Riemann hypothesis

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Let's consider $s = a + ib$ with $a, b \in \mathbb{R}$, we have :

$$\left(\frac{1}{n^a}+\frac{1}{n^{ib}}\right)^2 = \frac{1}{n^{2a}} + \frac{1}{n^{2ib}} + \frac{2}{n^{a+ib}} \iff \frac{1}{n^{a+ib}} = \frac{1}{2}\left(\left(\frac{1}{n^a}+\frac{1}{n^{ib}}\right)^2 - \frac{1}{n^{2a}} - \frac{1}{n^{2ib}}\right)$$

we conclude that: $$\sum_{n=1}^{+\infty} \frac{1}{n^s} = \frac{1}{2}\left(\sum_{n=1}^{+\infty}\left(\frac{1}{n^a}+\frac{1}{n^{ib}}\right)^2 - \sum_{n=1}^{+\infty}\frac{1}{n^{2a}} - \sum_{n=1}^{+\infty}\frac{1}{n^{2ib}}\right)$$

Consequently: $$\sum_{n=1}^{+\infty}\frac{1}{n^s} = 0 \iff \sum_{n=1}^{+\infty}\left(\frac{1}{n^a}+\frac{1}{n^{ib}}\right)^2 - \sum_{n=1}^{+\infty}\frac{1}{n^{2ib}} = \sum_{n=1}^{+\infty}\frac{1}{n^{2a}}$$

$$\iff \sum_{n=1}^{+\infty}\left(\left(\frac{1}{n^a}+\frac{1}{n^{ib}}\right)^2 - \left(\frac{1}{n^{ib}}\right)^2\right) = \sum_{n=1}^{+\infty}\frac{1}{n^{2a}}$$

$$\iff \sum_{n=1}^{+\infty}\left(\left(\frac{1}{n^a}+\frac{2}{n^{ib}}\right)\cdot\frac{1}{n^a}\right) = \sum_{n=1}^{+\infty}\frac{1}{n^{2a}}$$

II

* We have the equation : $\sum_{n=1}^{+\infty}\left(\left(\frac{1}{n^a}+\frac{2}{n^{ib}}\right)\cdot\frac{1}{n^a}\right)=\sum_{n=1}^{+\infty}\frac{1}{n^{2a}}$

* we know that: $\sum_{n=1}^{m}\frac{1}{n^{2a}}\in\mathbb{R}\quad\forall m,n\in\mathbb{N}^*$, so even if $\sum_{n=1}^{+\infty}\frac{1}{n^{2a}}=+\infty$, the imaginary part of $\sum_{n=1}^{+\infty}\left(\left(\frac{1}{n^a}+\frac{2}{n^{ib}}\right)\cdot\frac{1}{n^a}\right)$ equals zero, which means: $\sum_{n=1}^{+\infty}e^{-a\ln n}\times\sin(b\ln n)=0$

we have also: $\sum_{n=1}^{+\infty}e^{-a\ln n}\times\sin(b\ln n)=0\Rightarrow\lim_{n\to+\infty}e^{-a\ln n}\times\sin(b\ln n)=0$

$\Rightarrow\lim_{n\to+\infty}e^{-a\ln n}=0\Rightarrow a>0$

We proved that: $\sum_{n=1}^{+\infty}\frac{1}{n^{a+ib}}=0\Rightarrow a>0\Rightarrow\lim_{n\to+\infty}\frac{1}{n^a}=0$

we have also:

$\sum_{n=1}^{+\infty}e^{-a\ln n}\times\sin(b\ln n)=0\Rightarrow\sum_{n=1}^{+\infty}e^{-a\ln n+ib\ln n}=\sum_{n=1}^{+\infty}e^{-a\ln n-ib\ln n}=0$

$\Rightarrow\sum_{n=1}^{+\infty}e^{-a\ln n}\cdot\cos(b\ln n)=0$

since: $\sin(b\ln n)=\frac{e^{ib\ln n}-e^{-ib\ln n}}{2i}$

* If $\sum_{n=1}^{+\infty}\left(\left(\frac{1}{n^a}+\frac{2}{n^{ib}}\right)\cdot\frac{1}{n^a}\right)$ diverges than this series should never respect Abel's criterion and since $\lim_{n\to+\infty}\frac{1}{n^a}=0$ than we should have:

$a\leqslant\frac{1}{2}$ and $\exists\, l,m\in\mathbb{N}^*,\ \sum_{n=l}^{m}\left|\frac{1}{n^a}+\frac{2}{n^{ib}}\right|=+\infty$

we have $\left|\frac{1}{n^a}+\frac{2}{n^{ib}}\right|\leqslant\left|\frac{1}{n^a}\right|+2\left|\frac{1}{n^{ib}}\right|\leqslant 3$

hence $\sum_{n=l}^{m}\left|\frac{1}{n^a}+\frac{2}{n^{ib}}\right|=+\infty\Rightarrow\sum_{n=l}^{m}3=3(m-l+1)=+\infty$