# Proof that there are no odd perfect numbers

Kouji Takaki

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1. Abstract

For $y$ to be a perfect number, if one of the prime factors is $p$, the exponent of $p$ is an integer $n(n \geqq 1)$, the prime factors other than $p$ are $p_1$,$p_2$,$p_3$,$\cdots p_r$ and the even exponent of $p_k$ is $q_k$,

$$y/p^n = (1 + p + p^2 + \cdots + p^n)\prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k})\,/(2p^n) = \prod_{k=1}^{r}{p_k}^{q_k}$$

must be satisfied. Let $m$ be a non negative integer and $q$ be a positive integer,

$$n = 4m + 1$$

$$p = 4q + 1$$

Letting $a$, $b$ and $c$ be odd integers, satisfying following expressions,

$$a = \prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k})$$

$$b = \prod_{k=1}^{r}{p_k}^{q_k}$$

$$c = a/p^n$$

$$2b = c(p^n + \cdots + 1)$$

is established. This is a known content. By the consideration of this research paper, since it turned out that by the uniqueness of $a/b$ there is at most one solution that satisfies this equation for $p$. Since by the uniqueness of $a(p^n + \cdots + 1)/(bp^n)$ we proved that there is no solution to this equation other than $(a, b, p, n) = (1,1,1,1)$, we have obtained a conclusion that there are no odd perfect numbers.

2. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself, and the smallest perfect number is

$$1 + 2 + 3 = 6$$

It is 6. Whether an odd perfect number exists or not is currently an unsolved problem.

3. Proof

An odd perfect number is y, one of them is an odd prime number p, an exponent of p is an integer n (n ≧ 1). Let $p_1, p_2, p_3, \cdots p_r$ be the odd prime numbers of factors other than p, $q_k$ the index of $p_k$, and variable a be the sum of product combinations other than prime p.

$$a = \prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k}) \ \ldots ①$$

The number of terms N of variable a is

$$N = \prod_{k=1}^{r}(q_k + 1) \ \ldots ②$$

When y is a perfect number,

$$y = a(1 + p + p^2 + \cdots + p^n) - y \ (n > 0)$$

is established.

$$a \sum_{k=0}^{n} p^k /2 = y$$

$$a \sum_{k=0}^{n} p^k /(2p^n) = y/p^n \ \ldots ③$$

3.1. If $q_k$ has at least one odd integer

Letting the number of terms where $q_k$ is an odd integer be a positive integer u, because $y/p^n = \prod_{k=1}^{r} {p_k}^{q_k}$ is an odd integer, the denominator on the left side of the expression ③ has a prime factor 2, from the expression ② variable a has more than u prime factor 2 and variable a is an even integer. Therefore, $\sum_{k=0}^{n} p^k$ must be an odd integer, n is an even integer and u is 1.

3.2. When all $q_k$ are even integers

$y/p^n$ is an odd integer, the denominator on the left side of the expression ③ is an even integer, and since N is an odd integer when $q_k$ are all even integers, variable a is an odd integer. Therefore, $\sum_{k=0}^{n} p^k$ is necessary to include one prime factor 2, $\sum_{k=0}^{n} p^k \equiv 0 \ (mod\ 2)$ is established, and n must be an odd integer.

From 3.1, 3.2, in order to have an odd perfect number, only one exponent of the prime factor of y must be an odd integer and variable a must be an odd integer. We consider the case of 3.2 below.

In order for y to be a perfect number, the following expression must be established.

$$y/p^n = (1 + p + p^2 + \cdots + p^n)\prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k})/(2p^n) = \prod_{k=1}^{r}{p_k}^{q_k}$$

However, $q_1, q_2, \dots, q_r$ are all even integers.

Here, let b be an integer

$$b = \prod_{k=1}^{r}{p_k}^{q_k} \quad \dots④$$

A following expression is established.

$y/p^n = a(1 + p + p^2 + \cdots + p^n)/(2p^n) = b$

$a(p^{n+1} - 1)/(2(p - 1)p^n) = b$

$(a - 2b)p^{n+1} + 2bp^n - a = 0$ ...⑤

Because it is an $n + 1$ order equation of p, the solution of the odd prime p is $n + 1$ at most.

$(ap - 2bp + 2b)p^n = a$

Since $ap - 2bp + 2b$ is an odd integer, $a/p^n$ is an odd integer, which is c.

