Proof that there are no odd perfect numbers

Kouji Takaki

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1. Abstract

For y to be a perfect number, if one of the prime factors is p, the exponent of p is an integer $n(n \geqq 1)$, the prime factors other than p are $p_1, p_2, p_3, \cdots p_r$ and the even exponent of p_k is q_k,

$$y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} (1 + p_k + p_k{}^2 + \cdots + p_k{}^{q_k}) / (2p^n) = \prod_{k=1}^{r} p_k{}^{q_k}$$

must be satisfied. Let m be a non negative integer and q be a positive integer,

$$n = 4m + 1$$

$$p = 4q + 1$$

Letting b and c be odd integers, satisfying following expressions,

$$b = \prod_{k=1}^{r} p_k{}^{q_k}$$

$$c = \prod_{k=1}^{r} (1 + p_k + p_k{}^2 + \cdots + p_k{}^{q_k}) / p^n$$

$$2b = c(p^n + \cdots + 1)$$

is established. This is a known content. Let v be a rational number. If

$$v = \prod_{k} (1 + p_k + p_k{}^2 + \cdots + p_k{}^{q_k}) / \prod_{k} p_k{}^{q_k}$$

holds, assume that v is not an integer. By the consideration of this research paper, since it turned out that if the above supposition holds, the number of odd perfect numbers is one at most when $n \geqq 5$ since there is at most one solution that satisfies this equation for p and p is unique in the range of $n \geqq 5$. Then since two or more solutions are satisfied when n is fixed and p is changed and contradiction arises, we have obtained a conclusion that if the above supposition holds, there are no odd perfect numbers.

2. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself, and the smallest perfect number is

$$1 + 2 + 3 = 6$$

It is 6. Whether an odd perfect number exists or not is currently an unsolved problem.

3. Proof

An odd perfect number is y, one of them is an odd prime number p, an exponent of p is an integer n (n ≧ 1). Let $p_1,p_2,p_3,\cdots p_r$ be the odd prime numbers of factors other than p, q_k the index of p_k, and variable a be the sum of product combinations other than prime p.

$$a = \prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k}) \quad \ldots ①$$

The number of terms N of variable a is

$$N = \prod_{k=1}^{r}(q_k + 1) \quad \ldots ②$$

When y is a perfect number,

$$y = a(1 + p + p^2 + \cdots + p^n) - y \; (n > 0)$$

is established.

$$a \sum_{k=0}^{n} p^k /2 = y$$

$$a \sum_{k=0}^{n} p^k /(2p^n) = y/p^n \quad \ldots ③$$

3.1. If q_k has at least one odd integer

Letting the number of terms where q_k is an odd integer be a positive integer u, because $y/p^n = \prod_{k=1}^{r} {p_k}^{q_k}$ is an odd integer, the denominator on the left side of the expression ③ has a prime factor 2, from the expression ② variable a has more than u prime factor 2 and variable a is an even integer. Therefore, $\sum_{k=0}^{n} p^k$ must be an odd integer, n is an even integer and u is 1.

3.2. When all q_k are even integers

y/p^n is an odd integer, the denominator on the left side of the expression ③ is an even integer, and since N is an odd integer when q_k are all even integers, variable a is an odd integer. Therefore, $\sum_{k=0}^{n} p^k$ is necessary to include one prime factor 2, $\sum_{k=0}^{n} p^k \equiv 0 \; (mod\; 2)$ is established, and n must be an odd integer.

From 3.1, 3.2, in order to have an odd perfect number, only one exponent of the prime factor of y must be an odd integer and variable a must be an odd integer. We consider the case of 3.2 below.

In order for y to be a perfect number, the following expression must be established.

$$y/p^n = (1 + p + p^2 + \cdots + p^n)\prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k})/(2p^n) = \prod_{k=1}^{r}{p_k}^{q_k}$$

However, $q_1, q_2, \ldots, q_r$ are all even integers.

Here, let b be an integer

$$b = \prod_{k=1}^{r}{p_k}^{q_k} \ \ldots④$$

A following expression is established.

$y/p^n = a(1 + p + p^2 + \cdots + p^n)/(2p^n) = b$

$a(p^{n+1} - 1)/(2(p-1)p^n) = b$

$(a - 2b)p^{n+1} + 2bp^n - a = 0 \ \ldots⑤$

Because it is an $n + 1$ order equation of p, the solution of the odd prime p is $n + 1$ at most.

