

# Proof that there are no odd perfect numbers

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## 1. Abstract

For  $y$  to be a perfect number, if one of the prime factors is  $p$ , the exponent of  $p$  is an integer  $n(n \geq 1)$ , the prime factors other than  $p$  are  $p_1, p_2, p_3, \dots, p_r$  and the even exponent of  $p_k$  is  $q_k$ ,

$$y/p^n = (1 + p + p^2 + \dots + p^n) \prod_{k=1}^r (1 + p_k + p_k^2 + \dots + p_k^{q_k}) / (2p^n) = \prod_{k=1}^r p_k^{q_k}$$

must be satisfied. Let  $m$  and  $q$  be non-negative integers,

$$n = 4m + 1$$

$$p = 4q + 1$$

Letting  $b$  and  $c$  be odd integers, satisfying following expressions,

$$b = \prod_{k=1}^r p_k^{q_k}$$

$$c = \prod_{k=1}^r (1 + p_k + p_k^2 + \dots + p_k^{q_k}) / p^n$$

$$2b = c(p^n + \dots + 1)$$

is established. This is a known content. By the consideration of this research paper, since it turns out that there are no solutions that satisfy the condition of this problem in  $n$  order equation of  $p$ , we have obtained the conclusion that there are no odd perfect numbers.

## 2. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself, and the smallest perfect number is

$$1 + 2 + 3 = 6$$

It is 6. Whether an odd perfect number exists or not is currently an unsolved problem.

### 3. Proof

An odd perfect number is  $y$ , one of them is an odd prime number  $p$ , an exponent of  $p$  is an integer  $n$  ( $n \geq 1$ ). Let  $p_1, p_2, p_3, \dots, p_r$  be the odd prime numbers of factors other than  $p$ ,  $q_k$  the index of  $p_k$ , and variable  $a$  be the sum of product combinations other than prime  $p$ .

$$a = \prod_{k=1}^r (1 + p_k + p_k^2 + \dots + p_k^{q_k}) \dots \textcircled{1}$$

The number of terms  $N$  of variable  $a$  is

$$N = \prod_{k=1}^r (q_k + 1) \dots \textcircled{2}$$

When  $y$  is a perfect number,

$$y = a(1 + p + p^2 + \dots + p^n) - y \quad (n > 0)$$

is established.

$$a \sum_{k=0}^n p^k / 2 = y$$

$$a \sum_{k=0}^n p^k / (2p^n) = y/p^n \dots \textcircled{3}$$

#### 3.1. If $q_k$ has at least one odd integer

Letting the number of terms where  $q_k$  is an odd integer be a positive integer  $u$ , because  $y/p^n = \prod_{k=1}^r p_k^{q_k}$  is an odd integer, the denominator on the left side of expression  $\textcircled{3}$  has a prime factor 2, from expression  $\textcircled{2}$  variable  $a$  has more than  $u$  prime factor 2 and variable  $a$  is an even integer. Therefore  $\sum_{k=0}^n p^k$  must be an odd integer,  $n$  is an even integer and  $u$  is 1.

#### 3.2. When all $q_k$ are even integers

$y/p^n$  is an odd integer, the denominator on the left side of expression  $\textcircled{3}$  is an even integer, and since  $N$  is an odd integer when  $q_k$  are all even integers, variable  $a$  is an odd integer. Therefore  $\sum_{k=0}^n p^k$  is necessary to include one prime factor 2,  $\sum_{k=0}^n p^k \equiv 0 \pmod{2}$  is established, and  $n$  must be an odd integer.

From 3.1, 3.2, in order to have an odd perfect number, only one exponent of the prime factor of  $y$  must be an odd integer and variable  $a$  must be an odd integer. We consider the case of 3.2 below.

In order for  $y$  to be a perfect number, the following expression must be established.

$$y/p^n = (1 + p + p^2 + \dots + p^n) \prod_{k=1}^r (1 + p_k + p_k^2 + \dots + p_k^{q_k}) / (2p^n) = \prod_{k=1}^r p_k^{q_k}$$

However,  $q_1, q_2, \dots, q_r$  are all even integers.