$ap - 2bp + 2b = c\ (c > 0)$ ...⑥

$(2b - a)p = 2b - c$

Since variable a is an odd integer, $2b - a$ is an odd integer and $2b - a \neq 0$

$p = (2b - c)/(2b - a)$

Since $n \geqq 1$

$a - c = cp^n - c \geqq cp - c > 0$

$a > c$

is.

From the equation ⑥

$2b(p - 1) - (ap - c) = 0$

$2b - c(p^{n+1} - 1)/(p - 1) = 0$

$(p^n + \cdots + 1)/2$ is an odd integer, $n = 4m + 1$ is required with m as an integer.

$2b(p - 1) = c(p^{n+1} - 1)$

$2b = c(p^n + \cdots + 1)$

$2b = c(p + 1)(p^{n-1} + p^{n-3} + \cdots + 1)$ ...⑦

b is an odd integer when $p + 1$ is not a multiple of 4. It is necessary that $p - 1$ be a multiple of 4. A positive integer is taken as q.

$p = 4q + 1$

is established.

When $p > 1$

$p^n - 1 < p^n$

$(p^n - 1)/(p - 1) < p^n/(p - 1)$

$p^{n-1} + \cdots + 1 < p^n/(p - 1)$ ...⑧

Since p is an odd prime number satisfying $p = 4q + 1$ and $p \geqq 5$

$p^{n-1} + \cdots + 1 < p^n/4$

$2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1)$

$2b - a < cp^n/4 = a/4$

$2b < 5a/4$

$a > 8b/5$ ...⑨

Let $a_k$ and $b_k$ be integers and if

$a_k = 1 + p_k + p_k^2 + \cdots + p_k^{q_k}$, $b_k = p_k^{q_k}$,

$a_k - b_k < b_k/(p_k - 1)$

$a_k < b_k p_k/(p_k - 1)$

$$a = \prod_{k=1}^{r} a_k < \prod_{k=1}^{r} b_k p_k/(p_k - 1) = b \prod_{k=1}^{r} p_k/(p_k - 1)$$

$$a/b < \prod_{k=1}^{r} p_k/(p_k - 1)$$

When $r = 1$, since $a/b < 3/2$ is established, it becomes inappropriate contrary to inequality ⑨.

From the expression ⑦,

$b = c(p + 1)/2 \times (p^{n-1} + p^{n-3} + \cdots + 1)$

holds. Since $(p + 1)/2$ is the product of only prime numbers of b, let $d_k$ be the index,

$$(p + 1)/2 = \prod_{k=1}^{r} {p_k}^{d_k}$$

$$p = 2\prod_{k=1}^{r} {p_k}^{d_k} - 1$$

From $a = cp^n$ and the expression ⑦,

$2bp^n = a(p^n + \cdots + 1)$

$a(p^n + \cdots + 1)/(2bp^n) = 1$ ... (A)

When $r = 1$,

$a = ({p_1}^{q_1+1} - 1)/(p_1 - 1)$

$b = {p_1}^{q_1}$

The equation (A) does not hold since there is no odd perfect number when $r = 1$.

Let R be a rational number,

$R = a(p^n + \cdots + 1)/(2bp^n)$

Let b’ be a rational number and let A and B to be an integer,

$b' = (p_k{}^{q_k+1} - 1)/(p_k{}^{q_k}(p_k - 1)) > 1$

$A_k = (p_k{}^{q_k+1} - 1)/(p_k - 1)$

$B_k = p_k{}^{q_k}$

Multiplying R by b’, there are both cases that $p_k$ increases p or does not change. When multiplied by b’, the rate of change of R is $A_{r+1}p^n(p'^n + \cdots + 1)/(B_{r+1}p'^n(p^n + \cdots + 1))$, if p after variation is p’. If the rate of change of R is 1,

$A_{r+1}p^n(p'^n + \cdots + 1)/(B_{r+1}p'^n(p^n + \cdots + 1)) = 1$

$A_{r+1}p^n(p'^n + \cdots + 1) = B_{r+1}p'^n(p^n + \cdots + 1)$

This expression does not hold since the right side is not a multiple of p when $p' > p$, and $A_{r+1} > B_{r+1}$ holds when $p' = p$. Due to this operation, R may be larger or smaller than the original value since the rate of change of R does not become 1.

