Expanding Universe from Weyl Cosmology and Asymptotic Safety in Quantum Gravity *

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Abstract

A study of the simplest Jordan-Brans-Dicke-like action within the context of Weyl geometry, combined with the findings of Weinberg's Asymptotic Safety program in quantum gravity, leads to a plethora of nice numerical results : (i) like singling out the quartic potential from all others; (ii) having (Anti) de Sitter space as the most natural solution; (iii) furnishing the value of the observed vacuum energy density at the Hubble scale $\frac{3}{8\pi G_N R_H^2} \sim 10^{-122} M_P^4$; (iv) a $\frac{3}{8\pi} M_P^4$ vacuum energy density at the Planck scale; and (v) allowing the possibility that our universe is a Black Hole whose horizon coincides with the cosmological Hubble horizon. It is warranted to explore deeper the interplay among Weyl geometry, Asymptotic safety and Maldacena's AdS/CFT correspondence (holographic renormalization group flow). Also relevant is the work by Wetterich on the role of dilatation symmetry in higher dimensions and the vanishing of the cosmological constant. Last, but not least, we should also consider the implications of Penrose' Conformal Cyclic Cosmology and Nottale's Scale Relativity Theory with the key findings of this work.

Keywords: Weyl Geometry, Cosmology, Asymptotic Safety, Brans-Dicke-Jordan Gravity,

*Dedicated to the loving memory of Irina Novikova, a brilliant and heavenly creature who met a tragic death

1 Introduction : Why Weyl Geometry

The problem of dark energy and the solution to the cosmological constant problem is one of the most challenging problems facing Cosmology today. There are a vast numerable proposals for its solution. We refer to the monograph [9], [8], [27], [26] and many references therein for details. There have been many previous proposals [7] to explain dark matter (instead of dark energy) in terms of Brans-Dicke gravity. One purpose of this work is to show that it is not only necessary to include the Jordan-Brans-Dicke scalar field ϕ, but it is essential to include a Weyl geometric extension and generalization of Riemannian geometry (ordinary gravity). In doing so we shall see how one can recover a plethora of key numerical results in Cosmology.

Given the Lorentzian signature $(-,+,+,+)$, let us begin with an action in a curved *Riemannian* background

$$S = \int d^4x \sqrt{|g|} \left(\frac{R}{16\pi G_o} - \frac{g^{\mu\nu}}{2} (\partial_\mu \Phi) (\partial_\nu \Phi) - V(\Phi) \right) \tag{1.1}$$

and associated with a canonical real scalar field Φ with a known prescribed potential $V(\Phi)$. Varying the action with respect to the two fields $g_{\mu\nu}, \Phi$ yields

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R =$$

$$8\pi G_o \left((\partial_\mu\Phi)(\partial_\nu\Phi) - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta}(\partial_\alpha\Phi)(\partial_\beta\Phi) - g_{\mu\nu}V(\Phi) \right) \tag{1.2}$$

$$\frac{1}{\sqrt{|g|}}\partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu\Phi \right) - \frac{\partial V(\Phi)}{\partial\Phi} = 0 \tag{1.3}$$

The two equations (1.2, 1.3) are now *coupled* and induce a *nonlinear* Klein-Gordon-like equation for Φ after solving eq-(1.2) for the metric $g_{\mu\nu}$ in terms of Φ. Namely, a substitution of the form $g_{\mu\nu}[\Phi]$ into (1.3) yields a *nonlinear* Klein-Gordon-like equation.

The nonrelativistic limit of the two coupled equations (1.2, 1.3), when $V(\Phi) = 0$, furnish the *nonlinear* Newton-Schrödinger equation [23]

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} = - \frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) - \left(Gm^2 \int \frac{|\Psi(\vec{r}',t)|^2}{|\vec{r}-\vec{r}'|} d^3r' \right) \Psi(\vec{r},t) \tag{1.4}$$

which is obtained after solving the Poisson equation

$$\nabla^2U = 4\pi G_o m\rho = 4\pi G_o m \Psi^*\Psi \tag{1.5}$$

for the Newtonian potential $U = V(\Psi,\Psi^*)$ and substituting its value into the Schrödinger equation.

The immediate advantage of recurring to Weyl geometry, is that it will allow us to find exact solutions to the very complicated coupled system of equations

(1.2, 1.3). And, in doing so, it will furnish the value of the observed vacuum energy density at the Hubble scale $\frac{3}{8\pi G_N R_H^2} \sim 10^{-122} M_P^4$; and a value $\frac{3}{8\pi} M_P^4$ for the vacuum energy density at the Planck scale.

It is essential to emphasize that the Weyl geometric approach undertaken in this work is *not* the same as working with Conformal gravity based on the full Conformal group of translations, Lorentz boosts, dilations, and conformal boosts. The standard approach in Conformal gravity to recover ordinary Einstein gravity is based on the action corresponding to the kinetic terms of a real scalar field φ

$$S \;=\; \int d^4x \;\sqrt{|g|}\; \frac{1}{2}\; [\; \varphi D^c_\mu D^\mu_c \varphi \;] \tag{1.6}$$

the conformal Laplacian can be rewritten

$$D^c_\mu D^\mu_c \varphi \;=\; \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|}\; g^{\mu\nu}\; D^c_\nu \varphi \right) \;+\; A_\mu D^\mu_c \varphi \;+\; \frac{\varphi}{6}\; R \tag{1.7}$$

One can fix, firstly, the conformal boosts transformations by choosing the gauge $A_\mu = 0$. And, secondly, one can then fix the scaling symmetry by gauging φ to a constant $\varphi_o^2 = (16\pi G_N)^{-1}$. In this way one recovers the Einstein-Hilbert action from the last term of eq-(1.7).