$(ap - 2bp + 2b)p^n = a$

Since $ap - 2bp + 2b$ is an odd integer, a/p^n is an odd integer, which is c.

$ap - 2bp + 2b = c \ (c > 0) \ \ldots⑥$

$(2b - a)p = 2b - c$

Since variable a is an odd integer, $2b - a$ is an odd integer and $2b - a \neq 0$

$p = (2b - c)/(2b - a)$

Since $n \geqq 1$

$a - c = cp^n - c \geqq cp - c > 0$

$a > c$

is.

From the equation ⑥

$2b(p - 1) - (ap - c) = 0$

$2b - c(p^{n+1} - 1)/(p - 1) = 0$

$(p^n + \cdots + 1)/2$ is an odd integer, $n = 4m + 1$ is required with m as an integer.

$2b(p - 1) = c(p^{n+1} - 1)$

$2b = c(p^n + \cdots + 1)$

$2b = c(p + 1)(p^{n-1} + p^{n-3} + \cdots + 1)$ ...⑦

b is an odd integer when $p + 1$ is not a multiple of 4. It is necessary that $p - 1$ be a multiple of 4. A positive integer is taken as q.

$p = 4q + 1$

is established.

When $p > 1$

$p^n - 1 < p^n$

$(p^n - 1)/(p - 1) < p^n/(p - 1)$

$p^{n-1} + \cdots + 1 < p^n/(p - 1)$ ...⑧

Since p is an odd prime number satisfying $p = 4q + 1$ and $p \geqq 5$

$p^{n-1} + \cdots + 1 < p^n/4$

$2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1)$

$2b - a < cp^n/4 = a/4$

$2b < 5a/4$

$a > 8b/5$ ...⑨

Let a_k and b_k be integers and if

$a_k = 1 + p_k + p_k^2 + \cdots + p_k^{q_k}$, $b_k = p_k^{q_k}$,

$a_k - b_k < b_k/(p_k - 1)$

$a_k < b_k p_k/(p_k - 1)$

$$a = \prod_{k=1}^{r} a_k < \prod_{k=1}^{r} b_k p_k/(p_k - 1) = b \prod_{k=1}^{r} p_k/(p_k - 1)$$

$$a/b < \prod_{k=1}^{r} p_k/(p_k - 1)$$

When $r = 1$, since $a/b < 3/2$ is established, it becomes inappropriate contrary to inequality ⑨.

From the expression ⑦,

$b = c(p + 1)/2 \times (p^{n-1} + p^{n-3} + \cdots + 1)$

holds. Since $(p + 1)/2$ is the product of only prime numbers of b, let d_k be the index,

$$(p + 1)/2 = \prod_{k=1}^{r} {p_k}^{d_k}$$

$$p = 2\prod_{k=1}^{r} {p_k}^{d_k} - 1$$

From $a = cp^n$ and the expression ⑦,

$2bp^n = a(p^n + \cdots + 1)$

$a(p^n + \cdots + 1)/(2bp^n) = 1$ ... (A)

When $r = 1$,

$a = ({p_1}^{q_1+1} - 1)/(p_1 - 1)$

$b = {p_1}^{q_1}$

The equation (A) does not hold since there is no odd perfect number when $r = 1$.

Let R be a rational number,
$R = a(p^n + \cdots + 1)/(2bp^n)$
Let b' be a rational number and let A and B to be an integer,
$b' = (p_k{}^{q_k+1} - 1)/(p_k{}^{q_k}(p_k - 1)) > 1$
$A_k = (p_k{}^{q_k+1} - 1)/(p_k - 1)$
$B_k = p_k{}^{q_k}$

Multiplying R by b', there are both cases that p_k increases p or does not change. When multiplied by b', the rate of change of R is $A_{r+1}p^n(p'^n + \cdots + 1)/(B_{r+1}p'^n(p^n + \cdots + 1))$, if p after variation is p'. If the rate of change of R is 1,
$A_{r+1}p^n(p'^n + \cdots + 1)/(B_{r+1}p'^n(p^n + \cdots + 1)) = 1$
$A_{r+1}p^n(p'^n + \cdots + 1) = B_{r+1}p'^n(p^n + \cdots + 1)$
This expression does not hold since the right side is not a multiple of p when $p' > p$, and $A_{r+1} > B_{r+1}$ holds when $p' = p$. Due to this operation, R may be larger or smaller than the original value since the rate of change of R does not become 1.

