The Inconsistency of Arithmetic

Ralf Wüsthofen [1]

Abstract. Based on a strengthened form of the strong Goldbach conjecture, this paper presents an antinomy within the Peano arithmetic (PA). We derive two contradictory statements by using the same main instrument as in the proof [2] of the conjecture, i.e. a set that is a structuring of the natural numbers starting from 3.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Theorem. *The Peano arithmetic (PA) is inconsistent.*

Proof. We define the set

$$S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$$

and we consider the following two cases.

(G) The numbers m in the components mk take all integer values x ≥ 4.
¬(G) The numbers m in the components mk do not take all integer values x ≥ 4.

For each k ≥ 1, let S_g (m,k) denote the set of the middle components mk of the S_g triples. Then, by definition

(G) <=> $S_g (m,k) = k\mathbb{N}_4$ for every k ≥ 1
¬(G) <=> $S_g (m,k) \neq k\mathbb{N}_4$ for every k ≥ 1.

This implies that S_g does not contain the same triples in the cases (G) and ¬(G):

(I) ∃ sets S, S' such that S ≠ S' and (((G) => S_g = S) and (¬(G) => S_g = S')).

On the other hand, the case ¬(G) means that for each k ≥ 1 there is an nk, n ≥ 4, different from all the mk, where all pairs (p, q) of odd primes, that determine the numbers m, are used in S_g. For each k ≥ 1, such an nk can be written as some pk when n is prime, as some pk' when n is composite and not a power of 2, or as 4k' when n is a power of 2; p ∈ $\mathbb{P}_3$; k, k' ∈ $\mathbb{N}$.

[1] rwuesthofen@gmail.com

[2] http://vixra.org/abs/1702.0300

The expression pk' for nk with k' = k or k' ≠ k is a first component of S_g triples and the expression 4k' for nk is component of the triple (3k', 4k', 5k'). So, since nk equals some triple component pk' or 4k' that exists by definition of S_g, the S_g triples are the same in the case nk exists and in the case nk does not exist.

In other words, the S_g triples are always the same, regardless of whether nk as a component of them exists or not. Therefore, we obtain the contradiction that S_g contains the same triples in the cases (G) and ¬(G):

(∃ sets S, S' such that (((G) => S_g = S) and (¬(G) => S_g = S')))   =>   S = S'

<=>

(II) ∄ sets S, S' such that S ≠ S' and (((G) => S_g = S) and (¬(G) => S_g = S')).   □

The statement (II) is built on two properties of S_g, namely that nk, given by the case ¬(G), for each k ≥ 1 can be expressed by a S_g triple component and that nk, k = 1, cannot be the arithmetic mean of a pair of odd primes not used in S_g. We call these two properties of S_g '*covering*' and '*maximality*'. Without them, we could establish only the statement (I) and there would be no contradiction.

The proof uses a strengthened form of the strong Goldbach conjecture:

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

SSGB is equivalent to saying that all integers x ≥ 4 appear as m in a component mk of S_g. Therefore, SSGB is equivalent to the case (G) and the negation ¬SSGB is equivalent to the case ¬(G). We have seen above that the S_g triples are the same in these two cases. This means that both SSGB and ¬SSGB hold.