Weyl's geometry main feature is that the norm of vectors under parallel infinitesimal displacement going from x^μ to $x^\mu + dx^\mu$ change as follows $\delta||V|| \;\sim\; ||V|| A_\mu dx^\mu$ where A_μ is the Weyl gauge field of scale calibrations that behaves as a connection under Weyl transformations :

$$A'_\mu = A_\mu \;-\; \partial_\mu\; \Omega(x). \quad\; g_{\mu\nu} \;\rightarrow\; e^{2\Omega}\; g_{\mu\nu}. \tag{1.8}$$

involving the Weyl scaling parameter $\Omega(x^\mu)$. The Weyl covariant derivative operator acting on a tensor $\mathbf{T}$ is defined by $D_\mu \mathbf{T} = (\; \nabla_\mu \;+\; \omega(\mathbf{T}\;)\; A_\mu\;)\; \mathbf{T}$; where $\omega(\mathbf{T})$ is the Weyl weight of the tensor $\mathbf{T}$ and the derivative operator $\nabla_\mu \;=\; \partial_\mu + \Gamma_\mu$ involves a connection Γ_μ which is comprised of the ordinary Christoffel symbols $\{^{\rho}_{\mu\nu}\}$ *plus* the A_μ terms

$$\Gamma^\rho_{\mu\nu} \;=\; \{^{\rho}_{\mu\nu}\} \;+\; \delta^\rho_\mu\; A_\nu \;+\; \delta^\rho_\nu\; A_\mu \;-\; g_{\mu\nu}\; g^{\rho\sigma}\; A_\sigma \tag{1.9}$$

The Weyl gauge covariant operator $\partial_\mu + \Gamma_\mu + w(\mathbf{T}) A_\mu$ obeys the condition

$$D_\mu\; (g_{\nu\rho}) \;=\; \nabla_\mu\; (g_{\nu\rho}) \;+\; 2\; A_\mu\; g_{\nu\rho} \;=\; 0. \tag{1.10}$$

where $\nabla_\mu (g_{\nu\rho}) \;=\; -\,2\; A_\mu\; g_{\nu\rho} \;=\; Q_{\mu\nu\rho}$ is the non-metricity tensor. Torsion can be added [29] if one wishes but for the time being we refrain from doing so. The connection $\Gamma^\rho_{\mu\nu}$ is Weyl *invariant* so that the geodesic equation in Weyl spacetimes is Weyl-covariant under Weyl gauge transformations (scalings)

$$ds \rightarrow e^\Omega\; ds;\; \frac{dx^\mu}{ds} \rightarrow e^{-\Omega}\; \frac{dx^\mu}{ds};\; \frac{d^2x^\mu}{ds^2} \;\rightarrow\; e^{-2\Omega}\; [\; \frac{d^2x^\mu}{ds^2} \;-\; \frac{dx^\mu}{ds}\; \frac{dx^\nu}{ds}\; \partial_\nu \Omega\;].$$

$$g_{\mu\nu} \to e^{2\Omega}\, g_{\mu\nu};\;\; A_\mu \to A_\mu - \partial_\mu \Omega;\;\; A^\mu \to e^{-2\Omega}\, (A^\mu - \partial^\mu \Omega);\;\; \Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu}. \quad (1.11)$$

The Weyl connection and curvatures scale as

$$\Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu},\;\; \mathcal{R}^\rho_{\mu\nu\sigma} \to \mathcal{R}^\rho_{\mu\nu\sigma},\;\; \mathcal{R}_{\mu\nu} \to \mathcal{R}_{\mu\nu},\;\; \mathcal{R} \to e^{-2\Omega}\, \mathcal{R} \quad (1.12)$$

Thus, the Weyl covariant geodesic equation transforms under Weyl scalings as

$$\frac{d^2 x^\rho}{ds^2} \;+\; \Gamma^\rho_{\mu\nu}\, \frac{dx^\mu}{ds}\, \frac{dx^\nu}{ds} \;-\; A_\mu\, \frac{dx^\mu}{ds}\, \frac{dx^\rho}{ds} \;=\; 0 \;\to$$

$$e^{-2\Omega}\, [\; \frac{d^2 x^\rho}{ds^2} \;+\; \Gamma^\rho_{\mu\nu}\, \frac{dx^\mu}{ds}\, \frac{dx^\nu}{ds} \;-\; A_\mu\, \frac{dx^\mu}{ds}\, \frac{dx^\rho}{ds} \;] \;=\; 0. \quad (1.13)$$

The Weyl weight of the metric $g_{\nu\rho}$ is 2. The meaning of $D_\mu(g_{\nu\rho}) = 0$ is that the angle formed by two vectors remains the same under parallel transport despite that their lengths may change. This also occurs in conformal mappings of the complex plane. The Weyl covariant derivative acting on a scalar ϕ of Weyl weight $\omega(\phi) = -1$ is defined by

$$D_\mu \phi = \partial_\mu\, \phi \;+\; \omega(\phi) A_\mu\, \phi = \partial_\mu\, \phi \;-\; A_\mu\, \phi. \quad (1.14)$$

The Weyl scalar curvature in D dimensions and signature $(-,+,+,+....)$ is [1]

$$\mathcal{R}_{Weyl} \;=\; R_{Riemann} \;-\; (d-1)(d-2) A_\mu A^\mu \;-\; 2(d-1) \nabla_\mu A^\mu. \quad (1.15)$$

Having review very briefly the basics of Weyl geometry we shall embark with the crux of this work.

2 Weyl Cosmology and Asymptotic Safety

Having introduced the basics of Weyl geometry our starting action is the Weyl-invariant Jordan-Brans-Dicke-like action involving the scalar ϕ field and the scalar Weyl curvature $\mathcal{R}_{Weyl}$

$$S[g_{\mu\nu}, A_\mu, \phi] \;=\; S[g'_{\mu\nu}, A'_\mu, \phi'] \;\Rightarrow$$

$$\int d^4x\, \sqrt{|g|}\; [\; \phi^2\, \mathcal{R}_{Weyl}(g_{\mu\nu}, A_\mu) \;-\; \frac{1}{2} g^{\mu\nu}\, (D_\mu \phi)(D_\nu \phi) \;-\; V(\phi)\;] \;=$$

$$\int d^4x\, \sqrt{|g'|}\; [\; (\phi')^2\, \mathcal{R}_{Weyl}(g'_{\mu\nu}, A'_\mu) \;-\; \frac{1}{2} g'^{\mu\nu}\, (D'_\mu \phi')(D'_\nu \phi') \;-\; V(\phi')\;] \quad (2.1)$$

where under Wey scalings one has

[1] Some authors define their A_μ field with the opposite sign as $-A_\mu$ which changes the sign in the last term of the Weyl scalar curvature (1.15)

$$\phi' = e^{-\Omega}\,\phi; \;\; g'_{\mu\nu} = e^{2\Omega}\,g_{\mu\nu}; \;\; \mathcal{R}_{Weyl}(g'_{\mu\nu}, A'_\mu) = e^{-2\Omega}\,\mathcal{R}_{Weyl}(g_{\mu\nu}, A_\mu)$$

$$V(\phi') = e^{-4\Omega}\,V(\phi), \;\; \sqrt{|g'|} = e^{4\Omega}\,\sqrt{|g|}; \;\; D'_\mu\phi' = e^{-\Omega}\,D_\mu\phi; \;\; A'_\mu = A_\mu - \partial_\mu\Omega. \tag{2.2}$$

One could complicate matters by adding more terms to the most simple action (2.1) like

$$\int d^4x\,\sqrt{|g|}\,\left(\,\mathcal{R}^2_{Weyl} + F_{\mu\nu}\,F^{\mu\nu} + \cdots\,\right) \tag{2.3}$$

see [3], [2], [4]. However, it won't be necessary to modify the action (2.1) to recover fundamental results.