Here, let  $b$  be an integer

$$b = \prod_{k=1}^r p_k^{q_k} \dots \textcircled{4}$$

A following expression is established.

$$y/p^n = a(1 + p + p^2 + \dots + p^n) / (2p^n) = b$$

$$a(p^{n+1} - 1) / (2(p - 1)p^n) = b$$

$$(a - 2b)p^{n+1} + 2bp^n - a = 0 \dots \textcircled{5}$$

Because it is an  $n + 1$  order equation of  $p$ , the solution of the odd prime  $p$  is  $n + 1$  at most.

$$(ap - 2bp + 2b)p^n = a$$

Since  $ap - 2bp + 2b$  is an odd integer,  $a/p^n$  is an odd integer, which is  $c$ .

$$ap - 2bp + 2b = c \ (c > 0) \dots \textcircled{6}$$

$$(2b - a)p = 2b - c$$

Since variable  $a$  is an odd integer,  $2b - a$  is an odd integer and  $2b - a \neq 0$

$$p = (2b - c) / (2b - a)$$

Since  $n \geq 1$

$$a - c = cp^n - c \geq cp - c > 0$$

$$a > c$$

is.

From equation ⑥

$$2b(p - 1) - (ap - c) = 0$$

$$2b - c(p^{n+1} - 1)/(p - 1) = 0$$

$(p^n + \dots + 1)/2$  is an odd integer,  $n = 4m + 1$  is required with  $m$  as an integer.

$$2b(p - 1) = c(p^{n+1} - 1)$$

$$2b = c(p^n + \dots + 1)$$

$$2b = c(p + 1)(p^{n-1} + p^{n-3} + \dots + 1) \dots \textcircled{7}$$

$b$  is an odd integer when  $p + 1$  is not a multiple of 4. It is necessary that  $p - 1$  be a multiple of 4. A positive integer is taken as  $q$ .

$$p = 4q + 1$$

is established.

When  $p > 1$

$$p^n - 1 < p^n$$

$$(p^n - 1)/(p - 1) < p^n/(p - 1)$$

$$p^{n-1} + \dots + 1 < p^n/(p - 1) \dots \textcircled{8}$$

Since  $p$  is an odd prime number satisfying  $p = 4q + 1$  and  $p \geq 5$

$$p^{n-1} + \dots + 1 < p^n/4$$

$$2b - a = c(p^n + \dots + 1) - cp^n = c(p^{n-1} + \dots + 1)$$

$$2b - a < cp^n/4 = a/4$$

$$2b < 5a/4$$

$$a > 8b/5 \dots \textcircled{9}$$

Let  $a_k$  and  $b_k$  be integers and if

$$a_k = 1 + p_k + p_k^2 + \dots + p_k^{q_k}, \quad b_k = p_k^{q_k},$$

$$a_k - b_k < b_k/(p_k - 1)$$

$$a_k < b_k p_k/(p_k - 1)$$

$$a = \prod_{k=1}^r a_k < \prod_{k=1}^r b_k p_k/(p_k - 1) = b \prod_{k=1}^r p_k/(p_k - 1)$$

$$a/b < \prod_{k=1}^r p_k/(p_k - 1)$$

When  $r = 1$ , since  $a/b < 3/2$  is established, it becomes inappropriate contrary to inequality ⑨.

From expression ⑦,

$$b = c(p + 1)/2 \times (p^{n-1} + p^{n-3} + \dots + 1)$$

holds. Since  $(p + 1)/2$  is the product of only prime numbers of  $b$ , let  $d_k$  be the index,

$$(p + 1)/2 = \prod_{k=1}^r p_k^{d_k}$$

$$p = 2 \prod_{k=1}^r p_k^{d_k} - 1$$

From  $a = cp^n$  and expression ⑦,

$$2bp^n = a(p^n + \dots + 1)$$

$$a(p^n + \dots + 1)/(2bp^n) = 1 \dots (A)$$

When  $r = 1$ ,

$$a = (p_1^{q_1+1} - 1)/(p_1 - 1)$$

$$b = p_1^{q_1}$$

Equation (A) does not hold since there is no odd perfect number when  $r = 1$ .

