Refutation of provability logic GL and Japaridze's derived polymodal (GLP)

Abstract: We evaluate eight equations for provability logic (GL) and the derived polymodal logic of Japaridze (GLB, GLP). None is tautologous, hence refuting provability logic. These form a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓ ,·; \ Not And;
> Imply, greater than, →, ⇒ , ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇐⇒, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) N as non-contingency, Δ, ordinal 1; (%z<#z) C as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y); (A=B) (A~B); (B>A) (A⊢B); (B>A) (A⊨B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: Rineke (L.C.) Verbrugge (2017). Provability logic.
plato.stanford.edu/entries/logic-provability/ L.C.Verbrugge@rug.nl

These Löb conditions, as they are called nowadays, seem to cry out for a modal logical investigation, where the modality □ stands for provability in PA [Peano arithmetic]. Ironically, the first time that the formalized version of Löb's theorem was stated as the modal principle □(□*A*→*A*)→□*A* was in a paper … in 1963 about the logical basis of ethics, which did not consider arithmetic at all.

2.1 Axioms and rules Propositional provability logic is often called GL, after Gödel and Löb. (Alternative names found in the literature for equivalent systems are L, G, KW, K4W, and PrL). The logic GL results from adding the following axiom to the basic modal logic K: □(□*A*→*A*)→□*A*. (2.1.1)

LET p: p.

#(#p>p)>#p ; CTCT CTCT CTCT CTCT (2.1.2)

Remark 2.1.1: Eq. 2.1.2 reiterates that the Löb axiom is *no*t tautologous.

It is not difficult to see that the modal axiom □*A*→□□*A* (known as Axiom 4 of modal logic) is indeed provable in GL. To prove this, one uses the substitution *A*∧□*A* for *A* in the axiom (GL). (2.2.1)

LET p=(p&#p), to substitute into Eq. 2.1.2:

#(#(p&#p)>(p&#p))>#(p&#p) ; CTCT CTCT CTCT CTCT (2.2.2)

Remark 2.2.2: Eq. 2.2.2 is *not* tautologous. Axiom 4 of modal logic is not probable in GL by substitution. In fact, Eqs. 2.1.2 and 2.2.2 produce the equivalent values in truth table results, and hence are identical expressions.

2.2 The fixed point theorem The main “modal” result about provability logic is the fixed point theorem … . It says essentially that self-reference is not really necessary, in the following sense. Suppose that all occurrences of the propositional variable p in a given formula $A(p)$ are under the scope of the provability operator, for example $A(p)=\neg\Box p$, or $A(p)=\Box(p\rightarrow q)$. Then there is a formula B in which p does not appear, such that all propositional variables that occur in B already appear in $A(p)$, and such that $\mathrm{GL}\vdash B\leftrightarrow A(B)$. This formula B is called a *fixed point* of $A(p)$. (2.2.1.1)

LET p, q, r: p, A, B

(((q&p)=~#p)+((q&p)=#(p>q)))>(r=(q&r)) ;
TTTT N**F**TT TTTT N**F**TT (2.2.1.2)

Remark 2.2.1.2: Eq. 2.2.1.2 is *not* tautologous. Brouwer’s fixed point theorem is not proved by GL.

Moreover, the fixed point is unique, or more accurately, if there is another formula C such that $\mathrm{GL}\vdash C\leftrightarrow A(C)$, then we must have $\mathrm{GL}\vdash B\leftrightarrow C$. … For example, suppose that $A(p)=\neg\Box p$. Then the fixed point produced by such an algorithm is $\neg\Box\bot$, and indeed one can prove that $\mathrm{GL}\vdash\neg\Box\bot\leftrightarrow\neg\Box(\neg\Box\bot)$. (2.2.2.1)

~(#(s@s)=(s=s))=~(#(~(#(s@s)=(s=s))=(s=s))) ;
CCCC CCCC CCCC CCCC (2.2.2.2)

Remark 2.2.2.2: Eq. 2.2.2.2 is *not* tautologous. The truth table result of consistent C is the value for falsity. The fixed point is not proved as unique by GL. The assertion below of the second incompleteness theorem is also not proved to mean sufficiently strong consistent arithmetical theories can prove their own consistency.

If this is read arithmetically, the direction from left to right is just the formalized version of Gödel’s second incompleteness theorem: if a sufficiently strong formal theory T like Peano Arithmetic does not prove a contradiction, then it is not provable in T that T does not prove a contradiction. Thus, sufficiently strong consistent arithmetical theories cannot prove their own consistency.

5.3 Propositional quantifiers

Another way to extend the framework of propositional provability logic is to add propositional quantifiers, so that one can express principles like Goldfarb’s: $\forall p\forall q\exists r\Box((\Box p\vee\Box q)\leftrightarrow\Box r)$, (5.3.1.1)

#((##p+##q)=#%r)=(p=p) ; N**FFF** **F**NNN N**FFF** **F**NNN (5.3.1.2)

saying that for any two sentences there is a third sentence which is provable if and only if either of the first two sentences is provable. This principle is provable in Peano Arithmetic … . The set of sentences of GL with propositional quantifiers that is arithmetically valid turns out to be undecidable … .

5.4 Japaridze’s bimodal and polymodal provability logics Japaridze’s bimodal provability logic GLB … has three mixed axiom schemes … :

LET p, q, r: *A*, *k*, *m*, *n*.

$[m]A\rightarrow[n]A$, for $m\leq n$ (5.4.1.1)

~(s<r)>((r&p)>(s&p)) ; TTTT T**F**T**F** TTTT TTTT (5.4.1.2)

$\langle k\rangle A\rightarrow[n]\langle k\rangle A$, for $k<n$ (5.4.2.1)

~(s<q)>((q&p)>(s&(q&p))) ; TTT**F** TTTT TTT**F** TTTT (5.4.2.2)

$[m]A\rightarrow[n][m]A$, for $m\leq n$ (5.4.3.1)

~(s<r)>((r&p)>((s&p)&r)) ; TTTT T**F**T**F** TTTT T**F**T**F** (5.4.3.2)

Remark 5.4: GLB contains three axioms which are *not* tautologous. This serves to refute GLB and the derived GLP.

The Eqs. evaluated are *not* tautologous and deny GL, GLB, and GLP to refute provability logic.