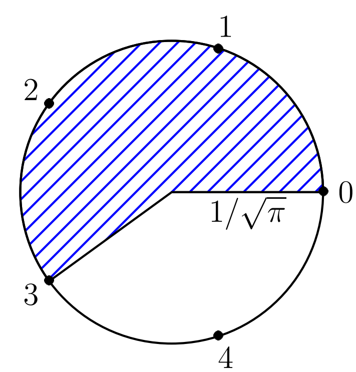
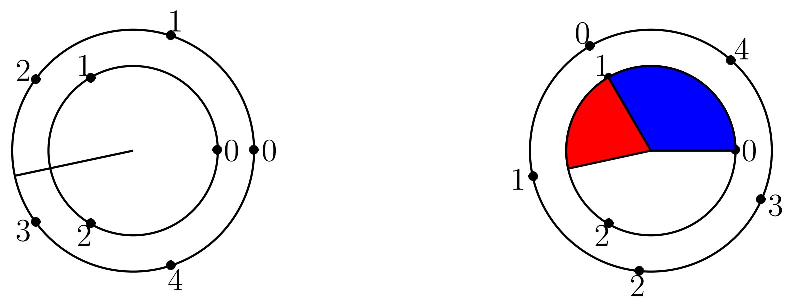


Figure 2: The shaded area is given by a modulo class.

Additional clocks can be added. In order to make them all sweep the same areas we use radii of $\sqrt{n/\pi}$. For example, in Figure 3(a) there are a 3-clock and a 5-clock. The radial given in this figure sweeps the same area in the inner circle and the annulus formed from the two circles. In this way the clocks can be used as a crude measurement device. We can infer from Figure 3(b) that the area associated with the sector given by the radial shown in Figure 3(a) measures neither thirds or fifths of the inner circle's area. It is in this sense that it is a very crude measuring device for sums of fractions: it doesn't say what the sum is equal to, but only what it is not equal to.

The circles can also be used to construct areas corresponding to the addition of fractions. In Figure 3(b) an addition method is given.



(a) (b)

Figure 3: Circles as measuring device (a) and construction device (b).

It is similar to the head to tail method of vector addition. The 5-clock is rotated so as to place its 0 (head) position at the 1 position (tail) of the 3-clock. The radius at the new 1 position of the 5-clock gives a sector area or $1/3 + 1/5$. The radial generated is the same as that in Figure 3(a). Thus we can infer that $1/3 + 1/5$ is not in the set $\{1/3, 2/3, 1/5, 2/5, 3/5, 4/5\}$ or any un-reduced form of these fractions. The clocks give both a way to construct addition of fractions and measure the result.

Sondow's proof

Here's Sondow's proof that e is irrational, using the CML idea as a visual aid. We omit the first two terms so the series converges to a number less than 1:

$$e - 2 = \sum_{k=2}^{\infty} \frac{1}{k!} < 1. \tag{2}$$

Figure 4 has a final radius that sweeps an arc giving a sector of area $e-2$. To see this note that the inner most circle has two sectors each of one half area: the first term in the series for $e - 2$ is $1/2! = 1/2$. So we sweep one half and then repeat the procedure to sweep another $1/3! = 1/6$ using CK_6; the annulus's blue band gives the next location of the series final radius. This procedure continues to infinity via adding $CK_{k!}$ clocks. As subsequent terms are fractional multiplies of each other, the sectors perpetually nest. The $CK\{a_k\}$, where a_k are the denominators of (2), covers the rationals: $p(q-1)!/q! = p/q$ with $p < q$. This implies that all possible rational convergence points are excluded.

Sondow, in his article, uses a series of lines representing intervals that give boundaries for possible convergence points. He doesn't drop the first two terms. Dropping the first two terms, as we do, makes the argument clearer; and, of course, if $e-2$ is irrational, so is e.