Let R be a rational number,

$$R = a(p^n + \dots + 1)/(2bp^n)$$

Let b' be a rational number and let A and B to be an integer,

$$b' = (p_k^{q_k+1} - 1)/(p_k^{q_k}(p_k - 1)) > 1$$

$$A = (p_k^{q_k+1} - 1)/(p_k - 1)$$

$$B = p_k^{q_k}$$

Multiplying R by b', there are both cases that p<sub>k</sub> increases p or does not change.

When multiplied by b', the rate of change of R is  $Ap^n(p'^n + \dots + 1)/(Bp'^n(p^n + \dots + 1))$ , if p after variation is p'. If the rate of change of R is 1,

$$Ap^n(p'^n + \dots + 1)/(Bp'^n(p^n + \dots + 1)) = 1$$

$$Ap^n(p'^n + \dots + 1) = Bp'^n(p^n + \dots + 1)$$

This expression does not hold, since the right side is not a multiple of p when p' > p, and A > B holds when p' = p. Due to this operation, R may be larger or smaller than the original value, since the rate of change of R does not become 1.

From  $R \neq 1$  and  $a = cp^n$  for some r, also multiplying fractions  $b' = A_1/B_1$ ,  $b'' = A_2/B_2$ ,  $\dots b'''\dots' = A_x/B_x$ , if  $R = 1$  holds finally,

$$, \text{ if } a(p^n + \dots + 1)/(2bp^n) \times A_1p^n(p_1^n + \dots + 1)/(B_1p_1^n(p^n + \dots + 1)) \times A_2p_1^n(p_2^n + \dots + 1)/(B_2p_2^n(p_1^n + \dots + 1)) \dots A_xp_{x-1}^n(p_x^n + \dots + 1)/(B_xp_x^n(p_{x-1}^n + \dots + 1)) = 1$$

$$a/(2b) \times A_1/B_1 \times A_2/B_2 \dots A_x(p_x^n + \dots + 1)/(B_xp_x^n) = 1$$

$$a(p_x^n + \dots + 1)A_1A_2 \dots A_x = 2bp_x^nB_1B_2 \dots B_x$$

$$cp^n(p_x^n + \dots + 1)A_1A_2 \dots A_x = 2bp_x^nB_1B_2 \dots B_x$$

When  $p_x > p$ , it becomes inconsistent since the right side of this expression does not include p as a factor.

When  $p_x = p$ ,

$$cp^n(p^n + \dots + 1)A_1A_2 \dots A_x = c(p^n + \dots + 1)p^n$$

$$A_1A_2 \dots A_x = 1$$

It becomes contradiction, since this expression is not established. Therefore,  $a = cp^n$  holds at one point where  $R = 1$ .

Assuming that  $R = 1$  in some  $r$  by multiplying fractions  $b' = A_1/B_1$ ,  $b'' = A_2/B_2$ ,  $\dots$   
 $b'''\dots' = A_x/B_x$ , if  $R = 1$  holds,

$$1 \times A_1 p^n (p_1^n + \dots + 1) / (B_1 p_1^n (p^n + \dots + 1)) \times A_2 p_1^n (p_2^n + \dots + 1) / (B_2 p_2^n (p_1^n + \dots + 1)) \dots A_x p_{x-1}^n (p_x^n + \dots + 1) / (B_x p_x^n (p_{x-1}^n + \dots + 1)) = 1$$

$$A_1 A_2 \dots A_x p^n (p_x^n + \dots + 1) = B_1 B_2 \dots B_x p_x^n (p^n + \dots + 1)$$

When  $p_x > p$ , it becomes inconsistent since the right side of this expression does not include  $p$  as a factor.

When  $p_x = p$ ,

$$A_1 A_2 \dots A_x = B_1 B_2 \dots B_x$$

is established. It becomes contradiction. Therefore, when  $n$  is fixed, the number of values of  $r$  for which  $R = 1$  is one or less.