The effective Newtonian coupling G is related to ϕ as follows $(16\pi G)^{-1} \equiv \phi^2$, and it is spacetime dependent in general and has a Weyl weight equal to 2. Despite that one has *not* introduced any explicit dynamics to the A_μ field (there are no $F_{\mu\nu}F^{\mu\nu}$ terms in the action (2.1)) one still has the equation obtained from the variation of the action in $d = 4$ w.r.t to the A^μ field and which leads to the pure-gauge configurations provided $\phi \neq 0$

$$\frac{\delta S}{\delta A_\mu} = 0 \Rightarrow \phi^2\,\frac{\delta\mathcal{R}_{Weyl}}{\delta A_\mu} + \frac{\delta(D_\mu\phi)}{\delta A_\mu}\,\frac{\delta S_{matter}}{\delta(D_\mu\phi)} = 0 \Rightarrow$$

$$g^{\mu\nu}D_\nu\phi^2 = 0 \Rightarrow D_\mu\phi = 0 \Rightarrow A_\mu = \partial_\mu\,ln\,(\phi). \tag{2.4}$$

Hence, a variation of the action w.r.t the A_μ field leads to the pure gauge solutions (2.4) which is tantamount to saying that the scalar ϕ is Weyl-covariantly *constant* $D_\mu\phi = 0$ in any gauge $D_\mu\phi = 0 \to e^{-\Omega}D_\mu\phi = D'_\mu\phi' = 0$ (for non-singular gauge functions $\Omega \neq \pm\infty$).

Therefore, the scalar ϕ does not have true local dynamical degrees of freedom from the Weyl spacetime perspective. Since the gauge field is a total derivative, under a local gauge transformation with a gauge function $\Omega = ln(\phi/\phi_o)$, one can gauge away (locally) the gauge field A_μ and have $A'_\mu = 0$ in the new gauge. Globally, however, this may not be the case because there may be *topological* obstructions. Therefore, the gauge $A'_\mu = 0$, implies that $\phi' = \phi_o =$ constant. Consequently $16\pi G' = \phi'^{-2}$ can be fixed to a constant, and one may set $G' = G_N$ where G_N is the observed Newtonian gravitational coupling.

The pure-gauge configurations leads to the Weyl integrability condition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0$ when $A_\mu = \partial_\mu\Omega$, and means physically that if we parallel transport a vector under a closed loop in a *flat* spacetime, as we come back to the starting point, the *norm* of the vector has not changed; i.e, the rate at which a clock ticks does not change after being transported along a closed loop back to the initial point; and if we transport a clock from A to B along different paths, the clocks will tick at the same rate upon arrival at the same point B. This will ensure, for example, that the observed spectral lines of identical atoms will not change when the atoms arrive at the laboratory after taking different paths (histories) from their coincident starting point. In this way on can by-pass Einstein's objections to Weyl. If $F_{\mu\nu} \neq 0$ the Weyl geometry is no longer integrable. This can occur if one adds explicit $F_{\mu\nu}F^{\mu\nu}$ terms to the action which may lead to true dynamical degrees of freedom for the gauge field A_μ.

This result $D_\mu\phi = 0$ also follows in other dimensions. Substituting

$$A_\mu \;=\; \frac{2}{d-2}\,\partial_\mu ln\phi \tag{2.5}$$

into

$$\mathcal{R}_{Weyl} \;=\; R_{Riemann} \;-\; (d-1)(d-2)A_\mu A^\mu \;-\; 2(d-1)\nabla_\mu A^\mu \tag{2.6}$$

gives

$$\mathcal{R}_{Weyl} \;=\; R_{Riemann} \;-\; 4\,\frac{d-1}{d-2}\,\frac{\nabla_\mu\nabla^\mu\phi}{\phi} \tag{2.7}$$

The last term in (2.7) has a similar functional form as Bohm's quantum potential [17], [18]. From now we shall denote R for the Riemannian scalar curvature $R_{Riemann}$. The covariant derivative ∇_μ appearing in (2.7) is the one defined in terms of the Christoffel conection $\{\}$, and *not* based on the Weyl connection Γ.

Given an action (2.1) in $d = 4$ the field equations are obtained after the variations of the action with respect to the 3 fields $g_{\mu\nu}, A_\mu, \phi$, respectively

$$\phi^2\,\left(\mathcal{R}^{Weyl}_{\mu\nu} \;-\; \frac{1}{2}\,g_{\mu\nu}\,\mathcal{R}_{Weyl}\right) \;+\; D_\mu D_\nu\phi^2 \;+\; g_{\mu\nu}\;g^{\alpha\beta}\;D_\alpha D_\beta\phi^2 \;=$$

$$\frac{1}{2}\,(D_\mu\phi)(D_\nu\phi) \;-\; \frac{1}{4}\,g_{\mu\nu}\;g^{\alpha\beta}(D_\alpha\phi)(D_\beta\phi) \;-\frac{1}{2}\,g_{\mu\nu}V(\phi) \tag{2.8}$$

$$D_\mu\phi^2 \;=\; 2\;D_\mu\phi \;=\; 0 \;\Rightarrow\; A_\mu \;=\; \partial_\mu ln(\phi) \tag{2.9}$$

$$2\phi\;\mathcal{R}_{Weyl} \;-\; \frac{\partial V(\phi)}{\partial\phi} \;+\; D_\mu D^\mu\phi \;=\; 0 \tag{2.10}$$

As stated earlier, the field equation $D_\mu\phi = 0$ just states the ϕ is Weyl-covariantly constant. This result when followed by taking the trace of (2.8) gives $\phi^2\mathcal{R}_{Weyl} = 2V(\phi)$ which allows to eliminate $\mathcal{R}_{Weyl} = 2\phi^{-2}V(\phi)$, and inserting it in eq-(2.10) yields $4\phi^{-1}V(\phi) - V'(\phi) = 0$, *singling* out the quartic potential $V(\phi) = \kappa\phi^4$ in $d = 4$, out of an infinity of possible choices for the potential. For example, one could have potentials of the form $V = \sum_n c_n M^{4-n}\phi^n$ where M is mass-like parameter (a scalar moduli parameter) which scales as $M \to e^{-\Omega}M$ in order to render the action Weyl invariant. To sum up, in this Weyl geometric approach the choice for the potential $V = \kappa\phi^4$ is *not* ad hoc but can be inferred from the field equations themselves.

