On the Ricci Scalar, Ricci Tensor and the Riemannian Curvature Tensor

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Abstract

The article in the first two sections proves decisively that the Ricci scalar and the norm of the Ricci tensor are constants on the manifold. In the subsequent sections Ricci tensor and Riemannian curvature tensor turn out to be null tensors. The Ricci scalar works out to zero.

Introduction

The Ricci scalar[1] and the norm of the Ricci tensor[2] are not only invariants but they are constants on a given manifold, independent of the space time coordinates. This idea is established in the initial stages of the article. Subsequent calculations show that the Ricci tensor and Riemannian curvature tensor are the null tensors. Consequently the Ricci scalar works out to zero.

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Ricci Scalar

First we write the Field equations[3][4]:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta} \ (1)$$

$R_{\alpha\beta}$:Ricci Tensor ;R Ricci Scalar ;

$g_{\alpha\beta}$:metric coefficients; $T_{\alpha\beta}$:stress energy tensor

$$g^{\alpha\beta} R_{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta} g^{\alpha\beta}$$

$$R - \frac{1}{2} R \times 4 = \frac{8\pi G}{c^4} T_{\alpha\beta} g^{\alpha\beta}$$

$$-R = \frac{8\pi G}{c^4} T_{\alpha\beta} g^{\alpha\beta} \ (2)$$

Equation (2) expresses a standard result

We differentiate each side of equation (2)with respect to $x^{\gamma}; \gamma = 0,1,2,3$

$$\frac{\partial R}{\partial x^{\gamma}} = -\frac{8\pi G}{c^4}\frac{\partial}{\partial x^{\gamma}}\left(T_{\alpha\beta}g^{\alpha\beta}\right)$$

We use the formula

$$\frac{\partial f}{\partial x^{i}} \equiv \nabla_i f$$

$$\nabla_\beta R = -\frac{8\pi G}{c^4}\nabla_\beta\left(g_{\alpha\beta}T^{\alpha\beta}\right) = -\frac{8\pi G}{c^4}\mathrm{T}^{\alpha\beta}\nabla_\beta g_{\alpha\beta} + g_{\alpha\beta}\nabla_\beta \mathrm{T}^{\alpha\beta}$$

$$\nabla_\beta R = 0; \beta = 0,1,2,3$$

$$-\frac{8\pi G}{c^4}\nabla_\beta\left(g_{\alpha\beta}T^{\alpha\beta}\right) = \frac{\partial R}{\partial x^{i}} = 0 \quad (3)$$

R is independent of space time coordinates.

<u>Detailed Explanation</u>

Covariant derivative of a scalar is equivalent to its partial derivative

We prove

$$\frac{\partial}{\partial x^{\gamma}}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = A_{\alpha\beta}\nabla_\gamma B^{\alpha\beta} + B_{\alpha\beta}\nabla_\gamma A_{\alpha\beta} \quad (4)$$

Proof:

We consider the following relations

$$\nabla_\gamma A^{\alpha\beta} = A^{\alpha\beta};_{\gamma} = \frac{\partial A^{\alpha\beta}}{\partial x^{\gamma}} + \Gamma_{\gamma s}{}^{\alpha}A^{s\beta} + \Gamma_{\gamma s}{}^{\beta}A^{\alpha s}$$

$$\nabla_\gamma B_{\alpha\beta} = B_{\alpha\beta};_{\gamma} = \frac{\partial B_{\alpha\beta}}{\partial x^{\gamma}} + \Gamma^{s}{}_{\gamma\alpha}B_{s\beta} + \Gamma^{s}{}_{\gamma\beta}B_{\alpha s}$$

[The above relations do not assume $A^{\alpha\beta}$ and $B_{\alpha\beta}$ as symmetric tensors]