Assuming that $R = 1$ in some r, letting x be an integer and by multiplying fractions $b' = A_{r+1}/B_{r+1}$, $b'' = A_{r+2}/B_{r+2}$, $\cdots b''^{\cdots\prime} = A_x/B_x$ to R. Furthermore, assuming that $A_{s+1}A_{s+2} \dots A_r$ is not a multiple of p, R is divided by A_{s+1}/B_{s+1}, A_{s+2}/B_{s+2},$\cdots A_r/B_r$ and it is assumed that finally $R = 1$. At this time, assuming that n changes, the change rate of R by this operation when multiplying by A_{r+1}/B_{r+1} is
$A_{r+1}p^n(p^{n_{r+1}} + \cdots + 1)/(B_{r+1}p^{n_{r+1}}(p^n + \cdots + 1))$

$$\begin{aligned}1 \times B_{s+1}p^n(p^{n_{s+1}} + \cdots + 1)/(A_{s+1}p^{n_{s+1}}(p^n + \cdots + 1)) \times \dots \times B_r p^{n_{r-1}}(p^{n_r} + \cdots \\ + 1)/(A_r p^{n_r}(p^{n_{r-1}} + \cdots + 1)) \times A_{r+1}p^{n_r}(p^{n_{r+1}} + \cdots + 1)/(B_{r+1}p^{n_{r+1}}(p^{n_r} \\ + \cdots + 1)) \times A_{r+2}p^{n_{r+1}}(p^{n_{r+2}} + \cdots + 1)/(B_{r+2}p^{n_{r+2}}(p^{n_{r+1}} + \cdots + 1)) \times \dots \\ \times A_x p^{n_{x-1}}(p^{n_x} + \cdots + 1)/(B_x p^{n_x}(p^{n_{x-1}} + \cdots + 1)) = 1\end{aligned}$$

$$\begin{aligned}B_{s+1}B_{s+2} \dots B_r A_{r+1}A_{r+2} \dots A_x p^{n-n_x}(p^{n_x} + \cdots + 1) \\ = A_{s+1}A_{s+2} \dots A_r B_{r+1}B_{r+2} \dots B_x(p^n + \cdots + 1) \quad \dots (B)\end{aligned}$$

When $n_x < n$, it becomes contradiction since the right side of above expression does not include factor p.
When $n_x = n$,
$B_{s+1}B_{s+2} \dots B_r A_{r+1}A_{r+2} \dots A_x = A_{s+1}A_{s+2} \dots A_r B_{r+1}B_{r+2} \dots B_x \quad \dots (C)$

Let v be a rational number. If

$$v = \prod_k (1 + p_k + p_k^2 + \cdots + p_k^{q_k}) \,/\, \prod_k p_k^{q_k}$$

holds, assume that v is not an integer. $\cdots$(D)

Let e_r, f_r be odd integers and g_r be a rational number,

$$e_r = \prod_{k=1}^{r} (p_k^{q_k} + \cdots + 1)\,, f_r = \prod_{k=1}^{r} p_k^{q_k}\,, g_r = e_r/f_r$$

holds.

$g_{r+1} = e_{r+1}/f_{r+1} = e_r/f_r \times (p_{r+1}{}^{q_{r+1}} + \cdots + 1)/p_{r+1}{}^{q_{r+1}} > e_r/f_r = g_r$

Let $q_1{}'$ be an even integer and $q_1' > q_1$ holds. Let g_r be $g_r{}'$ when q_1 becomes q_1',

$g_r' = (p_1{}^{q_1}(p_1{}^{q_1{}'} + \cdots + 1)/p_1{}^{q_1{}'}(p_1{}^{q_1} + \cdots + 1))g_r > g_r$

is established.

It is assumed that q_k becomes $q_k - h_k$ by changing q_k than before for g_r. h_k is an even integer. Then assume that r becomes $s(s > r)$, $g_s = g_r$ and g_s is not changed.

