Expanding Polynomials with Regular Polygons

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Abstract

Expanding the root form of a polynomial for large numbers of roots can be complicated. Such polynomials can be used to prove the irrationality of powers of π, so a technique for arriving at expanded forms is needed. We show here how roots of polynomials can generate regular polygons whose vertices considered as roots form known expanded polynomials. The product of these polynomials can be simple enough to yield the desired expanded form.

1 Introduction

The author of a recent paper on irrationality and transcendence proofs posed a challenge: use $z^5 - (\pi i)^5$ to show π^5 is irrational. In the paper the author shows the $j = 1$, 2, 3, and 4 cases of proving π^j is irrational. The $j = 5$ case is more difficult as a polynomial with as many as thirty-one roots needs to be generated. These thirty-one roots are derived from

$$\prod_{k=1}^{5}(1 + e^{r_k}), \tag{1}$$

where r_k, $k = 1 \ldots 5$, are the roots of $z^5 - (\pi i)^5$. These new exponents, designated by R_k, will be given by the sums of the r_k roots taken one through 5 at a time: thirty-one roots. To follow the pattern as in the earlier cases we need to find these roots and plot their coordinates. We will see that six pentagons are formed. The polynomials associated with each of these pentagons are easily discerned. The product of such polynomials reduces nicely to the expanded form desired. These tasks can be done with a spreadsheet such as Excel. We show this here.

2 The r_k roots

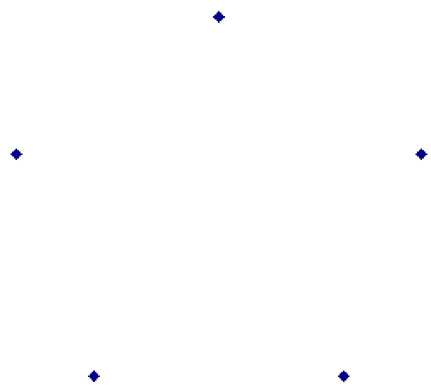

Figure 1: The roots form the vertices of a regular pentagon.

H4 | fx =COS(RADIANS(C4))

	B	C	D	E	F	G	H	I
1							cosine values	sine values
2	1	18	a				0.951056516	0.309016994
3	2	90	b				0	1
4	3	162	c				-0.951056516	0.309016994
5	4	234	d				-0.587785252	-0.809016994
6	5	306	e				0.587785252	-0.809016994

Figure 2: Excel calculates the coordinates for the vertices of the r_k roots.

The polynomial $z^j - (\pi i)^j$ will have j roots. The constant π^j can be ignored for the time being. The roots, then, of $g(z) = z^j - i^j$ depend upon $j \mod 4$. Our particular case gives the five roots:

$$[\cos(\pi/2 + 2k\pi) + i \sin(\pi/2 + 2k\pi)]^{1/5} \tag{2}$$

with $1 \leq k \leq 5$. These roots, designated as r_k, form a regular pentagon, see Figure 1, inscribed in a unit circle; using Excel numerical values for these five roots can be calculated; the results of such calculation are given in Figure 2.

3 The R_k roots

J16 =IMSUM(INDIRECT(TEXT(C16,"")),INDIRECT(TEXT(D16,"")))