Assuming that  $R = 1$  in some  $r$  by multiplying fractions  $b' = A_1/B_1$ ,  $b'' = A_2/B_2$ ,  $\dots$   
 $b'''\dots' = A_x/B_x$  and reciprocal of fraction previously multiplied, if  $R = 1$  holds. At this time, assuming that  $n$  also changes, the change rate when multiplying by  $A_1/B_1$  is

$$A_1 p^n (p_{r+1}^{n_{r+1}} + \dots + 1) / (B_1 p_{r+1}^{n_{r+1}} (p^n + \dots + 1))$$

The rate of change when multiplying  $p_{r+2}^{q_{r+2}} / (p_{r+2}^{q_{r+2}} + \dots + 1)$  after this is

$$p_1^{q_1} p_{r+1}^{n_{r+1}} (p_{r+2}^{n_{r+2}} + \dots + 1) / ((p_1^{q_1} + \dots + 1) p_{r+2}^{n_{r+2}} (p_{r+1}^{n_{r+1}} + \dots + 1))$$

$$1 \times A_1 p^n (p_{r+1}^{n_{r+1}} + \dots + 1) / (B_1 p_{r+1}^{n_{r+1}} (p^n + \dots + 1)) \times p_1^{q_1} p_{r+1}^{n_{r+1}} (p_{r+2}^{n_{r+2}} + \dots + 1) / ((p_1^{q_1} + \dots + 1) p_{r+2}^{n_{r+2}} (p_{r+1}^{n_{r+1}} + \dots + 1)) \dots = 1$$

When  $A_z$  and  $B_z$  are reduced when multiplied by the reciprocal, if the products excluding the reduced variable are expressed as  $A_1 A_2 \dots A_x$  and  $B_1 B_2 \dots B_x$ ,

$$A_1 A_2 \dots A_x p^n (p_x^{n_x} + \dots + 1) = B_1 B_2 \dots B_x p_x^{n_x} (p^n + \dots + 1)$$

When  $n = n_x$  it becomes contradiction like above proof.

When  $n_x < n$ , a contradiction arises in the case of  $p_x \neq p$  since prime number  $p$  is not included in the right side. It becomes inconsistent in the case of  $p_x = p$  since only the left side includes the prime number  $p$ . When  $n$  is arbitrary, the number of combinations  $(a, b, p, n)$  of solutions for  $R = 1$  is one or less.

When  $n = 0$ ,

$$R = a(p^n + \dots + 1) / (2bp^n) = a/2b = c/c = 1$$

holds. There are no solution that satisfies equation (A) in the range of  $n > 0$ .

Therefore, there are no odd perfect numbers.



4. Complement

From equation ⑤,

$$2bp^n(p-1) = a(p^{n+1}-1)$$

$$2 = a(p^{n+1}-1)/(bp^n(p-1))$$

$$2 = (p_1^{q_1+1}-1)(p_2^{q_2+1}-1) \dots (p_r^{q_r+1}-1)(p^{n+1}-1)$$

$$/(p_1^{q_1}p_2^{q_2} \dots p_r^{q_r}p^n(p_1-1)(p_2-1) \dots (p_r-1)(p-1))$$

$$2(p_1^{q_1+1}-p_1^{q_1})(p_2^{q_2+1}-p_2^{q_2}) \dots (p_r^{q_r+1}-p_r^{q_r})(p^{n+1}-p^n)$$

$$= (p_1^{q_1+1}-1)(p_2^{q_2+1}-1) \dots (p_r^{q_r+1}-1)(p^{n+1}-1)$$

We consider when  $r = 2$ .

$$(p_1^{q_1+1}-1)(p_2^{q_2+1}-1)(p^{n+1}-1) = 2(p_1^{q_1+1}-p_1^{q_1})(p_2^{q_2+1}-p_2^{q_2})(p^{n+1}-p^n)$$

Let  $s, t, u$  be integers,

$$s = p_1^{q_1+1} - 1$$

$$t = p_2^{q_2+1} - 1$$

$$u = p^{n+1} - 1$$

are.