$$\begin{aligned} g_s/g_r = {} & p_{r+1}{}^{q_{r+1}} \times \ldots \times p_s{}^{q_s}/((p_{r+1}{}^{q_{r+1}} + \cdots + 1) \times \ldots \times (p_s{}^{q_s} + \cdots + 1)) \times p_1{}^{q_1} \times \ldots \\ & \times p_r{}^{q_r}\left(p_1{}^{q_1-h_1} + \cdots + 1\right) \ldots (p_r{}^{q_r-h_r} + \cdots + 1)/(p_1{}^{q_1-h_1} \times \ldots \\ & \times p_r{}^{q_r-h_r}(p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1)) = 1 \end{aligned}$$

$$\begin{aligned} & p_{r+1}{}^{q_{r+1}} \times \ldots \times p_s{}^{q_s}/((p_{r+1}{}^{q_{r+1}} + \cdots + 1) \times \ldots \times (p_s{}^{q_s} + \cdots + 1)) \times p_1{}^{h_1} \times \ldots \\ & \times p_r{}^{h_r}\left(p_1{}^{q_1-h_1} + \cdots + 1\right) \ldots (p_r{}^{q_r-h_r} + \cdots \\ & + 1)/((p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1)) = 1 \end{aligned}$$

$$\begin{aligned} & p_{r+1}{}^{q_{r+1}} \times \ldots \times p_s{}^{q_s} \times p_1{}^{h_1} \times \ldots \times p_r{}^{h_r}\left(p_1{}^{q_1-h_1} + \cdots + 1\right) \ldots \left(p_r{}^{q_r-h_r} + \cdots + 1\right) \\ & = (p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1)(p_{r+1}{}^{q_{r+1}} + \cdots + 1) \ldots (p_s{}^{q_s} + \cdots + 1) \end{aligned}$$

$$\begin{aligned} & p_{r+1}{}^{q_{r+1}} \times \ldots \times p_s{}^{q_s}\left(p_1{}^{q_1} + \cdots + p_1{}^{h_1}\right) \ldots \left(p_r{}^{q_r} + \cdots + p_r{}^{h_r}\right) \\ & = (p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1)(p_{r+1}{}^{q_{r+1}} + \cdots + 1) \ldots (p_s{}^{q_s} + \cdots + 1) \end{aligned}$$

When $h_k < 0$, multiply both sides by $p_k{}^{-h_k}$ so that both sides become integers. If the supposition (D) holds, there is at least one prime number from p_{r+1} to p_s on the left side. $a = (p_1{}^{q_1} + \cdots + 1) \ldots (p_r{}^{q_r} + \cdots + 1) = cp^n$ holds and from the expression ⑦, c must be a product of primes from p_1 to p_r. Thereby, the above equation does not hold since it is inappropriate when there is even one prime number other than p_1 to p_r. When changing the value of p_k, it is equivalent to dividing by $p_k{}^{q_k}$ and then multiplying by new $p_k{}^{q_k}$, so it is sufficient to consider only the changes of q_k and r. From above, since g_r does not chord the original value when q_k or r is increased or decreased, it takes unique values for the variables p_k, q_k, r.

From the above proof,

$g_r = A_1A_2 \ldots A_s/B_1B_2 \ldots B_s \times A_{r+1}A_{r+2} \ldots A_x/B_{r+1}B_{r+2} \ldots B_x$

g_r must be represented uniquely, and the expression (C) does not satisfied. When dividing by the prime number in the expression of p, a contradiction arises since the prime number not included in b is in the expression of p. Therefore, when p holds $p \equiv 1 \pmod 4$ and $p \geqq 5$, the number of the solution (a, b, p, n) satisfying $R = 1$ is at most one.

Since $(a, b, p, n) = (1,1,1,1)$ is inappropriate solution and the expression (C) becomes contradiction, there is one solution when $n_x = n = 1$. Therefore, if the supposition (D) holds, there are no odd perfect numbers when $n = 1$.

Define the operation [multiplication] and the operation [division] as follows.

Assuming that p in the equation of R is replaced by p' by multiplying A_i/B_i, define operation [multiplication] to R as follows.

$$p' = 2\prod_{k=1}^{r} {p_k}^{d_k} \times {p_i}^{d_i} - 1$$

$0 \leqq d_i \leqq q_i$

Here, let i be $i > r$. Suppose operation [division] is division by A_j/B_j for R, and if p_j is included in p in the expression R, p_j is deleted as $d_j = 0$. Here, assuming that j satisfies $1 \leqq j \leqq r$.

In the proof of the expression (B), it is assumed that p changes on the way, and finally p becomes p_x.