Assuming that $R = 1$ in some r, letting x be an integer and by multiplying fractions $b' = A_{r+1}/B_{r+1}$, $b'' = A_{r+2}/B_{r+2}$, $\cdots b''^{\cdots\prime} = A_x/B_x$ to R. Furthermore, assuming that $A_{s+1}A_{s+2} \ldots A_r$ is not a multiple of p, R is divided by $A_{s+1}/B_{s+1}$, $A_{s+2}/B_{s+2}, \cdots A_r/B_r$ and it is assumed that finally $R = 1$. At this time, assuming that n changes, the change rate of R by this operation when multiplying by $A_{r+1}/B_{r+1}$ is

$A_{r+1}p^n(p^{n_{r+1}} + \cdots + 1)/(B_{r+1}p^{n_{r+1}}(p^n + \cdots + 1))$

$$
\begin{aligned}
1 \times B_{s+1}p^n(p^{n_{s+1}} + \cdots + 1)/(A_{s+1}p^{n_{s+1}}(p^n + \cdots + 1)) \times \ldots \times B_r p^{n_{r-1}}(p^{n_r} + \cdots \\
+ 1)/(A_r p^{n_r}(p^{n_{r-1}} + \cdots + 1)) \times A_{r+1}p^{n_r}(p^{n_{r+1}} + \cdots + 1)/(B_{r+1}p^{n_{r+1}}(p^{n_r} \\
+ \cdots + 1)) \times A_{r+2}p^{n_{r+1}}(p^{n_{r+2}} + \cdots + 1)/(B_{r+2}p^{n_{r+2}}(p^{n_{r+1}} + \cdots + 1)) \times \ldots \\
\times A_x p^{n_{x-1}}(p^{n_x} + \cdots + 1)/(B_x p^{n_x}(p^{n_{x-1}} + \cdots + 1)) = 1
\end{aligned}
$$

$$
\begin{aligned}
&B_{s+1}B_{s+2} \ldots B_r A_{r+1}A_{r+2} \ldots A_x p^{n-n_x}(p^{n_x} + \cdots + 1) \\
&= A_{s+1}A_{s+2} \ldots A_r B_{r+1}B_{r+2} \ldots B_x(p^n + \cdots + 1) \quad \ldots (B)
\end{aligned}
$$

When $n_x < n$, it becomes contradiction since the right side of above expression does not include factor p.

When $n_x = n$,

$B_{s+1}B_{s+2} \ldots B_r A_{r+1}A_{r+2} \ldots A_x = A_{s+1}A_{s+2} \ldots A_r B_{r+1}B_{r+2} \ldots B_x \quad \ldots (C)$

Let v be a rational number. If

$$v = \prod_k (1 + p_k + p_k^2 + \cdots + p_k^{q_k}) \,/ \prod_k p_k^{q_k}$$

holds, assume that v is not an integer. $\cdots$(D)

Let $e_r$, $f_r$ be odd integers and $g_r$ be a rational number,

$$e_r = \prod_{k=1}^{r} (p_k^{q_k} + \cdots + 1)\,, f_r = \prod_{k=1}^{r} p_k^{q_k}\,, g_r = e_r/f_r$$

holds.

$g_{r+1} = e_{r+1}/f_{r+1} = e_r/f_r \times (p_{r+1}^{q_{r+1}} + \cdots + 1)/p_{r+1}^{q_{r+1}} > e_r/f_r = g_r$

Let $q_1{}'$ be an even integer and $q_1' > q_1$ holds. Let $g_r$ be $g_r{}'$ when $q_1$ becomes $q_1'$,

$g_r' = (p_1^{q_1}(p_1^{q_1{}'} + \cdots + 1)/p_1^{q_1{}'}(p_1^{q_1} + \cdots + 1))g_r > g_r$

is established.

It is assumed that $q_k$ becomes $q_k - h_k$ by changing $q_k$ than before for $g_r$. $h_k$ is an even integer. Then assume that r becomes s(s > r), $g_s = g_r$ and $g_s$ is not changed.