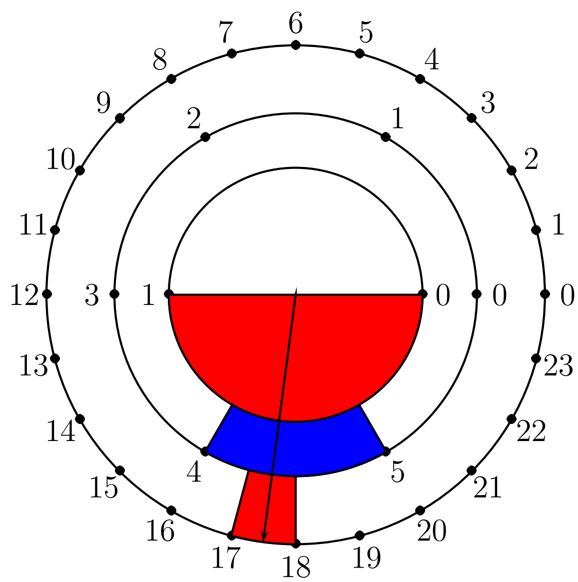


Figure 4: Sondow's proof that e is irrational using $CML\{k!\}$. The number of circles goes to infinity.

Attempt to use Sondow with $\zeta(2)$

In Figure 5, we attempt to use Sondow's strategy on

$$z_2 = \zeta(2) - 1 = \sum_{k=2}^{\infty} \frac{1}{k^2} < 1$$

It doesn't work. They don't nest. It's a mess.

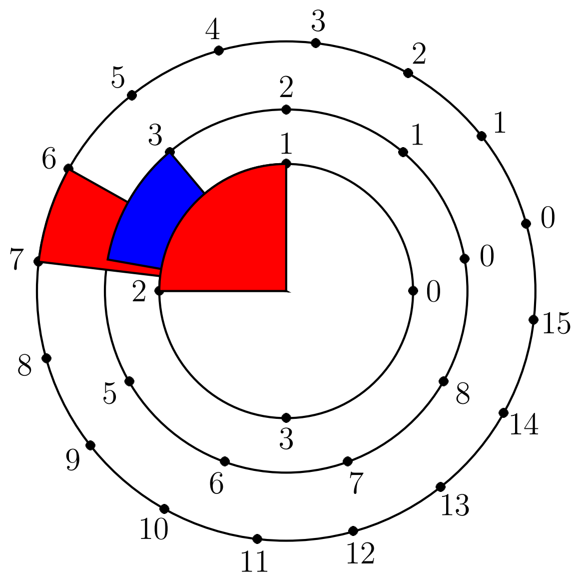


Figure 5: Sondow's interval technique fails for proving z_2 is irrational.

A pattern emerges

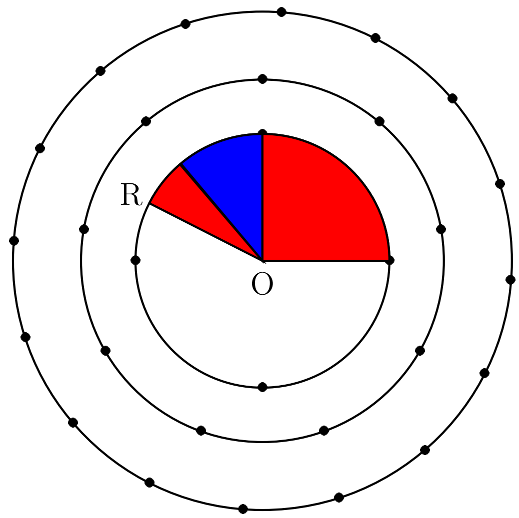


Figure 6: A partial sum for z_2 is constructed using $CML\{k^2\}$.

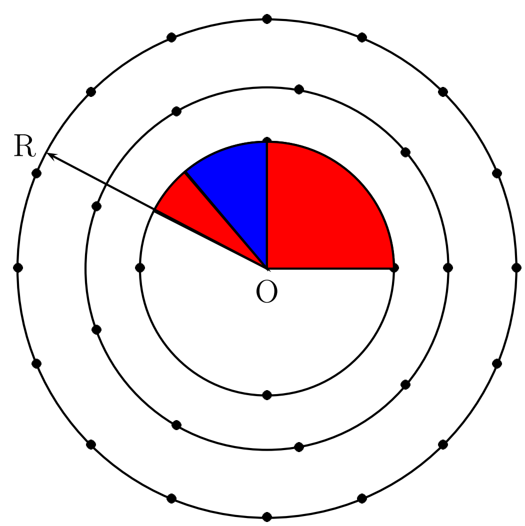


Figure 7: The radial for the partial doesn't intersect lattice points.