$$stu = 2(p_1^{q_1+1} - 1 - (p_1^{q_1} - 1))(p_2^{q_2+1} - 1 - (p_2^{q_2} - 1))(p^{n+1} - 1 - (p^n - 1))$$

$$stu = 2(s - (s+1)/p_1 + 1)(t - (t+1)/p_2 + 1)(u - (u+1)/p + 1)$$

$$pp_1p_2stu = 2((s+1)p_1 - (s+1))((t+1)p_2 + (t+1))((u+1)p + (u+1))$$

$$pp_1p_2stu = 2(s+1)(p_1-1)(t+1)(p_2-1)(u+1)(p-1)$$

$$stu/((s+1)(t+1)(u+1)) = 2(p_1-1)(p_2-1)(p-1)/(p_1p_2p)$$

Since  $stu/((s+1)(t+1)(u+1))$  is a monotonically increasing function for variables

$s, t$  and  $u$ , if

$$s \geq 3^{2+1} - 1 = 26, p_1 = 3, q_1 = 2$$

$$t \geq 7^{2+1} - 1 = 342, p_2 = 7, q_2 = 2$$

$$u \geq 5^2 - 1 = 24, p = 5, n = 1$$

holds,

$$stu/((s+1)(t+1)(u+1)) \geq 26 \times 342 \times 24 / (27 \times 343 \times 25) = 7904/8575$$

$$2(p_1-1)(p_2-1)(p-1)/(p_1p_2p) = 2 \times 2 \times 6 \times 4 / (3 \times 7 \times 5) = 32/35$$

Since  $stu/((s+1)(t+1)(u+1))$  is limited to 1 when  $s, t$  and  $u$  are infinite,  
 $stu/((s+1)(t+1)(u+1)) < 1$

If  $f(p_1, p_2, p) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$  holds, it is sufficient to consider a combination where  $f(p_1, p_2, p) < 1$ .

$$f(3,7,5) = 2 \times 2 \times 6 \times 4 / (3 \times 7 \times 5) = 32/35$$

$$f(3,11,5) = 2 \times 2 \times 10 \times 4 / (3 \times 11 \times 5) = 32/33$$

$$f(3,13,5) = 2 \times 2 \times 12 \times 4 / (3 \times 13 \times 5) = 64/65$$

$$f(3,17,5) = 2 \times 2 \times 16 \times 4 / (3 \times 17 \times 5) = 256/255$$

$$f(3,7,13) = 2 \times 2 \times 6 \times 12 / (3 \times 7 \times 13) = 96/91$$

$$f(3,5,17) = 2 \times 2 \times 4 \times 16 / (3 \times 5 \times 17) = 256/255$$

From the above, when  $r = 2$ , a combination  $(p_1, p_2, p) = (3,7,5), (3,11,5), (3,13,5)$  can be considered.

Let  $q_k$  be 2 and  $n = 1$ , if  $g(p_1, p_2, p) = (p_1^3 - 1)(p_2^3 - 1)(p^2 - 1)/(p_1^3 p_2^3 p^2)$ ,

$$g(3,7,5) = 26 \times 342 \times 24 / (3^3 7^3 5^2) = 7904/8575 > 32/35$$

$$g(3,11,5) = 26 \times 1330 \times 24 / (3^3 11^3 5^2) = 55328/59895$$

$$g(3,13,5) = 26 \times 2196 \times 24 / (3^3 13^3 5^2) = 3904/4225$$

Since the function  $g$  is the minimum in the case of  $q_k = 2$  and  $n = 1$ , there is no solution  $q_k$  and  $n$  when  $g > f$ , so the case of  $(p_1, p_2, p) = (3,7,5)$  becomes unsuitable.

$$\begin{aligned} stu/((s+1)(t+1)(u+1)) &= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) \\ (p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1)(p^{n+1} - 1) &/ (p_1^{q_1+1} p_2^{q_2+1} p^{n+1}) \\ &= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) \end{aligned}$$

If  $F(p_1, p_2, p) = (p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$ ,

$$F(p_1^{q_1+1}, p_2^{q_2+1}, p^{n+1}) = 2F(p_1, p_2, p)$$

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## 6. References

Hiroyuki Kojima "The world is made of prime numbers" Kadokawa Shoten, 2017

Fumio Sairaiji · Kenichi Shimizu "A story that prime is playing" Kodansha, 2015