Division by Zero Calculus and Pompe’s Theorem

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Abstract. In this paper, we will introduce the application of the division by zero calculus to geometry and it will show the power of the new calculus.

Keywords. Division by zero calculus, 0/0 = 1/0 = z/0 = 0, Laurent expansion, Pompe’s example.

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1. DIVISION BY ZERO CALCULUS

We will give the definition of the division by zero calculus. For any Laurent expansion around \( z = a \),

\[
f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n,
\]

we define the identity, by the division by zero

\[
f(a) = C_0.
\]

In addition, we will refer to the naturality of the division by zero calculus.

Recall the Cauchy integral formula for an analytic function \( f(z) \); for an analytic function \( f(z) \) around \( z = a \) and for a smooth simple Jordan closed curve \( \gamma \) enclosing one time the point \( a \), we have

\[
f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} \, dz.
\]

Even when the function \( f(z) \) has any singularity at the point \( a \), we assume that this formula is valid as the division by zero calculus. We define the value of the function \( f(z) \) at the singular point \( z = a \) with the Cauchy integral.

The division by zero calculus opens a new world since Aristotele-Euclid. See, in particular, [1] and also the references for recent related results.

On February 16, 2019 H. Okumura introduced the surprising news in Research Gate to Saitoh:

Jose Manuel Rodriguez Caballero
Added an answer
In the proof assistant Isabelle/HOL we have \( x/0 = 0 \) for each number \( x \). This is advantageous in order to simplify the proofs. You can download this proof assistant here: [https://isabelle.in.tum.de/](https://isabelle.in.tum.de/).

J.M.R. Caballero kindly showed surprisingly several examples by the system that

\[
\tan \frac{\pi}{2} = 0, \\
\log 0 = 0, \\
\exp \frac{1}{x} (x = 0) = 1,
\]

and others to Saitoh. Furthermore, for the presentation at the annual meeting of the Japanese Mathematical Society at the Tokyo Institute of Technology:

March 17, 2019; 9:45-10:00 in Complex Analysis Session, *Horn torus models for the Riemann sphere from the viewpoint of division by zero* with [1],

he kindly sent the message:

It is nice to know that you will present your result at the Tokyo Institute of Technology. Please remember to mention Isabelle/HOL, which is a software in which \( x/0 = 0 \). This software is the result of many years of research and a millions of dollars were invested in it. If \( x/0 = 0 \) was false, all these money was for nothing. Right now, there is a team of mathematicians formalizing all the mathematics in Isabelle/HOL, where \( x/0 = 0 \) for all \( x \), so this mathematical relation is the future of mathematics. [https://www.cl.cam.ac.uk/~lp15/Grants/Alexandria/](https://www.cl.cam.ac.uk/~lp15/Grants/Alexandria/)

Surprisingly enough, he sent his e-mail at 2019.3.30.18:42 as follows:

Nevertheless, you can use that \( x/0 = 0 \), following the rules from Isabelle/HOL and you will obtain no contradiction. Indeed, you can check this fact just downloading Isabelle/HOL: [https://isabelle.in.tum.de/](https://isabelle.in.tum.de/)

and copying the following code

theory DivByZeroSatoih imports Complex Main
begin
theorem T: \(?x/0 + 2000 = 2000? \) for \( x :: \) complex by simp
end

In this paper, from an example of Pompe ([16]), we will see the power of division by zero and division by zero calculus clearly.

2. Pompe’s theorem

Generalizing a sangaku problem, W. Pompe gave the following theorem (see Figure 1):
Theorem 1 ([16]). Let ABC be an equilateral triangle and let G be a point on the side AB. Points P and Q lie on the sides AC and BC, respectively, and satisfy $\angle PGC = \angle QGC = \pi/6$. Let $\alpha = \angle AGP$ and $\beta = \angle BGQ$. Denote by $r_1$ and $r_2$ the inradii of the triangles AGP and BGQ, respectively. Then

$$r_1/r_2 = \sin 2\alpha/\sin 2\beta.$$ 

We now concern with the case $\beta = \pi/2$ in the sense of division by zero and division by zero calculus. In this case the point $G$ coincides with $B$, then the triangle $BQG$ degenerates to the point $B$, i.e., $r_2 = 0$ (see Figure 2). In this case the left side of (1) equals $r_1/0 = 0$. Also the right side equals $\sin 2\alpha/\sin 2\beta = \sin 2\alpha/0 = 0$. Therefore (1) holds.

On the other hand the right side of (1) is a function of $\beta$; $\sin 2(2\pi/3 - \beta)/\sin 2\beta$ and

$$\sin 2(2\pi/3 - x)/\sin 2x = -\sqrt{3}/4x + 1/2 + x/\sqrt{3} + \cdots.$$ 

This implies that

$$r_1/r_2 = \sin 2\alpha/\sin 2\beta = 1/2$$

in the case $\beta = 0$ by division by zero calculus. The large circle in Figure 3 has radius $r_2 = 2r_1$ and center $B = Q$. It is orthogonal to the lines $AB$, $BC$ and the
perpendicular to $AB$ at $B$. Therefore the circle still touches the three lines, since $\tan \pi/2 = 0$, i.e., it is the circle of radius $2r_1$ touching the lines $AB$, $BC$ and the perpendicular to $AB$ at $B$.

Note that for many cases, we can calculate the division by zero calculus by Mathematica, because it is just a coefficient of Laurent expansion.

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REFERENCES


