Refutation of ordinal notation via simultaneous definition

Abstract: We evaluate five definitions as *not* tautologous. This refutes the conjecture that inductive-recursive definitions can give rise to ordinal notation systems that uniquely represent ordinals. Hence the definitions and conjecture are *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated proof value, **F** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓ , ·; \ Not And;
> Imply, greater than, →, ⇒ , ↦, ≻, ⊃, ↠ ;
< Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇐⇒, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) `T` as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) `N` as non-contingency, Δ, ordinal 1;
(%z<#z) `C` as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y); (A=B) (A~B); (B>A) (A⊢B); (B>A) (A⊨B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: Forsberg, F.N.; Xu, C. (2019). Ordinal notations via simultaneous definitions. arxiv.org/pdf/1904.10759.pdf fredrik.nordvall-forsberg@strath.ac.uk xu@math.lmu.de

1.1 Set-theoretic ordinals

$\alpha \cdot \beta$ is defined by transfinite recursion on β:

$\alpha \cdot (\beta + 1) \equiv \alpha \cdot \beta + \alpha$ (1.1.1.1)

LET p, q: a, b
(p&(q+(%s>#s)))=((p&q)+p) ; `TNTT TNTT TNTT TNTT` (1.1.1.2)

Def. 2:
The relation < on O is inductively defined by the following clauses:

(<1) If a ≠ 0 then 0 < a. (1.1.2.1)

(p@(s@s))>((s@s)<p) ; `T`**F**`T`**F** `T`**F**`T`**F** `T`**F**`T`**F** `T`**F**`T`**F** (1.1.2.2)

Def. 7:
Subtraction of O is defined as follows:

$0 - b \equiv 0$ (1.1.7.1.1)

((s@s)-q)=(s@s) ; **FF**TT **FF**TT **FF**TT **FF**TT (1.1.7.1.2)

a − 0 ≡ a (1.1.7.2.1)

(p-(s@s))=p ; **FFFF FFFF FFFF FFFF** (1.1.7.2.2)

4 The type-theoretic development of ordinal notations in Agda

_≥_: O → O → Set

a ≥ b = (b < a) ∨ (a ≡ b) (4.1)

(~q>p)=((q<p)+(p=q)) ; **FF**TT **FF**TT **FF**TT **FF**TT (4.2)

5 Concluding discussions

Hence we conjecture that actual use of inductive-recursive definitions can give rise to ordinal notation systems that uniquely represents ordinals …

Eqs. 1.1.1.2, ..2.2, 1.1.7.1.2, ..7.2.2, and 4.2 are *not* tautologous. This refutes the conjecture that inductive-recursive definitions can give rise to ordinal notation systems that uniquely represent ordinals.