Eq-(2.10) in $d = 4$ can be rewritten in terms of the Riemannian scalar curvature, after using $D_\mu \phi = 0$, as

$$2\ R\ \phi\ -\ \frac{12}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|}\ g^{\mu\nu}\ \partial_\nu \phi \right)\ -\ \frac{\partial V(\phi)}{\partial \phi}\ =\ 0 \tag{2.11}$$

Upon inserting the derived expression for $V(\phi) = \kappa \phi^4$ above, it gives

$$R\ \phi\ -\ \frac{6}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|}\ g^{\mu\nu}\ \partial_\nu \phi \right)\ -2\ \kappa\ \phi^3\ =\ 0 \tag{2.12}$$

It remains now to solve eq-(2.8) given $D_\mu \phi = 0$ and $V(\phi) = \kappa \phi^4$. After factoring out ϕ^2 and substituting $\mathcal{R}_{Weyl} = 2\kappa\phi^2$ leads to

$$\mathcal{R}^{Weyl}_{\mu\nu}\ =\ \frac{1}{2}\ g_{\mu\nu}\ \kappa \phi^2 \tag{2.13}$$

with

$$\mathcal{R}^{Weyl}_{\mu\nu}\ =\ R_{\mu\nu}\ -2\ \nabla_\mu A_\nu - g_{\mu\nu}\ g^{\alpha\beta}\ \nabla_\alpha A_\beta\ -2\ A_\mu\ A_\nu\ +2\ g_{\mu\nu}\ g^{\alpha\beta}\ A_\alpha A_\beta \tag{2.14}$$

Since the Weyl weight of $\mathcal{R}^{Weyl}_{\mu\nu}$ is 0, after some straightforward lengthy algebra, one can infer the transformation law of the Riemannian Ricci tensor $R_{\mu\nu}$ in $d = 4$ under scalings $g_{\mu\nu} \to e^{2\Omega} g_{\mu\nu}$

$$R'_{\mu\nu}\ =$$

$$R_{\mu\nu}\ -2\ \nabla_\mu \nabla_\nu \Omega - g_{\mu\nu}\ g^{\alpha\beta}\ (\nabla_\alpha \nabla_\beta \Omega)\ +2\ (\nabla_\mu \Omega)(\nabla_\nu \Omega)\ -2\ g_{\mu\nu}\ g^{\alpha\beta} (\nabla_\alpha \Omega)(\nabla_\beta \Omega) \tag{2.15}$$

From eq-(2.13) one arrives at the Weyl invariant field equation

$$\mathcal{R}^{Weyl}_{\mu\nu}(\mathbf{g}, \mathbf{A})\ =\ \mathcal{R}^{Weyl}_{\mu\nu}(\mathbf{g}', \mathbf{A}')\ =\ \frac{1}{2}\ g_{\mu\nu}\ \kappa\ \phi^2\ =\ \frac{1}{2}\ g'_{\mu\nu}\ \kappa\ \phi'^2 \tag{2.16}$$

with $\mathbf{g} = g_{\mu\nu}, \mathbf{A} = A_\mu, \cdots$. The dimensionless parameter κ is inert under scalings.

The zero gauge choice $A'_\mu = 0$ leads to

$$A'_\mu\ =\ \partial_\mu [ln(\frac{\phi'}{\phi_o})]\ =\ A_\mu\ -\ \partial_\mu \Omega\ =\ \partial_\mu [ln(\frac{\phi}{\phi_o})] - \partial_\mu \Omega\ = 0\ \Rightarrow$$

$$\phi' = \phi_o;\ \ e^\Omega\ =\ \frac{\phi}{\phi_o} \tag{2.17}$$

which resulted from $D'_\mu \phi' = D_\mu \phi = 0$.

Consequently, one arrives at

$$\mathcal{R}^{Weyl}_{\mu\nu}(\mathbf{g}', \mathbf{A}' = \mathbf{0}) =\ R'_{\mu\nu}\ =\ \frac{1}{2}\ g'_{\mu\nu}\ \kappa\ \phi_o^2\ \Rightarrow\ R'\ =\ 2\kappa\ \phi_o^2 \tag{2.18}$$

leading to a family of spacetime backgrounds which are all conformally equivalent to backgrounds of *constant* Riemannian scalar curvature : (Anti) de Sitter spaces. The solutions to the scalar ϕ field equation (2.12) defined in spacetime backgrounds which are conformally equivalent to a (Anti) de Sitter background

$$g_{\mu\nu} \; = \; e^{2\;\Omega'(x)}\; g'^{(A)dS}_{\mu\nu} \; = \; e^{-2\;\Omega(x)}\; g'^{(A)dS}_{\mu\nu}, \quad \Omega \; = - \; \Omega' \tag{2.19}$$

are of the form $\phi = e^{-\Omega'(x)}\phi_o = e^{\Omega(x)}\phi_o$; $\phi_o = (16\pi G_N)^{-1/2}$ is the constant directly related to the observed Newtonian coupling G_N.

Under Weyl scalings the constant Riemannian scalar curvature of (Anti) de Sitter space in $d = 4$ transforms as

$$g_{\mu\nu} \; = \; e^{2\;\Omega'(x)}\; g'^{(A)dS}_{\mu\nu} \; = \; e^{-2\;\Omega(x)}\; g'^{(A)dS}_{\mu\nu}, \quad \Rightarrow$$

$$R \; = \; e^{2\Omega(x)}\; \Big(\; R'_{(A)dS} \; + 6\; (\nabla_\mu \nabla^\mu \Omega) \; - 6\; (\nabla_\mu \Omega)\; (\nabla^\mu \Omega) \;\Big) \tag{2.20}$$

such that

$$R\;\phi \; - \; \frac{6}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|}\; g^{\mu\nu}\; \partial_\nu \phi \right) \; - 2\;\kappa\;\phi^3 \; =$$

$$e^{3\Omega(x)}\; \Big(\; R'_{(A)dS}\;\phi_o \; - 2\;\kappa\;\phi_o^3 \;\Big) \; = \; 0 \tag{2.21}$$

as expected.

