About Goldbach's Conjecture

JESÚS ESTEVE
JORGE ENRIQUE MARTÍNEZ
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Authors

Jesús Esteve Pérez. (Spain).
Taller de Acción Audiovisual, S.L.
editorial.eb@gmail.com

with the collaboration of Jorge Enrique Martínez. (Argentina).

Abstract

We proof Goldbach’s Conjecture. We use results obtained by Srinivāsa A. Rāmānujan (specifically in his paper *A Proof of Bertrand's Postulate*). A generalization of the conjecture is also proven for every natural not coprime with a natural $m > 1$ and greater or equal than $2m$.

Notation and observations

- Natural numbers

  $\mathbb{N}$ denotes the set of natural numbers, and all of them are here greater than zero (positive integers).

- Primorial:

  It is denoted by $p\#$ the primorial of the prime number $p$, meaning the product of all primes less or equal than $p$.

- Omission of subjects:

  To speed up the writing, we will omit sometimes the word *number* following others as *prime, composite, even, odd, natural, coprime*, since they are obvious here.

- In what follows, $p_i$ denotes the $i$-th prime.
Preliminary

The Bertrand’s postulate also known as Bertrand-Chebyshev theorem, was proposed by Joseph Bertrand in 1845 [1] and proven by Pafnuti L. Chebyshëv in 1850 [2], [5], Srinivāsa A. Rāmānujan in 1919 [3] and Paul Erdős in 1932 [4].

Here we will use the following result, obtained by Ramanujan’s in [3], after his proof of Bertrand’s postulate:

**Theorem 1. (Ramanujan)**

Let \( \pi(x) \) denote the number of primes not exceeding \( x \). Then \( \pi(x) - \pi(x/2) \geq 1, 2, 3, 4, 5, \ldots \), if \( x \geq 2, 11, 17, 29, 41, \ldots \) respectively.

**PROOF.** See [3].

The Bertrand’s postulate follows from the first case: if \( x \geq 2 \), then \( \pi(x) - \pi(x/2) \geq 1 \). Assigning to \( x \) a natural number \( n > 1 \), we have that \( \pi(n) - \pi(n/2) \geq 1 \) and we can write:

For every natural \( n > 1 \) there exists at least one prime number \( p \) such that \( n/2 < p \leq n \).

We will use here the less restrictive statement:

For every \( n > 1 \) there exists a prime number \( p \) such that \( n/2 \leq p \leq n \) \hspace{1cm} (01)

**Results**

**Lemma.**

Let’s be \( i \geq 1 \) and \( \ell \geq 2 \), natural numbers.

If \( \ell \) is coprime with \( p_i \# \) and \( p_{i+1} \leq \ell < (p_{i+1})^2 \) then \( \ell \) is prime.

**PROOF.**

We prove that the smallest composite coprime with \( p_i \# \) greater than \( p_{i+1} \) is \( (p_{i+1})^2 \). Since \( p_{i+1} \) is the prime immediately after \( p_i \), \( p_{i+1} \) cannot be a divisor of \( p_i \# \). Likewise, none of the the powers of \( p_{i+1} \) share any divisors with \( p_i \# \). The smallest of this powers greater than \( p_{i+1} \) is \( (p_{i+1})^2 \), consequently \( (p_{i+1})^2 \) is a composite number actually being coprime with \( p_i \# \). Note that \( (p_{i+1})^2 \) is coprime with \( p_i \# \) because \( (p_{i+1})^2 = (p_{i+1}) (p_{i+1}) \) and \( p_i \# = p_1 p_2 \ldots p_i \); the only prime divisor of \( (p_{i+1})^2 \) is \( p_{i+1} \), and as \( p_{i+1} > p_j \) for all \( j = 1, 2, \ldots, i \), the prime \( p_{i+1} \) cannot divide \( p_i \# \).

Any other coprime with \( p_i \# \) smaller than \( (p_{i+1})^2 \) and greater or equal than \( p_{i+1} \) will necessarily be a prime, because if it were a composite, it would be divisible by one of the \( p_1, p_2, \ldots, p_i \) or by \( p_{i+1} \). In the first case, it would contradict that it is coprime with \( p_i \# \) and in the second case there is only the possibility of being \( p_{i+1} \), which is a prime and therefore contradicts the hypothesis of being composite.

QED
Theorem 2.

Given a natural \( n > 1 \), there exists two primes \( p, q \), such that \( 2n = p + q \)

PROOF.

We will prove that for any natural \( n > 1 \), there exist one prime \( p \) such that \( 2n - p \) is also prime. Writing then \( q = 2n - p \), the statement of the theorem is obtained.

- The theorem is verified for every natural \( n \leq 7 \), since it has been shown to be true for all numbers less than \( 4 \times 10^{18} \) (Oliveira e Silva, 2013), see [6].

- For any natural \( n > 7 \), by application of (01), we have that there exists at least one prime \( p \) that verifies \( n/2 \leq p \leq n \).