$$\begin{aligned} g_s/g_r = {} & p_{r+1}^{q_{r+1}} \times \ldots \times p_s^{q_s}/((p_{r+1}^{q_{r+1}} + \cdots + 1) \times \ldots \times (p_s^{q_s} + \cdots + 1)) \times p_1^{q_1} \times \ldots \\ & \times p_r^{q_r}\left(p_1^{q_1-h_1} + \cdots + 1\right) \ldots (p_r^{q_r-h_r} + \cdots + 1)/(p_1^{q_1-h_1} \times \ldots \\ & \times p_r^{q_r-h_r}(p_1^{q_1} + \cdots + 1) \ldots (p_r^{q_r} + \cdots + 1)) = 1 \end{aligned}$$

$$\begin{aligned} & p_{r+1}^{q_{r+1}} \times \ldots \times p_s^{q_s}/((p_{r+1}^{q_{r+1}} + \cdots + 1) \times \ldots \times (p_s^{q_s} + \cdots + 1)) \times p_1^{h_1} \times \ldots \\ & \quad \times p_r^{h_r}\left(p_1^{q_1-h_1} + \cdots + 1\right) \ldots (p_r^{q_r-h_r} + \cdots \\ & \quad + 1)/((p_1^{q_1} + \cdots + 1) \ldots (p_r^{q_r} + \cdots + 1)) = 1 \end{aligned}$$

$$\begin{aligned} & p_{r+1}^{q_{r+1}} \times \ldots \times p_s^{q_s} \times p_1^{h_1} \times \ldots \times p_r^{h_r}\left(p_1^{q_1-h_1} + \cdots + 1\right) \ldots \left(p_r^{q_r-h_r} + \cdots + 1\right) \\ & \quad = (p_1^{q_1} + \cdots + 1) \ldots (p_r^{q_r} + \cdots + 1)(p_{r+1}^{q_{r+1}} + \cdots + 1) \ldots (p_s^{q_s} + \cdots + 1) \end{aligned}$$

$$\begin{aligned} & p_{r+1}^{q_{r+1}} \times \ldots \times p_s^{q_s}\left(p_1^{q_1} + \cdots + p_1^{h_1}\right) \ldots \left(p_r^{q_r} + \cdots + p_r^{h_r}\right) \\ & \quad = (p_1^{q_1} + \cdots + 1) \ldots (p_r^{q_r} + \cdots + 1)(p_{r+1}^{q_{r+1}} + \cdots + 1) \ldots (p_s^{q_s} + \cdots + 1) \end{aligned}$$

When $h_k < 0$, multiply both sides by $p_k^{-h_k}$ so that both sides become integers. If the condition (D) holds, there is at least one prime number from $p_{r+1}$ to $p_s$ on the left side. $a = (p_1^{q_1} + \cdots + 1) \ldots (p_r^{q_r} + \cdots + 1) = cp^n$ holds and from the expression ⑦, c must be a product of primes from $p_1$ to $p_r$. Thereby, the above equation does not hold since it is inappropriate when there is even one prime number other than $p_1$ to $p_r$. When changing the value of $p_k$, it is equivalent to dividing by $p_k^{q_k}$ and then multiplying by new $p_k^{q_k}$, so it is sufficient to consider only the changes of $q_k$ and r. From above, since $g_r$ does not chord the original value when $q_k$ or r is increased or decreased, it takes unique values for the variables $p_k$, $q_k$, r.

From the above proof,

$$p_{r+1}^{q_{r+1}} \times \ldots \times p_s^{q_s}\left(p_1^{q_1-h_1} + \cdots + 1\right) \ldots \left(p_r^{q_r-h_r} + \cdots + 1\right)\left(p^{n-h} + \cdots + 1\right)$$
$$= 2p_1^{q_1-h_1} \ldots p_r^{q_r-h_r} p^{n-h}(p_{r+1}^{q_{r+1}} + \cdots + 1) \ldots (p_s^{q_s} + \cdots + 1)$$

If the condition (D) holds,

$$(p_{r+1}^{q_{r+1}} + \cdots + 1) \ldots (p_s^{q_s} + \cdots + 1) = vp_{r+1}^{q_{r+1}} \times \ldots \times p_s^{q_s}$$

is established,

$$\left(p_1^{q_1-h_1} + \cdots + 1\right) \ldots \left(p_r^{q_r-h_r} + \cdots + 1\right)\left(p^{n-h} + \cdots + 1\right) = 2vp_1^{q_1-h_1} \ldots p_r^{q_r-h_r} p^{n-h}$$

From the expression (A),

$$(p_1^{q_1} + \cdots + 1) \ldots (p_r^{q_r} + \cdots + 1)(p^n + \cdots + 1) = 2p_1^{q_1} \ldots p_r^{q_r} p^n$$

holds. Dividing by the above two expressions,

$$(p_1^{q_1} + \cdots + 1) \ldots (p_r^{q_r} + \cdots + 1)(p^n + \cdots + 1)$$
$$/\left(\left(p_1^{q_1-h_1} + \cdots + 1\right) \ldots \left(p_r^{q_r-h_r} + \cdots + 1\right)\left(p^{n-h} + \cdots + 1\right)\right)$$
$$= p_1^{h_1} \ldots p_r^{h_r} p^h / v$$

$$v(p_1^{q_1} + \cdots + 1) \ldots (p_r^{q_r} + \cdots + 1)(p^n + \cdots + 1)$$
$$= \left(p_1^{q_1} + \cdots + p_1^{h_1}\right) \ldots \left(p_r^{q_r} + \cdots + p_r^{h_r}\right)\left(p^n + \cdots + p^h\right)$$