Partial sums don't equal fractions from the circles that are used to construct the partial sums. For z_2 we can see that at least the partial $1/4+1/9+1/16$ is not equal to any of the sector areas from the circles that are used to construct its radius. This is shown in Figure 6 which constructs the value of the partial and Figure 7 which shows that it doesn't go through any points on the circles used. For $e-2$, we notice that the partial associated with n can't be expressed with $CK_{(n-1)!}$.

Criterion for irrationality

Perhaps instead of using Sondow's nested interval proof, we can use the CML with a criterion that the partial sum radius always misses all the dots (values) on its constructing circles. These dots in the case of e give all possible rational convergence points.

Using sets of clocks associated with an infinite series, we can frame the question of convergence to an irrational point. In Figure 7 the partial sum $1/4 + 1/9 + 1/16$ for z_2 is depicted using the original, un-rotated clocks. The radial OR generates a sector of this sums area and it doesn't intersect any of the points on the three circles. This means $1/4 + 1/9 + 1/16$ doesn't have a reduced form associated with CK_4, CK_9, or CK_{16}. If this is always true, i.e., if the radial for z_2, the infinite series, doesn't go through a lattice point and all the lattice points give all the possible rational areas, then z_2 is irrational.

Here is the criterion in two definitions and a theorem.

Definition 2. *Given an infinite series with positive, strictly decreasing terms of the form $1/a_j$, $a_j \in \mathbb{N}$, let the set of all points on CK_{a_j} be called the circular moduli lattice for the series. Designate this set with $CML\{a_j\}$.*

Definition 3. *For a given series with terms $1/a_j$, if there exists for every m/n, with $0 < m < n$, CK_r and modulus class s such that $s/r = m/n$ then the CML associated with the series, $CML\{a_j\}$ is said to cover the rational numbers.*

Theorem 1. *If $CML\{a_k\}$ covers the rational numbers and partial sums for the series are such that*

$$\sum_{k=2}^{n} \frac{1}{a_k} \in \mathbb{R}(0,1) \setminus \bigcup_{k=2}^{\varphi(n)} CK_{a_k}, \tag{3}$$

where $\varphi(n)$ is a natural number, strictly increasing function, then the series converges to an irrational number.

Proof. Using (3),

$$\lim_{n\to\infty} \mathbb{R}(0,1) \setminus \bigcup_{k=2}^{\varphi(n)} CK_{a_k} = \mathbb{H}(0,1),$$

where $\mathbb{R}(0,1)$ are the reals in $(0,1)$ and $\mathbb{H}(0,1)$ are the irrational numbers in $(0,1)$. We have used

$$\mathbb{R}(0,1) = \mathbb{Q}(0,1) \cup \mathbb{H}(0,1),$$

where $\mathbb{Q}(0,1)$ are the rational numbers in $(0,1)$.

Now we have

$$\lim_{n\to\infty} \sum_{k=2}^{n} \frac{1}{a_k} = \sum_{k=2}^{\infty} \frac{1}{a_k} \in \lim_{n\to\infty} \mathbb{R}(0,1) \setminus \bigcup_{k=2}^{\varphi(n)} CK_{a_k} = \mathbb{H}(0,1)$$

and this implies the series converges to an irrational.

□

Examples

The technique given in section 2 replaces interval endpoint exclusions with general, in effect, denominator exclusions. It's easy. We need to move away from the $\epsilon - \delta$ world of point set topology and analysis and use just sets without a defined metric. Here are two examples that motivate the idea.

Consider the task of proving the limit of $1 - 1/n$ is not of the form m/n, $0 < m < n$ with $n > 4$. That is we want to show the limit is an integer and not a fraction, of one class and not another. Now

$$1 - 1/n = \frac{n-1}{n} \in \bigcup_{k=1}^{\infty} CK_k \setminus \bigcup_{k=2}^{n-1} CK_k$$

and, as $(n-1, n) = 1$, $(n-1)/n$ is a reduced fraction,

$$\lim_{n\to\infty} \bigcup_{k=1}^{\infty} CK_k \setminus \bigcup_{k=2}^{n-1} CK_k = CK_1.$$

This implies that the limit is in CK_1; CK_1 only has 1 as a non-zero element.

