Refutation of the correspondence theory of Sahlqvist

Abstract: We evaluate the example of mapping the Sahlqvist formula of p ∧ ◊p → □p into corresponding quantified expressions. The formula is not a theorem, but the corresponding quantified expressions are theorems. Hence the mapping refutes the Sahlqvist correspondence theory. Therefore these failures are *non* tautologous fragments of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated proof value, **F** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables.
(See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓ , ·; \ Not And;
\> Imply, greater than, →, ⇒ , ↦, ≻, ⊃, ↠ ;
< Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇐⇒, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) `T` as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) `N` as non-contingency, Δ, ordinal 1;
(%z<#z) `C` as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y); (A=B) (A~B); (B>A) (A⊢B); (B>A) (A⊨B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: Conradie, W.; Palmigiano, A.; Sourabh, S. (2018). Algebraic modal correspondence: Sahlqvist and beyond. arxiv.org/pdf/1606.06881.pdf palmigiano.appliedlogictudelft@gmail.com, willem.conradie@wits.ac.za, sumit.sourabh@gmail.com

Example 3.12. Let us consider the very simple Sahlqvist formula p ∧ ◊p → □p, (3.12.1.1)

Remark 3.12.1.1: The antecedent is [[p ∧ □p]], as in the text for Eq. 3.12.3.1.

LET p, q, r, u, x, y, z: z1, z2, r, u, x, y, z.

(p&%p)>#p ; `TNTN TNTN TNTN TNTN` (3.12.1.2)

Remark 3.12.1.2: Eq. 3.12.1.2 as rendered is *not* tautologous. However, we evaluate the steps of the algebraic modal correspondence in Eqs. 3.12.2-.6 below.

which locally corresponds to the property of having at most one R-successor12, i.e.
∀z∀u(Rxz ∧ Rxu → z = u). (3.12.2.1)
((r&(x&#z)) & (r&(x&#u))) > (#z=#u) ; `TTTT TTTT TTTT TTTT` (3.12.2.2)

Remark 3.12.2.2: Eq. 3.12.1.1 is supposed to map to 3.12.2.1, however the respective truth table results show the *non* tautology of the former is transformed into the tautology of the latter.

This on it's face refutes the algebraic modal correspondence of Sahlqvist, because 3.12.2.2 should logically match 3.12.1.2. We continue evaluating the correspondence approach.

The variable p occurs twice positively in the antecedent, making $[[p \wedge \Box p]]$ a 2-additive map. Hence, according to our reduction strategy, the monadic second-order quantification in the second-order translation $\forall P[P(x) \wedge \exists y(Rxy \wedge P(y)) \rightarrow \forall u(Rxu \rightarrow P(u))]$ (3.12.3.1)

```
((#p&x)&((r&(x&%y))&(p&%y)))>((r&(x&#u))>(#p&#u)) ;
                                        TTTT TTTT TTTT TTTT      (3.12.3.2)
```

can be equivalently restricted to subsets of size at most 2. Doing this yields the equivalent L_0-formula $\forall z1\forall z2[(x = z1 \vee x = z2) \wedge \exists y(Rxy \wedge (y = z1 \vee y = z2)) \rightarrow \forall u(Rxu \rightarrow (u = z1 \vee u = z2))]$. (3.12.4.1)

```
(((x=#p)+(x=#q))&((r&(x&%y))&((%y=#p)+(%y=#q))))>((#x&(r&#u))>((#u=p)+(#u=q))) ;
                                        TTTT TTTT TTTT TTTT      (3.12.4.2)
```

This can be simplified to
$\forall z1\forall z2[(x = z1 \vee x = z2) \wedge (Rxz1 \vee Rxz2)) \rightarrow \forall u(xRu \rightarrow (u = z1 \vee u = z2))]$, (3.12.5.1)

```
(((x=#p)+(x=#q))&((r&(x&#p))+(r&(x&#q))))>((#x&(r&#u))>((#u=p)+(#u=q))) ;
                                        TTTT TTTT TTTT TTTT      (3.12.5.2)
```

and reasoning a bit further this can be seen to be equivalent to
$\forall z1\forall z2[(Rxz1 \vee Rxz2)) \rightarrow \forall u(Rxu \rightarrow (u = z1 \vee u = z2))]$, (3.12.6.1)

```
(r&(x&#p))&(r&(x&#q)))>((r&(x&#u))>((#u=#p)+(#u=#q))) ;
                                        TTTT TTTT TTTT TTTT      (3.12.6.2)
```

which, in turn, is equivalent to $\forall z\forall u(Rxz \wedge Rxu \rightarrow z = u)$. (3.12.2.1)

Eqs. 3.12.2.2-6.2 are not equivalent to 3.12.1.2. This means the approach fails to map a modal correspondence as claimed. We again note that "the very simple Sahlqvist formula $p \wedge \Diamond p \rightarrow \Box p$" is not a theorem as the beginning conjecture.