To sum up, starting with a scalar-tensor theory within the context of Weyl's geometry, permits to *derive* the expression for the potential $V(\phi) = \kappa\phi^4$, instead of being introduced by hand, and find the solutions $g_{\mu\nu}, A_\mu$ to the field equations (2.8, 2.9, 2.10)

$$g_{\mu\nu}[\phi] \; = \; e^{-2\Omega}\; g'^{(A)dS}_{\mu\nu}[\phi_o] \; = \; (\frac{\phi_o}{\phi})^2\; g'^{(A)dS}_{\mu\nu}[\phi_o], \quad A_\mu[\phi] \; = \; \partial_\mu[ln(\frac{\phi}{\phi_o})] \tag{2.22}$$

The (Anti) de Sitter metric $g'^{(A)dS}_{\mu\nu}[\phi_o]$ has an explicit dependence on ϕ_o via the cosmological constant Λ : $R' = 4\Lambda = 2\kappa\phi_o^2$. $\kappa < 0$ for Anti de Sitter space; $\kappa > 0$ for de Sitter space. The solutions with $\kappa = 0$ lead, for example, to the Schwarzschild ($R'_{\mu\nu} = R' = 0$) and Reisnner-Nordstrom ($R' = 0$) metrics corresponding to static spherically symmetric backgrounds.

The prime example of a de Sitter background of constant Riemannian scalar curvature is the observed accelerated-expanding universe $R' = 12H_o^2 \; = \; \frac{12}{R_H^2}$ where R_H is the present day Hubble radius. Substituting $\phi'^2 = \phi_o^2 = (16\pi G_N)^{-1}$ and $R' = 12H_o^2$ into eq-(2.21) yields the numerical coefficient κ of the potential $V(\phi') = \kappa\phi'^4$,

$$12H_o^2 \; = \; 2\kappa\;\phi_o^2 \; \Rightarrow \; \kappa \; = \; \frac{6}{\phi_o^2 R_H^2} \tag{2.23}$$

therefore, by evaluating the potential at $\phi' = \phi_o$

$$V(\phi_o) \; = \; \kappa\;\phi_o^4 \; = \; \frac{6}{\phi_o^2 R_H^2}\;\phi_o^4 \; = \; 6\;\frac{\phi_o^2}{R_H^2} \; = \; \frac{6}{16\pi G_N R_H^2} \; = \; \frac{3}{8\pi G_N R_H^2} \; = \; \rho_{cr} \tag{2.24}$$

one recovers, in a straightforward fashion, the Universe's observed critical mass density with the precise numerical factor, and which agrees also with the observed vacuum energy density ρ_{vac}.

The boundary conditions at $t \rightarrow \infty$ for ϕ are $\phi \rightarrow \phi_o$, and $(d\phi/dt) \rightarrow 0$. Because there are an infinite number of choices for the scaling factor $\Omega(t)$ the physics and experimental evidence should dictate what are the suitable expressions one should have for $\Omega(t)$. In other words, by choosing the specific gauge $A'_\mu = 0 \Rightarrow A_\mu = (A_t = \partial_t \Omega(t), 0, 0, 0)$ one will break the Weyl scale invariance by fixing $\phi' = \phi_o = (16\pi G_N)^{-1/2}$ to a constant. In order to find $\Omega(t)$ we shall be guided by the results of the Asymptotic Safety program in quantum gravity [9] that there is a (nontrivial) interacting ultraviolet fixed point G_* for the dimensionless running gravitational coupling in the $k \rightarrow \infty$ (infinite energy) limit (short distance limit) defined as : $G_* = lim_{k\rightarrow\infty} \ G_k(r) k^2(r) = 0 \times \infty \neq 0$.

The limiting value of the running gravitational coupling $G_{k=\infty}(r)$ obtained in the dynamical renormalization of black-hole spacetimes turned out to be [11]

$$G_\infty(r) \ = \ G_N \ (1 - e^{-r^3/r_s l_{cr}^2}) \tag{2.25}$$

r_s is the classical Schwarzschild radius $2G_N M$, and $l_{cr} \sim L_P$ (Planck scale) represents a critical length scale below which the modifications by the running Newtonian coupling become important [11]. The end result of this running gravitational coupling is a "renormalized" black hole spacetime of the Dymnikova-type which is free from singularities [13] and given by the metric

$$(ds)^2 \ = \ - \ (1 - \frac{2G_\infty(r) M_o}{r})(dt)^2 \ + \ (1 - \frac{2G_\infty(r) M_o}{r})^{-1}(dr)^2 \ + \ r^2 (d\Omega_2)^2 \tag{2.26}$$

The $r \rightarrow 0$ limit of $G_\infty(r)/r$ is $G_o r^2/r_s l_{cr}^2$ leading to a *repulsive* de Sitter core at very short distances. Similar repulsive de Sitter core were found later on in [12] by using a Gaussian profile mass density function, and in infinite derivative gravity [15].

We shall borrow the physical implications of these results, giving the short distance $G_\infty(r = 0) = 0$, and large distance $G_\infty(r \rightarrow \infty) \rightarrow G_N$ behavior, and apply them to the *temporal* flow of $G(t) \sim \phi^{-1/2}(t)$, as t runs from 0 to ∞; i.e. as the Universe expands from the big-bang singularity (point) to the present size and beyond. In this fashion we will be able to find a judicious choice for the Weyl scaling function $\Omega(t)$ and which is tantamount of determining the expression for A_μ, since the gauge field does not have any true dynamics (it is pure gauge).