Consider next a less trivial example. Suppose we want to show that $.\overline{1}$ base 4 converges to a denominator that does not have a denominator of a power of 4. Using clocks, the set

$$\bigcup_{k=1}^{n-1} CK_{4^k}$$

gives all positive finite decimals of length $n-1$ or less. So $.1, .11, \ldots, .\overline{1}_{n-1}$ are in this set, where the last decimal represents $n-1$ repeated 1 decimal digits. The following set

$$\bigcup_{k=2}^{\infty} CK_k$$

gives all finite decimals in the unit interval, $[0,1]$, in all bases. Observing

$$.\overline{1}_n \in \bigcup_{k=2}^{\infty} CK_k \setminus \bigcup_{k=1}^{n-1} CK_{4^k},$$

we can infer

$$\lim_{n\to\infty} \bigcup_{k=2}^{\infty} CK_k \setminus \bigcup_{k=1}^{n-1} CK_{4^k} = F \neq \emptyset. \qquad (4)$$

As, in the limit, all finite decimals in base 4 are exhausted, the convergence point must not be a finite decimal in base 4. Note: $.\overline{1}$ in base 4 converges to $1/3$ and (4) is consistent with this: $1/3 \in F$.

If, instead of $.\overline{1}$ base 4, we used $\sqrt{2}$ given as an infinite, non-repeating decimal, we would still get a non-empty set, so the test is not sufficient to show rationality. We could not infer (4) as the decimals are non-repeating and not all ones, violating the antecedents of Theorem 1. The criterion demands that the terms of the series be complete or cover the rationals; that is that the denominators of the terms, taken as bases, allow all rationals to be expressed as finite decimals within a base given by such denominators.

Irrationality proofs

There are two essential steps necessary to use Theorem 1. First, the denominators of the series must cover the rationals; second, the partials must reside in a set given by a set difference; and third, taking the limit of this set difference results in an empty set; this last result should be automatic. We illustrate these steps to show e and z_2 are irrational in this section.

Irrationality of $e-2$

The denominators cover the rationals: given reduced p/q with $p<q$, $p(q-1)!/q! \in CK_{q!}$. That's step one. Step two: we must find a strictly increasing function $\phi(n)$, per Theorem 1, such that

$$\sum_{k=2}^{n} \frac{1}{k!} \in \mathbb{R}(0,1) \setminus \bigcup_{k=2}^{\phi(n)} CK_{k!}.$$

We observe that

$$(n-1)! \sum_{k=2}^{n} \frac{1}{k!} = K + \frac{1}{n},$$

where K is a positive integer. This implies that

$$\sum_{k=2}^{n} \frac{1}{k!} \in \mathbb{R}(0,1) \setminus \bigcup_{k=2}^{n-1} CK_{k!}. \tag{5}$$

We can set $\phi(n)=n-1$ with $n>3$. Taking the limit in (5),

$$\sum_{k=2}^{n} \frac{1}{k!} \in \mathbb{H}(0,1),$$

implying that $e-2$ is irrational.

Irrationality of z_n

The denominators cover the rationals: all candidate rational numbers, p/q, can be written as $pq/q^2 \in CK_{q^2}$. That's step one. Step two: we must find a strictly increasing function $\phi(n)$, per Theorem 1, such that

$$\sum_{k=2}^{n} \frac{1}{k^2} \in \mathbb{R} \setminus \bigcup_{k=2}^{\phi(n)} CK_{k^2}.$$

We need to show that the partials for z_2, $\sum_2^n 1/k^2$, require greater than n^2 denominators, for example. We show this in the next section.