$A_1 \ldots A_r = cp^n$

$2B_1 \ldots B_r = c(p^n + \cdots + 1)$

$A_1 \ldots A_x = c'{p_x}^{n_x}$

$2B_1 \ldots B_x = c'({p_x}^{n_x} + \cdots + 1)$

It is assumed that the above expressions are satisfied.

$$B_{s+1}B_{s+2} \ldots B_rA_{r+1}A_{r+2} \ldots A_xp^n({p_x}^{n_x} + \cdots + 1)$$
$$= A_{s+1}A_{s+2} \ldots A_rB_{r+1}B_{r+2} \ldots B_x{p_x}^{n_x}(p^n + \cdots + 1)$$
$$B_{s+1}B_{s+2} \ldots B_rA_1 \ldots A_rA_{r+1}A_{r+2} \ldots A_xp^n({p_x}^{n_x} + \cdots + 1)$$
$$= A_1 \ldots A_rA_{s+1}A_{s+2} \ldots A_rB_{r+1}B_{r+2} \ldots B_x{p_x}^{n_x}(p^n + \cdots + 1)$$
$$B_{s+1}B_{s+2} \ldots B_rc'{p_x}^{n_x}p^n({p_x}^{n_x} + \cdots + 1)$$
$$= A_1 \ldots A_rA_{s+1}A_{s+2} \ldots A_rB_{r+1}B_{r+2} \ldots B_x{p_x}^{n_x}(p^n + \cdots + 1)$$
$$B_{s+1}B_{s+2} \ldots B_rc'p^n({p_x}^{n_x} + \cdots + 1) = A_1 \ldots A_rA_{s+1}A_{s+2} \ldots A_rB_{r+1}B_{r+2} \ldots B_x(p^n + \cdots + 1)$$

$$B_1 \ldots B_r B_{s+1} B_{s+2} \ldots B_r c' p^n (p_x^{n_x} + \cdots + 1)$$
$$= A_1 \ldots A_r A_{s+1} A_{s+2} \ldots A_r B_1 \ldots B_r B_{r+1} B_{r+2} \ldots B_x (p^n + \cdots + 1)$$
$$B_1 \ldots B_r B_{s+1} B_{s+2} \ldots B_r c' p^n (p_x^{n_x} + \cdots + 1)$$
$$= A_1 \ldots A_r A_{s+1} A_{s+2} \ldots A_r c' (p_x^{n_x} + \cdots + 1)/2 \times (p^n + \cdots + 1)$$
$$B_1 \ldots B_r B_{s+1} B_{s+2} \ldots B_r p^n = A_1 \ldots A_r A_{s+1} A_{s+2} \ldots A_r/2 \times (p^n + \cdots + 1)$$

$$c(p^n + \cdots + 1)/2 \times B_{s+1} B_{s+2} \ldots B_r p^n = c p^n A_{s+1} A_{s+2} \ldots A_r/2 \times (p^n + \cdots + 1)$$
$$B_{s+1} B_{s+2} \ldots B_r = A_{s+1} A_{s+2} \ldots A_r$$
is established. It becomes contradiction since $A_k > B_k$ holds when the operations [division] are performed.

We consider in the case of $n \geqq 5$ as follows. Consider a tree whose vertex is $(a, b, p, n) = (1,1,1,1)$, and when the operations [multiplication] are performed, it becomes a child node. For example, consider a child node connected to a vertex as follows.
$(a, b, p, n) = (13,9,5,5)$ as $p_1 = 3$, $q_1 = 2$ and $d_1 = 1$
$(a, b, p, n) = (13,9,17,9)$ as $p_1 = 3$, $q_1 = 2$ and $d_1 = 2$
$(a, b, p, n) = (57,49,97,13)$ as $p_1 = 7$, $q_1 = 2$ and $d_1 = 2$

Suppose that the operations [multiplication] for changing the value of p are performed first, and then the operations [multiplication] for not changing the value of p are performed to create a tree structure. Here, when there is a solution in a certain p and there is a solution even in the other value p', considering a set of line segments connecting these two points in four-dimensional space (a, b, p, n). If the operations [multiplication] that do not change p from both points are the same, by reversing the order of operations [multiplication], there is a case that the two points can be connected only by operations [multiplication] that changes p without performing operations [division] from the vertex. In this case and when there are no operations [multiplication] that do not change p for both, if $R = 1$ holds again when performing operation [multiplication] from one point where $R = 1$,
$$1 \times A_{r+1} p^n (p_{r+1}^{n_{r+1}} + \cdots + 1)/(B_{r+1} p_{r+1}^{n_{r+1}} (p^n + \cdots + 1)) \times A_{r+2} p_{r+1}^{n_{r+1}} (p_{r+2}^{n_{r+2}} + \cdots$$
$$+ 1)/(B_{r+2} p_{r+2}^{n_{r+2}} (p_{r+1}^{n_{r+1}} + \cdots + 1)) \times \ldots \times A_x p_{x-1}^{n_{x-1}} (p_x^{n_x} + \cdots$$
$$+ 1)/(B_x p_x^{n_x} (p_{x-1}^{n_{x-1}} + \cdots + 1)) = 1$$
$$A_{r+1} A_{r+2} \ldots A_x/(B_{r+1} B_{r+2} \ldots B_x) = p_x^{n_x} (p^n + \cdots + 1)/(p^n (p_x^{n_x} + \cdots + 1))$$
$$A_1 A_2 \ldots A_x (p_x^{n_x} + \cdots + 1)/(B_1 B_2 \ldots B_x p_x^{n_x}) = A_1 A_2 \ldots A_r (p^n + \cdots + 1)/(B_1 B_2 \ldots B_r p^n) \quad \ldots (E)$$