Since

$$(p_1^{q_1} + \cdots + 1) \ldots (p_r^{q_r} + \cdots + 1)(p^n + \cdots + 1)$$
$$> \left(p_1^{q_1} + \cdots + p_1^{h_1}\right) \ldots \left(p_r^{q_r} + \cdots + p_r^{h_r}\right)\left(p^n + \cdots + p^h\right)$$

is established,

$$0 < v < 1$$

holds. And v is not an integer. Therefore, the condition (D) holds.

From the above proof,

$$g_r = A_1A_2 \ldots A_s/B_1B_2 \ldots B_s \times A_{r+1}A_{r+2} \ldots A_x/B_{r+1}B_{r+2} \ldots B_x$$

$g_r$ must be represented uniquely, and the expression (C) does not satisfied. When dividing by the prime number in the expression of p, a contradiction arises since the prime number not included in b is in the expression of p. Therefore, when p holds $p \equiv 1 \pmod 4$ and $p \geqq 5$, the number of the solution $(a, b, p, n)$ satisfying $R = 1$ is at most one.

Since $(a, b, p, n) = (1,1,1,1)$ is inappropriate solution and the expression (C) becomes contradiction, there is one solution when $n_x = n = 1$. Therefore, there are no odd perfect numbers when $n = 1$.

Define the operation [multiplication] and the operation [division] as follows.

Assuming that p in the equation of R is replaced by p' by multiplying $A_i/B_i$, define operation [multiplication] to R as follows.

$$p' = 2\prod_{k=1}^{r} p_k^{d_k} \times p_i^{d_i} - 1$$

$$0 \leqq d_i \leqq q_i$$

Here, let i be $i > r$. Suppose operation [division] is division by $A_j/B_j$ for R, and if $p_j$ is included in p in the expression R, $p_j$ is deleted as $d_j = 0$. Here, assuming that j satisfies $1 \leqq j \leqq r$.

In the proof of the expression (B), it is assumed that p changes on the way, and finally p becomes $p_x$.

$$A_1 \dots A_r = cp^n$$

$$2B_1 \dots B_r = c(p^n + \cdots + 1)$$

$$A_1 \dots A_x = c' p_x^{n_x}$$

$$2B_1 \dots B_x = c'(p_x^{n_x} + \cdots + 1)$$

It is assumed that the above expressions are satisfied.

$$B_{s+1}B_{s+2} \dots B_r A_{r+1} A_{r+2} \dots A_x p^n (p_x^{n_x} + \cdots + 1)$$
$$= A_{s+1}A_{s+2} \dots A_r B_{r+1} B_{r+2} \dots B_x p_x^{n_x} (p^n + \cdots + 1)$$

$$B_{s+1}B_{s+2} \dots B_r A_1 \dots A_r A_{r+1} A_{r+2} \dots A_x p^n (p_x^{n_x} + \cdots + 1)$$
$$= A_1 \dots A_r A_{s+1} A_{s+2} \dots A_r B_{r+1} B_{r+2} \dots B_x p_x^{n_x} (p^n + \cdots + 1)$$

$$B_{s+1}B_{s+2} \dots B_r c' p_x^{n_x} p^n (p_x^{n_x} + \cdots + 1)$$
$$= A_1 \dots A_r A_{s+1} A_{s+2} \dots A_r B_{r+1} B_{r+2} \dots B_x p_x^{n_x} (p^n + \cdots + 1)$$

$$B_{s+1}B_{s+2} \dots B_r c' p^n (p_x^{n_x} + \cdots + 1) = A_1 \dots A_r A_{s+1} A_{s+2} \dots A_r B_{r+1} B_{r+2} \dots B_x (p^n + \cdots + 1)$$

$$B_1 \dots B_r B_{s+1} B_{s+2} \dots B_r c' p^n (p_x^{n_x} + \cdots + 1)$$
$$= A_1 \dots A_r A_{s+1} A_{s+2} \dots A_r B_1 \dots B_r B_{r+1} B_{r+2} \dots B_x (p^n + \cdots + 1)$$