	A	B	C	D	E	F	G	H	I	J	K	L
1								cosine values	sine values	complex sum	length	
2	1	1	18	a				0.951056516	0.309016994	0.95105651629	1	
3	2	2	90	b				0	1	i	1	
4	3	3	162	c				-0.951056516	0.309016994	-0.95105651629	1	
5	4	4	234	d				-0.587785252	-0.809016994	-0.58778525229	1	
6	5	5	306	e				0.587785252	-0.809016994	0.58778525229	1	
7	6	1	a	b				0.951056516	1.309016994	0.95105651629	1.618	
8	7	2	a	cc				0	0.618033989	0.61803398874	0.618	
9	8	3	a	d				0.363271264	-0.5	0.36327126400	0.618	
10	9	4	a	e				1.538841769	-0.5	1.53884176858	1.618	
11	10	5	b	cc				-0.951056516	1.309016994	-0.95105651629	1.618	
12	11	6	b	d				-0.587785252	0.190983006	-0.58778525229	0.618	
13	12	7	b	e				0.587785252	0.190983006	0.58778525229	0.618	
14	13	8	cc	d				-1.538841769	-0.5	-1.53884176858	1.618	
15	14	9	cc	e				-0.363271264	-0.5	-0.36327126400	0.618	
16	15	10	d	e				0	-1.618033989	-1.61803398874	1.618	
17	16	1	a	b	cc			0	1.618033989	1.61803398874	1.618	
18	17	2	a	b	d			0.363271264	0.5	0.36327126400	0.618	
19	18	3	a	b	e			1.538841769	0.5	1.53884176858	1.618	
20	19	4	a	cc	d			-0.587785252	-0.190983006	-0.58778525229	0.618	
21	20	5	a	cc	e			0.587785252	-0.190983006	0.58778525229	0.618	
22	21	6	a	d	e			0.951056516	-1.309016994	0.95105651629	1.618	
23	22	7	b	cc	d			-1.538841769	0.5	-1.53884176858	1.618	
24	23	8	b	cc	e			-0.363271264	0.5	-0.36327126400	0.618	
25	24	9	b	d	e			0	-0.618033989	-0.61803398874	0.618	
26	25	10	cc	d	e			-0.951056516	-1.309016994	-0.95105651629	1.618	
27	26	1	a	b	cc	d		-0.587785252	0.809016994	-0.58778525229	1	
28	27	2	a	b	cc	e		0.587785252	0.809016994	0.58778525229	1	
29	28	3	a	b	d	e		0.951056516	-0.309016994	0.95105651629	1	
30	29	4	a	cc	d	e		0	-1	-i	1	
31	30	5	b	cc	d	e		-0.951056516	-0.309016994	-0.95105651629	1	
32	31	1	a	b	cc	d	e	0	0	-1.11022302462	0	

Figure 3: The formula for cell J16 shows Excel functions giving the sum of the roots named "d" and "e".

Per the transcendence proof for π, we need to find the exponents of e in

$$\prod_{k=1}^{5}(1+e^{r_k}). \tag{3}$$

These exponents are designated with R_k. As multiplication goes to addition in exponentiation, we need to find all the sums of r_k roots taken one through five at a time. We can use Excel to do this in a particularly elegant way. The technique uses a combination of the indirect function and cell naming: our r_k roots are named a, b, cc, d, and e; these references are used in a list of all the combinations of roots. The number of R_k roots is a function of the number of r_k roots, in our case we have $2^5 - 1 = 31$ roots. The reasoning is the same as the more familiar sums of products taken one through n at a time in forming polynomial coefficients. These roots are given in Figure 3.