Assume that $g_r = A_1A_2 \dots A_r(p^n + \cdots + 1)/(B_1B_2 \dots B_x p^n)$ holds. Here, it is assumed that q_k becomes $q_k - h_k$ by changing q_k than before and n becomes $n - h(n - h > 0)$ for g_r. h_k is an even integer and h is a non-negative integer that is a multiple of 4. Then assuming that r becomes $s(s > r)$, $g_s = g_r$ and g_s is not changed, by the same calculation as the proof on page 7,

$$g_s/g_r = p_{r+1}{}^{q_{r+1}} \times \dots \times p_s{}^{q_s}/((p_{r+1}{}^{q_{r+1}} + \cdots + 1) \times \dots \times (p_s{}^{q_s} + \cdots + 1)) \times p_1{}^{q_1} \times p_2{}^{q_2} \times \dots$$
$$\times p_r{}^{q_r} p^n \left(p_1{}^{q_1 - h_1} + \cdots + 1\right) \dots (p_r{}^{q_r - h_r} + \cdots + 1)(p^{n-h} + \cdots + 1)/(p_1{}^{q_1 - h_1}$$
$$\times \dots \times p_r{}^{q_r - h_r} p^{n-h}(p_1{}^{q_1} + \cdots + 1) \dots (p_r{}^{q_r} + \cdots + 1)(p^n + \cdots + 1)) = 1$$

$$p_{r+1}{}^{q_{r+1}} \times \dots \times p_s{}^{q_s}\left(p_1{}^{q_1} + \cdots + p_1{}^{h_1}\right) \dots \left(p_r{}^{q_r} + \cdots + p_r{}^{h_r}\right)(p^n + \cdots + p^h)$$
$$= (p_1{}^{q_1} + \cdots + 1) \dots (p_r{}^{q_r} + \cdots + 1)(p^n + \cdots + 1)(p_{r+1}{}^{q_{r+1}} + \cdots + 1) \dots (p_s{}^{q_s}$$
$$+ \cdots + 1)$$

Since $(p_1{}^{q_1} + \cdots + 1) \dots (p_r{}^{q_r} + \cdots + 1) = cp^n$ holds,

$$p_{r+1}{}^{q_{r+1}} \times \dots \times p_s{}^{q_s}\left(p_1{}^{q_1} + \cdots + p_1{}^{h_1}\right) \dots \left(p_r{}^{q_r} + \cdots + p_r{}^{h_r}\right)(p^{n-h} + \cdots + 1)$$
$$= cp^{n-h}(p^n + \cdots + 1)(p_{r+1}{}^{q_{r+1}} + \cdots + 1) \dots (p_s{}^{q_s} + \cdots + 1)$$

When $h_k < 0$, multiply both sides by $p_k{}^{-h_k}$ so that both sides become integers. If the supposition (D) holds, there is at least one prime number from p_{r+1} to p_s on the left side. Because c and $p^n + \cdots + 1$ are products of prime numbers from p_1 to p_r and in the case of $s > r + 1$, the left side has prime numbers that is not on the right side as a factor, this expression does not hold. In the case of $s = r + 1$, when $p \neq p_s$, this expression does not hold in the same way. When $p = p_s$ and $q_s > n - h$, since there is a prime factor p only on the left side, this expression does not hold. Therefore, since except for the case of $s = r + 1$, $p = p_s$ and $q_s < n - h$ g_r must be uniquely expressed, the expression (E) does not hold. When $s = r + 1$, $p = p_s$ and $q_s < n - h$, substituting $B_x = p^{q_s}$ into the expression (E) as $x = r + 1$,

