A Positivity-Based Approach to Delay-Dependent Stability of Systems of Second Order Equations

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Abstract

In this paper, new explicit tests for exponential stability of systems of second order equations are proposed. Our approach is based on nonoscillation of solutions of the corresponding diagonal scalar second order delay differential equations.

Keywords: time-delay systems, exponential stability, positive systems

1. Introduction

In this paper, we develop the positivity-based stability analysis for the system of second order equations with delay.

$$\begin{gathered} x_i''(t) = q_i(t)x_i'(t-\tau_i(t)) + \sum_{j=1}^{n} p_{ij}(t)x_j(t-\theta_{ij}(t)), \\ i = 1, ..., n, \quad t \in [0, +\infty), \end{gathered} \tag{1.1}$$

where $q_i, p_{ij} \in L_\infty$ (the space of essentially bounded functions), $\tau_i(t)$ and $\theta_{ij}(t)$ measurable nonnegative bounded functions. The positivity-based approach to the study of stability was used for systems of the first order equations

$$x_i'(t) = \sum_{j=1}^{n} p_{ij}(t)x_j(t-\theta_{ij}(t)), \; i = 1, ..., n, \quad t \in [0, +\infty), \tag{1.2}$$

for example, in the books [1, 2]. Denote the matrix of the coefficients $P(t) = \{p_{ij}(t)\}_{i,j=1}^{n}$.

Definition 1.1. *The matrix P is Metzler if all its off-diagonal elements are nonnegative for $t \geq 0$, i.e. $p_{ij}(t) \geq 0$ for every $i \neq j$, $i, j = 1, ..., n$.*

Consider the autonomous system of ordinary differential equations

$$x'(t) = Px(t), \quad t \in [0, +\infty), \tag{1.3}$$

here $P(t) = P$ is an $n \times n$ matrix. It is clear that system (1.3) is asymptotically stable (and also exponentially stable) if and only if the matrix P is Hurwitz. The matrix is said to be Hurwitz if all eigenvalues have negative real part.

Proposition 1.1. (see, for example, [3, 4]). *If matrix P is Metzler, the following 4 facts are equivalent:*

A) P is Hurwitz,

B) there exists a constant-vector $z = col\{z_1, ..., z_n\}$ with all positive components such that all components of the constant vector Pz are negative,

C) the matrix $(-P)^{-1}$ exists and all its entries are nonnegative,

D) system of ordinary differential equations (1.3) is exponentially stable.

It is well-known (Remark 2.1 from [5]) that (1.2) with a Hurwitz matrix P can be unstable for sufficiently large delays. It was demonstrated in [6, 7] that under the condition on a smallness of the products

$$|p_{ii}|\,\theta^*_{ii} \le \frac{1}{e},\ i = 1, ..., n, \tag{1.4}$$

where θ^*_{ii} =esssup$_{t\ge 0}\theta_{ii}(t)$, the equivalence of the assertions *A),B),C)* and *D)* is preserved for delay systems of first order equations (1.2). For the more complicated system

$$x_i'(t) = \sum_{j=1}^{n}\sum_{k=1}^{m} p^k_{ij}(t)x_j(t - \theta^k_{ij}(t)), \quad t \in [0, +\infty), \tag{1.5}$$

sufficient conditions of the exponential stability, which become necessary and sufficient in the case of constant coefficients, are obtained in the recent paper [5].

We propose an analogue of Proposition 1.1 for the system

$$x_i^{''}(t) = q_i x_i'(t - \tau_i(t)) + \sum_{j=1}^{n} p_{ij}x_j(t - \theta_{ij}(t)),\ \ i = 1, ..., n,\ \ t \in [0, +\infty), \tag{1.6}$$

with constant coefficients q_i and p_{ij}.

Theorem 1.1. *If matrix P is Metzler,*

$$p_{ii} < 0,\ q_i < 0,\ |q_i|\,\theta^*_{ii} \le \frac{1}{e},\ \theta_{ii}(t) \le \tau_i(t) \le \tau^*_i < \infty,\ i = 1, ..., n, \tag{1.7}$$

and

$$4\,|p_{ii}| < q_i^2,\ i = 1, ..., n, \tag{1.8}$$

then 4 facts A),B),C) and E) are equivalent for equation (1.6), where

E) system (1.6) is exponentially stable.

For system (1.1) with variable coefficients and delays we propose sufficient conditions of the exponential stability.