4 Plotting the roots

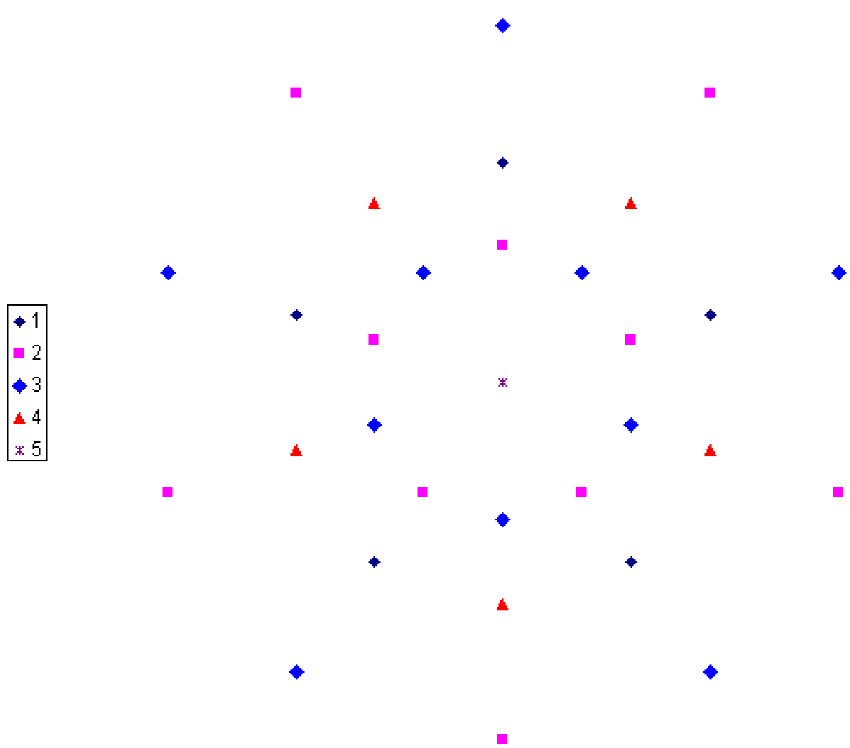

Figure 4: The R_k roots shown are r_k roots summed one through five at a time.

It is an easy matter to plot these roots using Excel's scatter plot graphing capability. We seek to find patterns in the regular pentagons that are formed. This can be done by using different series for each of the five natural groups of the roots: one at a time, two at a time, etc.. Figure 4 shows all roots and Figure 5 shows the three decagons that are formed from the six pentagons generated. One root is zero. Note: Excel by default uses different units for the x and y axis. This can be adjusted with a macro.

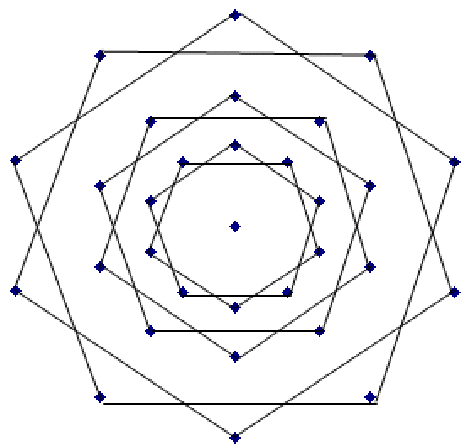

Figure 5: Connecting the roots yields three regular decagons.

5 Expanding the polynomial

If one attempted to expand

$$\prod_{k=1}^{30}(z - R_k), \tag{4}$$

in Maple, say, the task would be laborious and the results uncertain. The author tried such using Maple and was unable to obtain an answer. Can we use the six pentagons formed from the roots to phantom the expanded form of this polynomial? The answer is yes.

The six pentagons have vertices that are the roots of five degree polynomials of the form

$$z^5 - (ri)^5, \tag{5}$$

where r is the radius of the circle that inscribes each pentagon. We need to find the radius for each of the three cases. One case is simple: the radius is 1. The other two cases are

$$\frac{\sqrt{5}+1}{2} \text{ and } \frac{\sqrt{5}-1}{2}, \tag{6}$$

the first is known as the golden mean or ratio. Multiplying pairs of these pentagons that have the same radius yields three polynomials of degree ten. Although these are not difficult hand calculations, using Maple's expand function the result can be built up in three phases: form the polynomials associated with the decagons and multiply all of them. The final result is given, via Maple's expand function, in Figure 6.

```
expand(%);
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$$z^{30} + 124\, z^{20}\, \pi^{10} + 124\, z^{10}\, \pi^{20} + \pi^{30}$$

Figure 6: The desired expanded form.

References

[1] P. Eymard and J.-P. Lafon, *The Number* π, American Mathematical Society, Providence, RI, 2004.

[2] E. W. Hobson, *Squaring the Circle: A History of the Problem*, Merchant Books, 2007.

[3] T.W. Jones, Discovering and proving that π is irrational, this Monthly, **117** (2010) 553–557.

[4] F. Lindemann, Über die Zahl π, *Mathematische Annalen* **20** (1882) 213–225.

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