Since \( -1 + (8n + 5)^{1/2} < n \) for all \( n > 7 \), we can say that for every natural \( n > 7 \), there exists at least one prime, be \( p_{i+1} \) for some natural \( i > 1 \), which verifies the inequalities:

\[
\frac{-1 + (8n + 5)^{1/2}}{2} \leq p_{i+1} \leq n, \tag{02}
\]

from where it follows, operating:

From the first inequality, \( (8n + 5)^{1/2} \leq 2p_{i+1} + 1 \); squaring, \( 8n + 5 \leq 4(p_{i+1})^2 + 4p_{i+1} + 1 \); rearranging, \( 8n - 4p_{i+1} \leq 4(p_{i+1})^2 - 4 \), and simplifying, \( 2n - p_{i+1} \leq (p_{i+1})^2 - 1 \).

From the second inequality, \( p_{i+1} \leq n \), then \( 2p_{i+1} \leq 2n \), and \( p_{i+1} \leq 2n - p_{i+1} \), so we have:

\[
p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1 \tag{03}
\]

Then we can write:

For every natural \( n > 7 \), there exists a natural \( i > 1 \) such that \( p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1 \), \( \tag{04} \)

if \( 2n - p_{i+1} \) is coprime with \( p_i \# \), by application of the previous lemma it will be necessary prime.

The proof concludes by demonstrating that:

For every natural \( n > 7 \), there exists a natural \( i > 1 \) such that \( 2n - p_{i+1} \) is coprime with \( p_i \# \) and \( p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1 \). \( \tag{05} \)

Note that for all natural \( n > 7 \), always exists at least one prime \( p_{i+1} \), for some \( i > 1 \), verifying (03), as we showed before.

We proceed by reduction to absurdity. The reduction hypothesis consists in supposing that:

There exists a natural \( n > 7 \), such that for all natural \( i > 1 \), it’s not true that:

\( 2n - p_{i+1} \) is coprime with \( p_i \# \) and \( p_{i+1} \leq 2n - p_{i+1} \) and \( 2n - p_{i+1} \leq (p_{i+1})^2 - 1 \) \( \tag{06} \)
The statement \((2n - p_{i+1} \text{ is coprime with } p_i\#)\) implies \((p_{i+1} \leq 2n - p_{i+1})\), since if \(2n - p_{i+1} < p_{i+1}\), then the number \(2n - p_{i+1}\) is necessarily not coprime with \(p_i\#, since in such a case, it must be a prime less or equal to \(p_i\) or a number divisible by at least any of the primes \(p_j\) with \(1 < j \leq i\). Applying this to (06), we can write:

\[
\text{There exists a natural } n > 7, \text{ such that for all natural } i > 1, \text{ it's not true that:} \\
(p_{i+1} \leq 2n - p_{i+1} \text{ and } 2n - p_{i+1} \leq (p_{i+1})^2 - 1) \\
\]  

(07)

Unifying the two inequalities in only one, we obtain (03):

\[
\text{There exists a natural } n > 7, \text{ such that for all natural } i > 1, \text{ it's not true that:} \\
p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1 \\
\]  

(08)

but (08) contradicts (04) (not only (05)). As we saw before, in the deduction of these inequalities, for any \(n > 7\) and some \(i > 1\), \(p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1\) if Ramanujan's theorem (01) is true.

A contradiction is reached, so that the reduction hypothesis leads to an absurdity and the starting proposition is true.

QED


For \(m > 1\), every not coprime with \(m\) greater or equal than \(2m\) can be written as the sum of \(m\) primes.

PROOF.

Let \(n\) be a natural not coprime with \(m\), with \(n \geq 2m\), for some \(m > 1\).

For \(m = 2\), the proposition is the Goldbach’s Conjecture, previously proven.

For \(m > 2\):

- If \(n/m = p \in \mathbb{P}\), then \(n = mp\) and \(n\) is expressed as the sum of \(m\) primes \(p\).

- If \(n/m \notin \mathbb{P}\), then we can write \(n = 2m + r\), for any natural \(r\).

- If \(r\) is even, then \(r + 4\) is also even and we can write: \(n = 2 (m - 2) + (r + 4)\). Applying Theorem 2 to the even number \(r + 4\), there exist two primes \(p, q\), such that \(r + 4 = p + q\). Therefore we can write \(n = 2 (m - 2) + p + q\), which is the sum of \(m\) primes, since \(2 (m - 2)\) is equal to \((m - 2)\) times the sum of the prime number 2.

- If \(r\) is odd, then we write: \(n = 2 (m - 3) + r + 6 = 2 (m - 3) + 3 + (r + 3)\). Thus, being odd \(r\), the number \((r + 3)\) is even and applying Theorem 2 can be written as the sum of two primes, \(r + 3 = p + q\). Thus, \(n = 2 (m - 3) + 3 + p + q\), which is, as above, the sum of \(m\) primes.

QED
Corollary.
Let $m, n > 1$ two natural numbers. Then $m \times n$ can be written as the sum of $m$ primes.

PROOF.
The natural number $n \times m$ is greater than 1, is not coprime with $m$ and is greater or equal than $2m$.
We apply the Theorem 3 to the numbers $n \times m$ and $m$.

QED

Corollary 2.
Let $m, n > 1$ two natural numbers. Then $m^n$ can be written as the sum of $m$ primes.

PROOF.
By applying the Theorem 3 to the numbers $m^n$ and $m$.

QED

References