$$B_1 \dots B_r B_{s+1} B_{s+2} \dots B_r c' p^n (p_x^{n_x} + \cdots + 1)$$
$$= A_1 \dots A_r A_{s+1} A_{s+2} \dots A_r c'(p_x^{n_x} + \cdots + 1)/2 \times (p^n + \cdots + 1)$$

$$B_1 \dots B_r B_{s+1} B_{s+2} \dots B_r p^n = A_1 \dots A_r A_{s+1} A_{s+2} \dots A_r / 2 \times (p^n + \cdots + 1)$$

$$c(p^n + \cdots + 1)/2 \times B_{s+1} B_{s+2} \dots B_r p^n = cp^n A_{s+1} A_{s+2} \dots A_r / 2 \times (p^n + \cdots + 1)$$

$$B_{s+1} B_{s+2} \dots B_r = A_{s+1} A_{s+2} \dots A_r$$

is established. It becomes contradiction since $A_k > B_k$ holds when the operations [division] are performed.

Consider a tree whose vertex is $(a, b, p, n) = (1,1,1,1)$, and when the operations [multiplication] are performed, it becomes a child node. For example, consider a child node connected to a vertex as follows.

$(a, b, p, n) = (13,9,5,5)$ as $p_1 = 3$, $q_1 = 2$ and $d_1 = 1$

$(a, b, p, n) = (13,9,17,9)$ as $p_1 = 3$, $q_1 = 2$ and $d_1 = 2$

$(a, b, p, n) = (57,49,97,13)$ as $p_1 = 7$, $q_1 = 2$ and $d_1 = 2$

Suppose that the operations [multiplication] for changing the value of p are performed first, and then the operations [multiplication] for not changing the value of p are performed to create a tree structure. Here, when there is a solution in a certain p and there is a solution even in the other value p', considering a set of line segments connecting these two points in four-dimensional space $(a, b, p, n)$. If $R = 1$ holds again when performing operation [multiplication] from one point where $R = 1$,

$$1 \times A_{r+1}p^n({p_{r+1}}^{n_{r+1}} + \cdots + 1)/(B_{r+1}{p_{r+1}}^{n_{r+1}}(p^n + \cdots + 1)) \times A_{r+2}{p_{r+1}}^{n_{r+1}}({p_{r+2}}^{n_{r+2}} + \cdots + 1)/(B_{r+2}{p_{r+2}}^{n_{r+2}}({p_{r+1}}^{n_{r+1}} + \cdots + 1)) \times \ldots \times A_x{p_{x-1}}^{n_{x-1}}({p_x}^{n_x} + \cdots + 1)/(B_x{p_x}^{n_x}({p_{x-1}}^{n_{x-1}} + \cdots + 1)) = 1$$

$$A_{r+1}A_{r+2} \ldots A_x/(B_{r+1}B_{r+2} \ldots B_x) = {p_x}^{n_x}(p^n + \cdots + 1)/(p^n({p_x}^{n_x} + \cdots + 1))$$

$$A_1A_2 \ldots A_x({p_x}^{n_x} + \cdots + 1)/(B_1B_2 \ldots B_x{p_x}^{n_x}) = A_1A_2 \ldots A_r(p^n + \cdots + 1)/(B_1B_2 \ldots B_rp^n) \quad \ldots (E)$$

Assume that $g_r = A_1A_2 \ldots A_r(p^n + \cdots + 1)/(B_1B_2 \ldots B_xp^n)$ holds. Here, it is assumed that $q_k$ becomes $q_k - h_k$ by changing $q_k$ than before and n becomes $n - h(n - h > 0)$ for $g_r$. $h_k$ is an even integer and h is a non-negative integer that is a multiple of 4. Then assuming that r becomes $s(s > r)$, $g_s = g_r$ and $g_s$ is not changed, by the same calculation as the proof on page 7,

$$g_s/g_r = {p_{r+1}}^{q_{r+1}} \times \ldots \times {p_s}^{q_s}/(({p_{r+1}}^{q_{r+1}} + \cdots + 1) \times \ldots \times ({p_s}^{q_s} + \cdots + 1)) \times {p_1}^{q_1} \times {p_2}^{q_2} \times \ldots \times {p_r}^{q_r}p^n\left({p_1}^{q_1-h_1} + \cdots + 1\right) \ldots ({p_r}^{q_r-h_r} + \cdots + 1)(p^{n-h} + \cdots + 1)/({p_1}^{q_1-h_1} \times \ldots \times {p_r}^{q_r-h_r}p^{n-h}({p_1}^{q_1} + \cdots + 1) \ldots ({p_r}^{q_r} + \cdots + 1)(p^n + \cdots + 1)) = 1$$