We obtain,

$$\frac{\partial}{\partial x^{\gamma}}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = B_{\alpha\beta}\left(\nabla_\gamma A^{\alpha\beta} - \Gamma_{\gamma s}{}^{\alpha}A^{s\beta} - \Gamma_{\gamma s}{}^{\beta}A^{\alpha s}\right) + A^{\alpha\beta}\left(\nabla_\gamma B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha}B_{s\beta} + \Gamma^{s}{}_{\gamma\beta}B_{\alpha s}\right)$$

$$\frac{\partial}{\partial x^{\gamma}}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = B_{\alpha\beta}\left(-\Gamma_{\gamma s}{}^{\alpha}A^{s\beta} - \Gamma_{\gamma s}{}^{\beta}A^{\alpha s}\right) + A^{\alpha\beta}\left(\Gamma^{s}{}_{\gamma\alpha}B_{s\beta} + \Gamma^{s}{}_{\gamma\beta}B_{\alpha s}\right) + A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta}$$

$$= -\Gamma_{\gamma s}{}^{\alpha} g^{s\beta} B_{\alpha\beta} - \Gamma_{\gamma s}{}^{\beta} g^{\alpha s} B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha} A^{\alpha\beta} T_{s\beta} + \Gamma^{s}{}_{\gamma\beta} A^{\alpha\beta} B_{s\alpha} + A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta}$$

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = \left(-\Gamma_{\gamma s}{}^{\alpha} A^{s\beta} B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha} A^{\alpha\beta} B_{s\beta}\right) + \left(\Gamma^{s}{}_{\gamma\beta} A^{\alpha\beta} B_{\alpha s} - \Gamma_{\gamma s}{}^{\beta} A^{\alpha s} B_{\alpha\beta}\right) + A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta} \ (5)$$

[In the above α, s, β are dummy indices]

We work out the two parentheses separately.

With the second term in the first parenthesis to the right we interchange as follows

$$\alpha \leftrightarrow s$$

$$\left(-\Gamma_{\gamma s}{}^{\alpha} A^{s\beta} B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha} A^{\alpha\beta} B_{s\beta}\right) = \left(-\Gamma_{\gamma s}{}^{\alpha} A^{s\beta} T_{\alpha\beta} + \Gamma^{\alpha}{}_{\gamma s} A^{s\beta} B_{\alpha\beta}\right) = 0$$

We do not have to worry about reflections on the left side of (5)because alpha and beta on the left side also disappear on contraction.

Indeed recalling (5) and using the relation:$B_{\alpha\beta}A^{\alpha\beta} = B_{\mu\nu}A^{\mu\nu}$we may rewrite it [equation (5)] in the following form :

$$\frac{\partial}{\partial x^\gamma}\left(B_{\mu\nu}A^{\mu\nu}\right) = B_{\alpha\beta}\left(\nabla_\gamma A^{\alpha\beta} - \Gamma_{\gamma s}{}^{\alpha} A^{s\beta} - \Gamma_{\gamma s}{}^{\beta} A^{\alpha s}\right) + A^{\alpha\beta}\left(\nabla_\gamma B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha} B_{s\beta} + \Gamma^{s}{}_{\gamma\beta} B_{\alpha s}\right) + A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta}$$

There is no α, β on the left side of the above.

With the second term in the second parenthesis

$$\beta \leftrightarrow s$$

$$\left(\Gamma^{s}{}_{\gamma\beta} A^{\alpha\beta} B_{\alpha s} - \Gamma_{\gamma s}{}^{\beta} A^{\alpha s} B_{\alpha\beta}\right) = \left(\Gamma^{s}{}_{\gamma\beta} A^{\alpha\beta} B_{\alpha s} - \Gamma_{\gamma\beta}{}^{s} A^{\alpha\beta} B_{\alpha s}\right) = 0$$

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta} \ (6)$$

From(3) R is independent of space and time coordinates. Incidentally with the Einstein Hilbert Action[5] we do not treat the Ricci scalar as a constant: it depends on the space-time coordinates through the metric coefficients[implicit dependence].