$$A_1A_2 \dots A_r(p^{q_s} + \cdots + 1)(p_x{}^{n_x} + \cdots + 1)/(B_1B_2 \dots B_r p^{q_s} p_x{}^{n_x})$$
$$= A_1A_2 \dots A_r(p^n + \cdots + 1)/(B_1B_2 \dots B_r p^n)$$

$$(p^{q_s} + \cdots + 1)(p_x{}^{n_x} + \cdots + 1)/(p^{q_s} p_x{}^{n_x}) = (p^n + \cdots + 1)/p^n$$

$$(p^{q_s} + \cdots + 1)(p_x{}^{n_x} + \cdots + 1)p^{n-q_s} = (p^n + \cdots + 1)p_x{}^{n_x}$$

Since the right side does not have a prime number p as a factor, this expression does not hold.

From the above, if there is an odd perfect number at one point, when two points are connected only by operations [multiplication] that change p or when there are no operations [multiplication] that does not change p for both, there is no other point that becomes an odd perfect number.

Therefore, considering the case except the above cases, because there is a at most one solution for arbitrary p and the operations [multiplication] that do not change p from both points are different, the operations [division] must be performed to return to the bifurcation where the value of p changes. At this time, since it becomes inconsistent when the operations [division] are performed according to the above proof, p must be unique. Therefore, since there is at most one solution with $R = 1$ for p satisfying $p \geqq 5$, the number of odd perfect number is one at most where $n \geqq 5$.

When $n \geqq 5, R = 1$ at a certain r, and fixing a value of n, $R = 1$ even with p_x of a value different from p.

$A_1 \dots A_r = cp^n$

$2B_1 \dots B_r = c(p^n + \cdots + 1)$

$A_1 \dots A_x = c' {p_x}^n$

$2B_1 \dots B_x = c'({p_x}^n + \cdots + 1)$

$2B_1 \dots B_r p^n = A_1 \dots A_r(p^n + \cdots + 1)$

$2B_1 \dots B_x {p_x}^n = A_1 \dots A_x({p_x}^n + \cdots + 1)$

$B_{r+1} \dots B_x \times {p_x}^n/p^n = A_{r+1} \dots A_x \times ({p_x}^n + \cdots + 1)/(p^n + \cdots + 1)$

$B_{r+1} \dots B_x {p_x}^n(p^n + \cdots + 1) = A_{r+1} \dots A_x p^n({p_x}^n + \cdots + 1)$

Let t be an odd integer,

$A_{r+1} \dots A_x = t{p_x}^n \dots (F)$

$B_{r+1} \dots B_x(p^n + \cdots + 1) = tp^n({p_x}^n + \cdots + 1)$

If $B_x = p^{q_x}$,

$B_{r+1} \dots B_{x-1} p^{q_x - n}(p^n + \cdots + 1) = t({p_x}^n + \cdots + 1)$

From the expression (F),

$A_1 \dots A_x = ctp^n {p_x}^n = c' {p_x}^n$

$c' = ctp^n$

$B_{r+1} \dots B_{x-1} p^{q_x - n}(p^n + \cdots + 1) = c'/(cp^n) \times ({p_x}^n + \cdots + 1)$

$cB_{r+1} \dots B_{x-1} p^{q_x}(p^n + \cdots + 1) = c'({p_x}^n + \cdots + 1)$

$B_1 \dots B_r B_{r+1} \dots B_{x-1} p^{q_x} = B_1 \dots B_x$

When $B_x = p^{q_x}$, above expression holds. Therefore, if there is one odd perfect number, there are always odd perfect numbers of different value. This contradicts the proposition that the number of odd perfect numbers is at most one when $n \geqq 5$.

From the above, if the supposition (D) holds, there are no odd perfect numbers.

4. Complement

From the equation ⑤,

$2bp^n(p-1) = a(p^{n+1}-1)$

$2 = a(p^{n+1}-1)/(bp^n(p-1))$

$2 = (p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)\dots(p_r{}^{q_r+1}-1)(p^{n+1}-1)$
$\quad /(p_1{}^{q_1}p_2{}^{q_2}\dots p_r{}^{q_r}p^n(p_1-1)(p_2-1)\dots(p_r-1)(p-1))$

$2(p_1{}^{q_1+1}-p_1{}^{q_1})(p_2{}^{q_2+1}-p_2{}^{q_2})\dots(p_r{}^{q_r+1}-p_r{}^{q_r})(p^{n+1}-p^n)$
$\quad = (p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)\dots(p_r{}^{q_r+1}-1)(p^{n+1}-1)$

We consider when $r = 2$.