There are many different expressions to describe the de Sitter metric depending on the coordinates being used. A flat slicing of the four-dim de Sitter space is

$$(ds)^2_{dS} \ = \ - \ (dt)^2 + e^{2H_o t} \ \sum_{i=1}^{3} (dy_i)^2 \ = \ e^{2H_o t} \left(- e^{-2H_o t} \ \ (dt)^2 \ + \ \sum_{i=1}^{3} (dy_i)^2 \right) \tag{2.27}$$

After performing the change of coordinates

$$\int_{\xi_\infty}^{\xi} \ d\xi \ = \ \int_{\infty}^{t} \ e^{-Ht} \ dt \ \Rightarrow \ \xi_\infty \ - \ \xi \ = \ R_H \ e^{-H_o t}, \ \ R_H \equiv H_o^{-1} \tag{2.28}$$

yields a conformally flat metric

$$(ds)^2_{dS} \ = \ \frac{R_H^2}{(\xi_\infty - \xi)^2} \ \left(\ - \ (d\xi)^2 \ + \ \sum_{i=1}^{3} (dy_i)^2 \right) \tag{2.29}$$

The next step is to select the appropriate conformal scaling factor of the above metric as indicated by eq-(2.22). Inspired by the functional form of $G_\infty(r)$ in eq-(2.25), we can deduce the temporal dependence $G(t)$ from the following correspondence ($c = 1$)

$$r \ \leftrightarrow \ t, \ \ l_{cr} = L_P \ \leftrightarrow \ t_P, \ \ r_s = 2G_N M \ \leftrightarrow \ R_H \tag{2.30}$$

where the Schwarzschild radius r_s (black hole horizon) corresponds now to the cosmological horizon (Hubble radius), and the Planck scale L_P corresponds to the Planck time t_P. At the end of this work we will say more about the above correspondence within the framework of black hole Cosmology [16].

Therefore, the choice of the scaling factor $\Omega(t)$ which follows from the correspondence (2.30) associated to $G_\infty(r)$ (2.25) is given by

$$e^{2\Omega(t)} \ = \ (\frac{\phi}{\phi_o})^2 \ \equiv \ \frac{1}{1 \ - \ e^{-t^3/R_H t_P^2}} \ \Rightarrow \ \Omega(t) \ = \ -\frac{1}{2} \ ln(1 - e^{-t^3/R_H t_P^2}) \tag{2.31}$$

from the latter expression one can deduce the temporal flow of the Weyl's gauge field

$$A_t[\phi(t)] \ = \ \partial_t [ln(\frac{\phi(t)}{\phi_o})] \ = \ \partial_t \Omega(t) \ = \ \ - \ \frac{3}{2} \ \frac{(t^2/R_H t_P^2) \ e^{-t^3/R_H t_P^2}}{1 - e^{-t^3/R_H t_P^2}}, \tag{2.32}$$

and $A_i = 0 (i = 1, 2, 3)$, giving $A_t(t) \rightarrow 0$, as $t \rightarrow \infty$, and $A_t \rightarrow -3/2t \rightarrow -\infty$, as $t \rightarrow 0$. By construction, the flow of the gravitational coupling is

$$G(t) \ = \ G_N \ (1 - e^{-t^3/R_H t_P^2}), \ \ G(t = 0) = 0, \ \ G(t \rightarrow \infty) \rightarrow G_N \tag{2.33}$$

which is compatible with the results [11] in the ultraviolet and infrared regimes.

An important remark is in order. One should *not* view the end net result of the temporal-dependent Weyl scaling factor in (2.31) as the introduction of an $effective$ Hubble function $H(t)$ of the form

$$H(t) \ = \ H_o \ + \frac{3}{2} \ \frac{(t^2/R_H t_P^2) \ e^{-t^3/R_H t_P^2}}{1 - e^{-t^3/R_H t_P^2}} \tag{2.34}$$

because the integral

$$2 \int_0^t \ H(t') \ dt' \ = \ 2H_o t \ + \ ln[1 - e^{-t^3/R_H t_P^2}] \ - \ ln[0] \ = \ \infty \tag{2.35}$$

$diverges$. Therefore, the Weyl scaling factor of the de Sitter metric is roughly speaking given by the regularized version of the expression $exp\{2 \int_0^t [H(t') - H_o] dt'\}$ corresponding to an effective Hubble function $H(t)$.

Finally, from eqs-(2.23, 2.31) one obtains

$$V(\phi(t)) \ = \ \kappa \ \phi^4(t) \ = \ \frac{6}{\phi_o^2 R_H^2} \ \frac{\phi_o^4}{(1 - e^{-t^3/R_H t_P^2})^2} \tag{2.36}$$

and one finds that as $t \to \infty$, $V(\phi) \to V(\phi_o) = (3/8\pi G_N R_H^2) = \rho_{vac}$, recovering the observed vacuum energy density. Whereas at Planck's time $t = t_P$, one finds after a Taylor expansion of the exponential, in $c = 1$ units, the expected very large result

$$V(\phi(t_P)) \ = \ \frac{6}{\phi_o^2 R_H^2} \ \frac{\phi_o^4}{(1 - e^{-t_P/R_H})^2} \ \simeq \ \frac{6}{\phi_o^2 R_H^2} \ \frac{\phi_o^4}{(t_P/R_H)^2} \ = \ \frac{3}{8\pi} \ M_P^4 \tag{2.37}$$

simply by substituting $16\pi\phi_o^2 = G_N^{-1} = M_P^2$ ($\hbar = c = 1$). As the bubble expands it borrows energy from the vacuum, thus depleting its energy density to the extremely low value currently observed.

Note that other choices for $\Omega(t)$ like

$$e^{-2\Omega(t)} \ = \ (\frac{\phi_o}{\phi})^2 = \ 1 \ - \ e^{-(t^3/R_H t_P^2)^\beta}, \ \ \beta > 1 \tag{2.38}$$

would lead to a huge vacuum energy at the Planck scale (Planck time)

$$\rho \ = \frac{3}{8\pi} \ (\frac{R_H}{L_P})^{\beta - 1} \ M_P^4 \ >> \ M_P^4 \tag{2.39}$$

Therefore, $\beta = 1$ is the right value consistent with $\rho \sim M_P^4$, and the $G_\infty(r) \leftrightarrow G(t)$ correspondence.

The scalar curvature $R \sim e^{2\Omega}(4\Lambda) + \cdots$ blows up at $t = 0$ because $e^{2\Omega}$ (2.31) diverges at $t = 0$, and which is consistent with the Big-Bang singularity. The fact that the universe emerges from a "point" is also consistent with the fact that the metric $g_{\mu\nu} = e^{-2\Omega} g'^{dS}_{\mu\nu}$ degenerates/pinches-off to *zero* at $t = 0$. In the late time period $t \to \infty$, the scaling factor $e^{-2\Omega} \to 1$, such that one recovers the observed de Sitter metric, with $R = 4\Lambda = 12H_o^2 = 12(R_H)^{-2}$, and the Newtonian gravitational coupling G_N.