Ricci Tensor

Recalling equation (1)

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

$$\Rightarrow \; R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{2} R R^{\alpha\beta} g_{\alpha\beta} = \frac{8\pi G}{c^4} R^{\alpha\beta} T_{\alpha\beta}$$

$$\Rightarrow R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{2} R^2 = \frac{8\pi G}{c^4} R^{\alpha\beta} T_{\alpha\beta}$$

$$\Rightarrow \; R^{\alpha\beta} R_{\alpha\beta} = \frac{8\pi G}{c^4} R^{\alpha\beta} T_{\alpha\beta} + \frac{1}{2} R^2$$

$$\frac{\partial}{\partial x^\beta}(R^{\alpha\beta} R_{\alpha\beta}) = \frac{8\pi G}{c^4} \frac{\partial}{\partial x^\beta}\left(R^{\alpha\beta} T_{\alpha\beta}\right) + \frac{1}{2} \frac{\partial}{\partial x^\beta}(R^2)$$

R being a constant

$$\frac{\partial}{\partial x^\beta}(R^2) = 0$$

Therefore

$$\frac{\partial}{\partial x^\beta}(R^{\alpha\beta} R_{\alpha\beta}) = \frac{8\pi G}{c^4} \frac{\partial}{\partial x^\beta}\left(R^{\alpha\beta} T_{\alpha\beta}\right)$$

$$\Rightarrow \nabla_\beta\left(R^{\alpha\beta} T_{\alpha\beta}\right) = \frac{8\pi G}{c^4}\left(R^{\alpha\beta} \nabla_\beta T_{\alpha\beta} + T^{\alpha\beta} \nabla_\beta R_{\alpha\beta}\right)$$

$$\Rightarrow \frac{\partial}{\partial x^i}\left(R^{\alpha\beta} T_{\alpha\beta}\right) = \frac{8\pi G}{c^4} R^{\alpha\beta} \nabla_\beta T_{\alpha\beta} + \frac{8\pi G}{c^4} T^{\alpha\beta} \nabla_\beta R_{\alpha\beta} \; (7)$$

Applying covariant differentiation on (1)and remembering that R is a constant and that $\nabla_\beta g^{\alpha\beta} = 0$

$\nabla_\beta T_{\alpha\beta} = 0$, we have ,

$$\nabla_\beta R_{\alpha\beta} - \frac{1}{2} R \nabla_\beta g_{\alpha\beta} = \frac{8\pi G}{c^4} \nabla_\beta T_{\alpha\beta}$$

$$\Rightarrow \nabla_\beta R_{\alpha\beta} = 0$$

Therefore from(7) we have,

$$\frac{\partial}{\partial x^\beta}(R^{\alpha\beta} R_{\alpha\beta}) = \frac{8\pi G}{c^4} \frac{\partial}{\partial x^\beta}\left(R^{\alpha\beta} T_{\alpha\beta}\right) = 0 \;\; (8)$$

Norm of the Ricci tensor is a constant on the manifold, independent of space and time coordinates..

Riemannian Curvature Tensor

We start with the consideration of the Riemannian curvature tensor, $R_{\alpha\beta\gamma\delta}$

$$2R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + R_{\alpha\beta\gamma\delta}$$

$$= R_{\alpha\beta\gamma\delta} + R_{\gamma\delta\alpha\beta}$$

$$= R_{\alpha\beta\gamma\delta} + R_{\delta\gamma\beta\alpha}$$

$$R_{\alpha\beta\gamma\delta} = \frac{R_{\alpha\beta\gamma\delta} + R_{\delta\gamma\beta\alpha}}{2} \quad (9)$$

$R_{\alpha\beta\gamma\delta}$is symmetric with respect to the interchange of both α, δ and β, γ[or with respect to the interchange of either]

Now

$$R^{k}{}_{\beta\gamma\delta} = g_{k\alpha} R^{k}{}_{\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}$$

$$R^{k}{}_{\gamma\beta\delta} = g_{k\alpha} R^{k}{}_{\gamma\beta\delta} = R_{\alpha\gamma\beta\delta}$$

But from (8)

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\gamma\beta\delta}$$

Therefore,

$$R^{k}{}_{\beta\gamma\delta} = R^{k}{}_{\gamma\beta\delta} \quad (10)$$

Since the product of a symmetric and an antisymmetric tensor is zero we have,

$$g^{p\gamma} R_{p\gamma\beta\delta} = 0 \quad (11)$$

Since $R_{p\gamma\beta\delta} = g_{\alpha p} R^{\alpha}{}_{\gamma\beta\delta}$

$$g^{p\gamma} g_{\alpha p} R^{\alpha}{}_{\gamma\beta\delta} = 0;$$

But by (10)

$$g^{p\gamma} g_{\alpha p} R^{\alpha}{}_{\beta\gamma\delta} = \delta^{\gamma}{}_{a} R^{\alpha}{}_{\beta\gamma\delta}$$