$(p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)(p^{n+1}-1) = 2(p_1{}^{q_1+1}-p_1{}^{q_1})(p_2{}^{q_2+1}-p_2{}^{q_2})(p^{n+1}-p^n)$

Let s, t, u be integers,

$s = p_1{}^{q_1+1}-1$

$t = p_2{}^{q_2+1}-1$

$u = p^{n+1}-1$

are.

$stu = 2(p_1{}^{q_1+1}-1-(p_1{}^{q_1}-1))(p_2{}^{q_2+1}-1-(p_2{}^{q_2}-1))(p^{n+1}-1-(p^n-1))$

$stu = 2(s-(s+1)/p_1+1)(t-(t+1)/p_2+1)(u-(u+1)/p+1)$

$pp_1p_2stu = 2((s+1)p_1-(s+1))((t+1)p_2+(t+1))\big((u+1)p+(u+1)\big)$

$pp_1p_2stu = 2(s+1)(p_1-1)(t+1)(p_2-1)(u+1)(p-1)$

$stu/((s+1)(t+1)(u+1)) = 2(p_1-1)(p_2-1)(p-1)/(p_1p_2p)$

Since $stu/((s+1)(t+1)(u+1))$ is a monotonically increasing function for variables s, t and u, if

$s \geqq 3^{2+1}-1 = 26,\ p_1 = 3,\ q_1 = 2$

$t \geqq 7^{2+1}-1 = 342,\ p_2 = 7,\ q_2 = 2$

$u \geqq 5^2-1 = 24,\ p = 5,\ n = 1$

holds,

$stu/((s+1)(t+1)(u+1)) \geqq 26\times 342\times 24/(27\times 343\times 25) = 7904/8575$

$2(p_1-1)(p_2-1)(p-1)/(p_1p_2p) = 2\times 2\times 6\times 4/(3\times 7\times 5) = 32/35$

Since $stu/((s+1)(t+1)(u+1))$ is limited to 1 when s, t and u are infinite,

$stu/((s+1)(t+1)(u+1)) < 1$

If $f(p_1, p_2, p) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p)$ holds, it is sufficient to consider a combination where $f(p_1, p_2, p) < 1$.

$f(3,7,5) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35$

$f(3,11,5) = 2 \times 2 \times 10 \times 4/(3 \times 11 \times 5) = 32/33$

$f(3,13,5) = 2 \times 2 \times 12 \times 4/(3 \times 13 \times 5) = 64/65$

$f(3,17,5) = 2 \times 2 \times 16 \times 4/(3 \times 17 \times 5) = 256/255$

$f(3,7,13) = 2 \times 2 \times 6 \times 12/(3 \times 7 \times 13) = 96/91$

$f(3,5,17) = 2 \times 2 \times 4 \times 16/(3 \times 5 \times 17) = 256/255$

From the above, when $r = 2$, a combination $(p_1, p_2, p) = (3,7,5), (3,11,5), (3,13,5)$ can be considered.

Let q_k be 2 and $n = 1$, if $g(p_1, p_2, p) = (p_1{}^3 - 1)(p_2{}^3 - 1)(p^2 - 1)/(p_1{}^3p_2{}^3p^2)$,

$g(3,7,5) = 26 \times 342 \times 24/(3^3 7^3 5^2) = 7904/8575 > 32/35$

$g(3,11,5) = 26 \times 1330 \times 24/(3^3 11^3 5^2) = 55328/59895$

$g(3,13,5) = 26 \times 2196 \times 24/(3^3 13^3 5^2) = 3904/4225$

Since the function g is the minimum in the case of $q_k = 2$ and $n = 1$, there is no solution q_k and n when $g > f$, so the case of $(p_1, p_2, p) = (3,7,5)$ becomes unsuitable.

$stu/((s + 1)(t + 1)(u + 1)) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p)$

$$(p_1{}^{q_1+1} - 1)(p_2{}^{q_2+1} - 1)(p^{n+1} - 1)/(p_1{}^{q_1+1}p_2{}^{q_2+1}p^{n+1})$$
$$= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p)$$

If $F(p_1, p_2, p) = (p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p)$,

$F(p_1{}^{q_1+1}, p_2{}^{q_2+1}, p^{n+1}) = 2F(p_1, p_2, p)$

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6. References

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