$${p_{r+1}}^{q_{r+1}} \times \ldots \times {p_s}^{q_s}\left({p_1}^{q_1} + \cdots + {p_1}^{h_1}\right) \ldots \left({p_r}^{q_r} + \cdots + {p_r}^{h_r}\right)(p^n + \cdots + p^h) = ({p_1}^{q_1} + \cdots + 1) \ldots ({p_r}^{q_r} + \cdots + 1)(p^n + \cdots + 1)({p_{r+1}}^{q_{r+1}} + \cdots + 1) \ldots ({p_s}^{q_s} + \cdots + 1)$$

Since $(p_1^{q_1} + \cdots + 1) \dots (p_r^{q_r} + \cdots + 1) = cp^n$ holds,

$$p_{r+1}^{q_{r+1}} \times \dots \times p_s^{q_s}\left(p_1^{q_1} + \cdots + p_1^{h_1}\right) \dots \left(p_r^{q_r} + \cdots + p_r^{h_r}\right)(p^{n-h} + \cdots + 1)$$
$$= cp^{n-h}(p^n + \cdots + 1)(p_{r+1}^{q_{r+1}} + \cdots + 1) \dots (p_s^{q_s} + \cdots + 1)$$

When $h_k < 0$, multiply both sides by $p_k^{-h_k}$ so that both sides become integers. If the condition (D) holds, there is at least one prime number from $p_{r+1}$ to $p_s$ on the left side. Because c and $p^n + \cdots + 1$ are products of prime numbers from $p_1$ to $p_r$ and in the case of $s > r + 1$, the left side has prime numbers that is not on the right side as a factor, this expression does not hold. In the case of $s = r + 1$, when $p \neq p_s$, this expression does not hold in the same way. When $p = p_s$ and $q_s > n - h$, since there is a prime factor p only on the left side, this expression does not hold. Therefore, since except for the case of $s = r + 1$, $p = p_s$ and $q_s < n - h$ $g_r$ must be uniquely expressed, the expression (E) does not hold. When $s = r + 1$, $p = p_s$ and $q_s < n - h$, substituting $B_x = p^{q_s}$ into the expression (E) as $x = r + 1$,

$$A_1A_2 \dots A_r(p^{q_s} + \cdots + 1)(p_x^{n_x} + \cdots + 1)/(B_1B_2 \dots B_r p^{q_s} p_x^{n_x})$$
$$= A_1A_2 \dots A_r(p^n + \cdots + 1)/(B_1B_2 \dots B_r p^n)$$
$$(p^{q_s} + \cdots + 1)(p_x^{n_x} + \cdots + 1)/(p^{q_s} p_x^{n_x}) = (p^n + \cdots + 1)/p^n$$
$$(p^{q_s} + \cdots + 1)(p_x^{n_x} + \cdots + 1)p^{n-q_s} = (p^n + \cdots + 1)p_x^{n_x}$$

Since the right side does not have a prime number p as a factor, this expression does not hold.

If one point is $(a, b, p, n) = (1,1,1,1)$, when $s > r + 1$ or $p \neq p_s$ $g_s \neq g_r$ holds similarly and when $s = r + 1$ and $p = p_s$ it becomes inappropriate, since prime number $p_s$ is 1. Therefore, except for $(a, b, p, n) = (1,1,1,1)$, there is no solution with $g_r = 2$. From the above, there are no odd perfect numbers.

4. Complement

From the equation ⑤,

$2bp^n(p-1) = a(p^{n+1}-1)$

$2 = a(p^{n+1}-1)/(bp^n(p-1))$

$2 = (p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)\ldots(p_r{}^{q_r+1}-1)(p^{n+1}-1)$
$\qquad /(p_1{}^{q_1}p_2{}^{q_2}\ldots p_r{}^{q_r}p^n(p_1-1)(p_2-1)\ldots(p_r-1)(p-1))$

$2(p_1{}^{q_1+1}-p_1{}^{q_1})(p_2{}^{q_2+1}-p_2{}^{q_2})\ldots(p_r{}^{q_r+1}-p_r{}^{q_r})(p^{n+1}-p^n)$
$\qquad = (p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)\ldots(p_r{}^{q_r+1}-1)(p^{n+1}-1)$

We consider when $r = 2$.