Therefore

$$g^{p\gamma} g_{\alpha p} R^{\alpha}{}_{\beta\gamma\delta} = 0 \Rightarrow \delta^{\gamma}{}_{a} R^{\alpha}{}_{\beta\gamma\delta} = 0$$

$$\Rightarrow R^{\gamma}{}_{\beta\gamma\delta} = 0$$

The Ricci Tensor,

$$R_{\beta\delta} = 0 \quad (12)$$

Ricci scalar

$$R = g^{\beta\delta} R_{\beta\delta} = 0$$

From the Field Equations,

$$T_{\beta\delta} = 0 \quad (13)$$

The Riemannian curvature tensor

Now we will consider the first Bianchi identity:

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$$

Using the symmetricity result proved towards the beginning of the section, we interchange $\delta$and β in the middle term on the left of the last equation. We also interchange δ, βin the third term

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\beta\delta\gamma} - R_{\alpha\gamma\beta\delta} = 0$$

Lastly we interchange γ, δin the second term and γ, β in the third term above [considering the symmetricity proved earlier for the third term]

$$R_{\alpha\beta\gamma\delta} - R_{\alpha\beta\gamma\delta} - R_{\alpha\beta\gamma\delta} = 0$$

$$\Rightarrow R_{\alpha\beta\gamma\delta} = 0 \quad (14)$$

The Riemannian tensor being zero, the Ricci tensor is also a null tensor and the Ricci scalar stands zero.

That the Ricci tensor is zero has been proved by an alternative method towards the beginning of the section.

Dot Product Preserving Transport

In parallel transport [6] dot product is preserved. We consider here a transport where dot product is preserved but the two vectors individually are not transported parallel to themselves

We have due to the preservation of dot product,

$$t^i \nabla_i \left(g_{\alpha\beta} u^\alpha v^\beta \right) = 0 \quad (15)$$

We have

$$t^i \nabla_i u^\alpha \neq 0; t^i \nabla_i v^\beta \neq 0 \quad (16)$$

since each vector is not transported parallel to itself.

We transform to a frame of reference where t^i has only one non zero component.

$t^{k'}\nabla_{k'}\left(g_{\alpha\beta}{}'u^{\alpha}{}'v^{\beta'}\right) = 0$[no summation on k': prime denotes the new frame of reference and not differentiation]

$$\nabla'_k\left(g_{\alpha\beta}{}'u^{\alpha}{}'v^{\beta}{}'\right) = 0 \ (17)$$

$$u^{\alpha}{}'v^{\beta}{}'\nabla_{i'}\left(g_{\alpha\beta}{}'\right) + g_{\alpha\beta}{}'\nabla_{i'}\left(u^{\alpha'}v^{\beta}{}'\right) = 0$$

Since $\nabla_i\left(g_{\alpha\beta}\right) = 0$,we have,

$$g_{\alpha\beta}{}'\nabla_i\left(u^{\alpha'}v^{\beta'}\right) = 0 \ \ (18)$$

The vectors $u^{\alpha'}$and $v^{\beta}{}'$ and consequently their individual components are arbitrary. Therefore $g'_{\alpha\beta} = 0 \Rightarrow g_{\alpha\beta} = 0$[the null tensor remains null in all frames of reference]That implies that the Riemann tensor , Ricci tensor and the Ricci scalar are all zero valued objects.

Conclusion

We have unexpected constants on the manifold like the Ricci Scalar and the norm of the Ricci Tensor. They are independent of the space time coordinates. They are not only invariants but they are also constants. Analysis of the Ricci tensor brings us to a result that stands in contradiction to a conventional beliefs. The article renders the fact that the Ricci tensor and the Riemannian curvature tensor are the null tensors. There is a requirement for restructuring the subject.

References

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