Assuming this for the moment, we have

$$\sum_{k=2}^{n} \frac{1}{k^2} \in \mathbb{R}(0,1) \setminus \bigcup_{k=2}^{n} CK_{k^2} \tag{6}$$

and taking the limit, we get $z_n \in \mathbb{H}(0,1)$: z_n is irrational.

z_n's partials

Our aim in this section is to show that the reduced fractions that give the partial sums of z_n require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums of z_n can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms. Lemma 1 is similar to Apostol's chapter 1, problem 30. See [5] for a solution to this problem.

We use the following notation: for integers n, $n > 1$, partial sums of z_n are given by

$$s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

Lemma 1. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{7}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (7) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. $\square$

Lemma 2. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k,p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^n}{(k!)^n}\sum_{j=2}^{k}\frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{8}$$

As $(k,p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. $\square$

Theorem 2. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$.*

Proof. Bertrand's postulate states that for any $k \geq 2$, there exists a prime p such that $k < p < 2k$ [4]. For even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Bertrand's postulate, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1 and 2, we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. $\square$

In light of this result we give the following definitions and corollary.

Definition 4.

$$D_{j^n} = \{0, 1/j^n, \ldots, (j^n-1)/j^n\} = \{0, .1, \ldots, .(j^n-1)\} \textit{ base } j^n$$

Definition 5.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

Corollary 1.

$$s_k^n \notin \Xi_k^n$$

Proof. Reduced fractions are unique. Suppose, to obtain a contradiction, that there exists $a/b \in \Xi_k^n$ such that $a/b = r/s$ then $b < s$ by Theorem 1. If a/b is not reduced, reduce it: $a/b = a_1/b_1$. A reduced fraction must have a smaller denominator than the unreduced form so $b_1 \leq b < s$ and this contradicts the uniqueness of the denominator of a reduced fraction. □

Counter-examples

It may be of interest to consider examples of where just one of the two criteria of Theorem 1 apply. The telescoping series below gives an example of when the terms of a series cover the rationals, but the partials don't escape these terms. The infinite geometric series given by $.\overline{1}$ base 4 gives an example of the terms not covering the rationals, but the partials do escape the terms.

Consider the telescoping series:

$$\frac{1}{2} = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \dots.$$

The terms of this series are of the form $1/n(n-1)$, $n > 1$. They cover the rationals: $p/q = p(q-1)/q(q-1) \in CK_{q(q-1)}$. Is there a $\phi(n)$ such that

$$\sum_{k=3}^{n} \frac{1}{k(k-1)} \in \mathbb{R} \setminus \bigcup_{k=3}^{\phi(n)} CK_{k(k-1)}?$$

If there was this series would give a counter-example. But the partials don't force an increasing function. Using upper limits of 3, 4, 5, and 6, the partials sum to 1/6, 1/4, 2/7, 1/3.

Consider the geometric series:

$$\sum_{k=1}^{\infty} \frac{1}{4^k} = .\overline{1} \text{ base } 4.$$

If $(4, q) = 1$ then $0 < p/q < 1$ can't be expressed as a finite decimal in base 4. This means the terms don't cover the rationals. But the partials do escape the terms:

$$\sum_{k=1}^{n} \frac{1}{4^k} \in \mathbb{R} \setminus \bigcup_{k=1}^{n-1} CK_{4^k}.$$

That is $.\overline{1}_n$ can't be expressed with less than n decimal places.

Conclusion

The set theory used in this article seems to not be the standard type. What does

$$\lim_{n\to\infty} \mathbb{R} \setminus \bigcup_{k=2}^{\varphi(n)} CK_{a_k} = \emptyset, \tag{9}$$

mean, if one insists on an epsilon/delta type idea? There doesn't seem to be any metric involved.

But, isn't it obvious: if one has a glass full of fluid and drains it to nothing, nothing is left. The oddness of the mathematics is that at any moment the number of fractions is countably infinite. How can it go from countably infinite to the empty set in a gradual way? But set theory does address this. The metric of number in a set gives the idea: in the finite domain, removing marbles of a certain number from a glass full of marbles, one can at any moment say how many remain to go. But this finite world is not the world of fluids or abstract numbers that remain infinitely divisible: an interval, no matter how small, will have a uncountable cardinality, for example. And yet $(0, 1/n)$ will drain to zero elements, as in the empty set, with increasing n.

Perhaps (9) requires an axiom in set theory: its true.

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