$(p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)(p^{n+1}-1) = 2(p_1{}^{q_1+1}-p_1{}^{q_1})(p_2{}^{q_2+1}-p_2{}^{q_2})(p^{n+1}-p^n)$

Let s, t, u be integers,

$s = p_1{}^{q_1+1}-1$

$t = p_2{}^{q_2+1}-1$

$u = p^{n+1}-1$

are.

$stu = 2(p_1{}^{q_1+1}-1-(p_1{}^{q_1}-1))(p_2{}^{q_2+1}-1-(p_2{}^{q_2}-1))(p^{n+1}-1-(p^n-1))$

$stu = 2(s-(s+1)/p_1+1)(t-(t+1)/p_2+1)(u-(u+1)/p+1)$

$pp_1p_2stu = 2((s+1)p_1-(s+1))((t+1)p_2+(t+1))\big((u+1)p+(u+1)\big)$

$pp_1p_2stu = 2(s+1)(p_1-1)(t+1)(p_2-1)(u+1)(p-1)$

$stu/((s+1)(t+1)(u+1)) = 2(p_1-1)(p_2-1)(p-1)/(p_1p_2p)$

Since $stu/((s+1)(t+1)(u+1))$ is a monotonically increasing function for variables s, t and u, if

$s \geqq 3^{2+1}-1 = 26,\ p_1 = 3,\ q_1 = 2$

$t \geqq 7^{2+1}-1 = 342,\ p_2 = 7,\ q_2 = 2$

$u \geqq 5^2-1 = 24,\ p = 5,\ n = 1$

holds,

$stu/((s+1)(t+1)(u+1)) \geqq 26\times342\times24/(27\times343\times25) = 7904/8575$

$2(p_1-1)(p_2-1)(p-1)/(p_1p_2p) = 2\times2\times6\times4/(3\times7\times5) = 32/35$

Since $stu/((s+1)(t+1)(u+1))$ is limited to 1 when s, t and u are infinite,

$stu/((s+1)(t+1)(u+1)) < 1$

If $f(p_1, p_2, p) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$ holds, it is sufficient to consider a combination where $f(p_1, p_2, p) < 1$.

$f(3,7,5) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35$

$f(3,11,5) = 2 \times 2 \times 10 \times 4/(3 \times 11 \times 5) = 32/33$

$f(3,13,5) = 2 \times 2 \times 12 \times 4/(3 \times 13 \times 5) = 64/65$

$f(3,17,5) = 2 \times 2 \times 16 \times 4/(3 \times 17 \times 5) = 256/255$

$f(3,7,13) = 2 \times 2 \times 6 \times 12/(3 \times 7 \times 13) = 96/91$

$f(3,5,17) = 2 \times 2 \times 4 \times 16/(3 \times 5 \times 17) = 256/255$

From the above, when $r = 2$, a combination $(p_1, p_2, p) = (3,7,5), (3,11,5), (3,13,5)$ can be considered.

Let $q_k$ be 2 and $n = 1$, if $g(p_1, p_2, p) = (p_1{}^3 - 1)(p_2{}^3 - 1)(p^2 - 1)/(p_1{}^3 p_2{}^3 p^2)$,

$g(3,7,5) = 26 \times 342 \times 24/(3^3 7^3 5^2) = 7904/8575 > 32/35$

$g(3,11,5) = 26 \times 1330 \times 24/(3^3 11^3 5^2) = 55328/59895$

$g(3,13,5) = 26 \times 2196 \times 24/(3^3 13^3 5^2) = 3904/4225$

Since the function g is the minimum in the case of $q_k = 2$ and $n = 1$, there is no solution $q_k$ and n when $g > f$, so the case of $(p_1, p_2, p) = (3,7,5)$ becomes unsuitable.

$$stu/((s + 1)(t + 1)(u + 1)) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$$

$$(p_1{}^{q_1+1} - 1)(p_2{}^{q_2+1} - 1)(p^{n+1} - 1)/(p_1{}^{q_1+1} p_2{}^{q_2+1} p^{n+1})$$
$$= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$$

If $F(p_1, p_2, p) = (p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$,

$F(p_1{}^{q_1+1}, p_2{}^{q_2+1}, p^{n+1}) = 2F(p_1, p_2, p)$

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6. References

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