Concluding, the most salient feature of all these results is that they relied solely on Weyl's geometry and the short/large distance behavior of the running gravitational coupling in the Asymptotic Safety program of quantum gravity.

The Renormalization group flow of the cosmological constant in Asymptotic Safety was studied by [10] The scale dependence of $\lambda(k)$ in the de Sitter case was found to be [10]

$$\Lambda(k) \ = \ \Lambda_0 \ + \ \frac{b \ G(k)}{4} \ k^4, \quad \Lambda_0 > 0 \tag{2.40}$$

where b is positive numerical constant. In $d = 4$, the dimensionless gravitational coupling has a nontrivial fixed point $g = G(k)k^2 \to g_*$ in the $k \to \infty$ limit, and the dimensionless variable $\lambda = \Lambda(k)k^{-2}$ has also a nontrivial ultraviolet fixed point $\lambda_* \neq 0$ [10]. The infrared limits are $\Lambda(k \to 0) = \Lambda_0 > 0, \ G(k \to 0) = G_N$. Where the ultraviolet limit are $\Lambda(k = \infty) = \infty; G(k = \infty) = 0$.

In this work the temporal flow of the potential $V(\phi(t))$ recaptures the same effects as the above Renormalization group flow of the cosmological constant and leads to the flow of the vacuum energy density, from the large value of M^4_{Planck} to the extremely small present value $10^{-122}M^4_{Planck}$, and which could provide important clues to the resolution of the cosmological constant problem.

2.1 Static Spherically Symmetric Solutions and Black Hole Cosmology

Having analyzed the deep cosmological implications of the *temporal* behavior of the solutions to the field equations associated with the simplest Weyl invariant action, and corresponding to a Jordan-Brans-like scalar field ϕ, a metric $g_{\mu\nu}$ and Weyl's field A_μ, in this section we shall focus on the *spatial* behavior of the solutions, in the static spherically symmetric case.

Let us start with the renormalization-group improved black-hole metric [10]

$$(ds)^2 \ = \ - \ (1 - \frac{2G(r)M_o}{r})(dt)^2 \ + (1 - \frac{2G(r)M_o}{r})^{-1}(dr)^2 \ + \ r^2(d\Omega_2)^2 \tag{2.41}$$

based on the Renormalization group flow of $G(r)$ in the Asymptotic Safety program [9],

The Einstein field equations corresponding to the above metric are $G^\mu_\nu = 8\pi G(r)T^\mu_\nu$, where T^μ_ν is in this case the effective stress energy tensor associated to the graviton quantum-self energy resulting from vacuum polarizations effects [14]. Note the presence of the running Newtonian coupling $G(r)$ in the right hand side. A small variation of Newton's constant triggers a ripple effect of successive back reactions of the semiclassical background spacetime [11].

The limiting value of the running gravitational coupling $G_{k=\infty}(r)$ obtained in the dynamical renormalization of the black-hole spacetime represented by the metric (2.41) turned out to be [11]

$$G_\infty(r) \ = \ G_o \ (1 - e^{-r^3/r_s l_{cr}^2}) \tag{2.42}$$

It *is* this value of $G_\infty(r) \sim \phi^{-2}(r)$ that provides the radial dependence of the scalar field $\phi(r)$, which in turn, will select the expression for the Weyl scaling as follows

$$e^{-2\Omega(r)} \ = \ (\frac{\phi_o}{\phi(r)})^2 \ = \ \frac{G_\infty(r)}{G_o} \ = \ 1 \ - \ e^{-r^3/r_s l_{cri}^2} \tag{2.42}$$

Given the de Sitter metric in static coordinates

$$(ds)^2 \ = \ - \ (1 \ - \ \frac{\Lambda}{3} \ r^2)(dt)^2 \ + \ (1 \ - \ \frac{\Lambda}{3} \ r^2)^{-1}(dr)^2 \ + \ r^2 \ (d\Omega_2)^2 \tag{2.43}$$

the rescaled de Sitter metric metric is $g_{\mu\nu} = e^{-2\Omega(r)} g'^{dS}_{\mu\nu}$. And by construction, eq-(2.21) will be satisfied for the scalar field $\phi(r)$ whose functional dependence can can be read directly from eq-(2.42), and where the de Sitter metric is displayed in eq-(2.43).

As $r \to 0$, $G_\infty(r) \to 0$. As $r \to \infty$, $G_\infty(r) \to G_o = G_N$. The scaling factor vanishes at $r = 0$, so the rescaled de Sitter metric pinches off to zero, to a "point", and the scalar curvature blows up, like in the Big-Bang at $t = 0$. At $r = \infty$ one recovers the de Sitter metric (2.43) with a constant scalar Riemannian curvature $R = 4\Lambda$. In this fashion one obtains similar results as in section **2.1**.

In passing, we deem it important to point out that the metric (2.43) is not the only one which furnishes a constant scalar curvature. To proceed let us focus on the following spacetime background (in natural $c = 1$ units)

$$(ds)^2 \ = \ - \ (1 - \frac{2G_o\mathcal{M}(r)}{r})(dt)^2 \ + (1 - \frac{2G_o\mathcal{M}(r)}{r})^{-1}(dr)^2 + r^2(d\Omega_2)^2 \tag{2.44}$$

related to an anisotropic self gravitating fluid droplet [12].

The energy-momentum tensor corresponding to the Einstein equations $G^\mu_\nu = 8\pi G_o T^\mu_\nu$, and associated to the metric (2.44) is given by

$$T^\mu_\nu \ \equiv \ diag \ (-\rho(r), \ p_r(r), \ p_\theta(r), \ p_\varphi(r)) \tag{2.45}$$

physically it represents a self-gravitating anisotropic fluid (bubble) whose mass density and pressure components are

$$\rho \ = \ - \ p_r \ = \ \frac{1}{4\pi r^2} \ \frac{d\mathcal{M}}{dr}, \ \ p_\theta \ = \ p_\varphi \ = \ - \ \frac{1}{8\pi r} \ \frac{d^2\mathcal{M}}{dr^2} \tag{2.46}$$

From eqs-(2.44,2.45) one learns that

$$p_\theta \;=\; p_\varphi \;=\; -\;\rho\;-\;\frac{r}{2}\,\frac{d\rho}{dr} \tag{2.47}$$

and which is consistent with the local conservation law $\nabla_\mu T^\mu_\nu = 0$

In the case of a point mass M_o the above equations lead to

$$\rho \;=\; \frac{M_o\delta(r)}{4\pi r^2} \;=\; -\,p_r,\;\; p_\theta \;=\; p_\varphi \;=\; -\;\frac{M_o}{8\pi r}\;\delta'(r) \;=\; \frac{M_o\delta(r)}{8\pi r^2} \tag{2.48}$$

after using the relation $r\delta'(r) = -\delta(r)$. The scalar curvature is

$$R \;=\; -\;2G_o\;\left(\;\frac{(d^2\mathcal{M}/dr^2)}{r}\;+\;2\;\frac{(d\mathcal{M}/dr)}{r^2}\;\right) \tag{2.49}$$

consistent with taking the trace of the field equations $R = -8\pi G_o T$.

When the metric background is provided by (2.44), the scalar curvature is

$$R[\mathcal{M}(r)] \;=\; -\;2G_o\;\left(\;\frac{(d^2\mathcal{M}(r)/dr^2)}{r}\;+\;2\;\frac{(d\mathcal{M}(r)/dr)}{r^2}\;\right) \tag{2.50}$$

and the differential equation to solve for $\mathcal{M}(r)$ is the following Euler-Cauchy equation

$$\frac{(d^2\mathcal{M}(r)/dr^2)}{r}\;+\;2\;\frac{(d\mathcal{M}(r)/dr)}{r^2}\;=\;C\;=\;-\;\frac{4\Lambda}{2G_o} \tag{2.51}$$

after setting the scalar curvature to a constant equal to 4Λ.

The solution to (2.51) is

$$\mathcal{M}(r) \;=\; \frac{C}{12}\,r^3\;+\;\frac{A}{r}\;+\;B \tag{2.52}$$

and the metric (2.44) ends up by having the same functional $form$ as the Schwarzschild-(Anti)de Sitter-Reisnner-Nordstrom metric, whose temporal g_{tt} and radial components g_{rr} are

$$g_{tt} \;=\; -\,(1-\frac{2G_oM_o}{r}\;+\;\frac{Q^2G_o}{r^2}\;\pm\;\frac{\Lambda}{3}r^2),\;\;\; g_{rr} = -g_{tt}^{-1} \tag{2.53}$$

after the following identification among the parameters is made

$$B \;=\; M_o,\;\;\; -\,\frac{G_oC}{2}\;=\;\pm\,\Lambda,\;\;\; -\,2A\;=\;Q^2 \tag{2.54}$$

Since there is no mass nor charge in the real scalar-tensor field action (2.1), by setting $M_o = 0, Q^2 = 0$ one recovers the (Anti) de Sitter metric. The existence of a cosmological de Sitter horizon at the Hubble scale $1 - \frac{\Lambda}{3}R_H^2 = 0$ fixes then the value of $\Lambda = \frac{3}{R_H^2}$. Had $M_o \neq 0$, and/or $Q^2 \neq 0$, the value of Λ would have been different from $\frac{3}{R_H^2}$ if R_H still remained as the cosmological horizon.

To finalize, we will discuss the connection that the Dymnikova-type of metric [13] has with black hole cosmology. Although there is a mathematical equivalence in writing $G_\infty(r) M_o \leftrightarrow G_o \mathcal{M}(r)$, when the mass function is $\mathcal{M}(r) = M_o (1 - exp(-r^3/r_s L_P^2))$, there is a clear physical difference. In the former, one has a point mass M_o at $r = 0$ and a running gravitational coupling [11]. In the latter [13] one has the Newtonian coupling $G_o = G_N$ and a mass profile distribution as if the point mass M_o were smeared all over space. The importance of the Dymnikova solution is that one can rewrite

$$\frac{2 G_o \mathcal{M}(r)}{r} \ = \ \frac{\Lambda(r)}{3} \ r^2 \tag{2.55}$$

in terms of a running cosmological "constant" $\Lambda(r)$. At the Planck scale, one has a de Sitter core with a Planck size throat size and a with a very large value of $\Lambda(L_P) \sim \frac{1}{L_P^2}$. Whereas at the Hubble scale, one may rewrite the asymptotic Schwarzschild behavior of the metric in the form

$$\frac{2 G_o M_o}{R_H} \ = \ \frac{2 G_o M_o}{R_H} \ \frac{1}{R_H^2} \ R_H^2 \ = \ \frac{\Lambda(R_H)}{3} \ R_H^2 \tag{2.56}$$

with $\Lambda(R_H) = \frac{3}{R_H^2}$, if, and only if, M_o obeys $\frac{2 G_o M_o}{R_H} = 1$, and which is tantamount to viewing the Universe as a black hole whose mass $M_U = M_o$ is enclosed inside the cosmological horizon R_H, that matches also its Schwarzschild radius $r_s = 2 G_o M_o = R_H$. When this occurs, the uniform density over a spherical ball of radius R_H given by $M_o/(4\pi/3) R_H^3 = \frac{3}{8\pi G_o R_H^2}$ coincides precisely with the critical density (also vacuum density). For more details of black hole cosmology see [16].

Concluding, although the numerical findings of Asymptotic safety have been confined to $d = 4$, in principle we should expect the results of this work to admit an extension to other dimensions. If not this would signal four dimensions $d = 4$ as special. In a few words, the study of a Jordan-Brans-Dicke-like action (2.1) within the context of Weyl geometry, combined with the findings of Asymptotic Safety in quantum gravity, leads to a plethora of nice numerical results : (i) like singling out the quartic potential from all others; (ii) having (Anti) de Sitter space as the most natural solution; (iii) furnishing the value of the observed vacuum energy density at the Hubble scale $10^{-122} M_P^4$; (iv) a M_P^4 vacuum energy density at the Planck scale; (v) allowing the possibility that our universe is a black hole whose horizon coincides with the cosmological Hubble horizon.

Since (Anti) de Sitter space was an instrumental solution to the field equations (2.8-2.10), it is warranted to explore deeper the interplay among Weyl geometry, Asymptotic safety and the AdS/CFT correspondence (holographic renormalization group flow). Also relevant is the work by [31] on the role of dilatation symmetry in higher dimensions and the vanishing of the cosmological constant. Last, but not least, we should also consider the implications of Conformal Cyclic Cosmology [28] and Scale Relativity Theory [22] with the key findings of this work.

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