2. Preliminaries

Let us define the Cauchy function $c_i(t, s)$ of the scalar diagonal equation

$$x_i^{''}(t) = q_i(t)x_i'(t - \tau_i(t)) + p_{ii}(t)x_i(t - \theta_{ii}(t)), \quad t \in [0, +\infty), \tag{2.1}$$

$$x_i(\xi) = 0,\ x_i'(\xi) = 0 \text{ for } \ \xi < 0, \tag{2.2}$$

as follows [8]: for every fixed $s \geq 0$, as a function of the variable t, it satisfies the equation

$$(c_i)''_{tt}(t,s) = q_i(t)(c_i)'_t(t - \tau_i(t), s) + p_{ii}(t)c_i(t - \theta_{ii}(t), s), \quad t \in [s, +\infty), \tag{2.3}$$

$$c_i(\xi, s) = 0, \text{ for } \ \xi < s, \tag{2.4}$$

and the initial conditions

$$c_i(s,s) = 0, \frac{\partial c_i}{\partial t}(s,s) = 1. \tag{2.5}$$

The general solution of the scalar diagonal equation

$$x_i''(t) = q_i(t)x_i'(t - \tau_i(t)) + p_{ii}(t)x_i(t - \theta_{ii}(t)) + f_i(t), \quad t \in [0, +\infty), \tag{2.6}$$

$$x_i(\xi) = 0,\ x_i'(\xi) = 0 \text{ for } \ \xi < 0,$$

where $f_i \in L_\infty$, can be represented in the form [8]

$$x_i(t) = \int_0^t c_i(t,s)f_i(s)ds + x_{1i}(t)x(0) + x_{2i}(t)x'(0), \tag{2.7}$$

where $x_{1i}(t)$ and $x_{2i}(t)$ are solutions of the homogeneous equation (2.1) satisfying the initial conditions

$$x_{1i}(0) = 1,\ x_{1i}'(0) = 0,\ x_{2i}(0) = 0,\ x_{2i}'(0) = 1, \tag{2.8}$$

respectively.

Consider now the system of n second order equations

$$\begin{gathered} x_i''(t) = q_i(t)x_i'(t - \tau_i(t)) + \sum_{j=1}^n p_{ij}(t)x_j(t - \theta_{ij}(t)) + f_i(t), \\ t \in [0, +\infty),\ i = 1, ..., n, \end{gathered} \tag{2.9}$$

$$x_i(\xi) = 0,\ x_i'(\xi) = 0 \text{ for } \ \xi < 0.$$

We can rewrite it in the form of system of $2n$ first order equations

$$\begin{gathered} y_{2i-1}'(t) = q_i(t)y_{2i-1}(t - \tau_i(t)) + \sum_{j=1}^n p_{ij}(t)y_{2j}(t - \theta_{ij}(t)) + f_i(t), \\ y_{2i}'(t) = y_{2i-1}(t),\ t \in [0, +\infty),\ i = 1, ..., n, \end{gathered} \tag{2.10}$$

$$y_j(\xi) = 0 \text{ for } \ \xi < 0,\ j = 1, ..., 2n, \tag{2.11}$$

The general solution $y(t) = \operatorname{col}\{y_1(t), ..., y_{2n}(t)\}$ of the system (2.10) can be represented in the form

$$y(t) = \int_0^t C(t,s)g(s)ds + C(t,0)y(0), \tag{2.12}$$

where the $2n-$vector $g(t) = \mathrm{col}\{0, f_1(t), ..., 0, f_n(t)\}$. Its kernel $C(t,s)$ is called the Cauchy matrix of system (2.10).

Definition 2.1. *The Cauchy matrix $C(t,s)$ is said to satisfy the exponential estimate if there exist positive numbers N and α such that for all the entries of the Cauchy matrix $C(t,s) = \{c_{i,j}(t,s)\}_{i,j=1,...,n}$*

$$|c_{i,j}(t,s)| \leq N \exp\{-\alpha(t-s)\}, \quad i,j = 1,...,n, \ 0 \leq s \leq t < +\infty. \tag{2.13}$$

In this case we say that system (2.9) is exponentially stable.

Our main results are based on the following extension of the classical Bohl-Perron theorem:

Proposition 2.2[8]. *In the case of bounded delays $\theta_{ij}(t), \tau_i(t)$ and coefficients in the matrices $P(t)$ $(i,j = 1,...,n)$, the fact that for every bounded right-hand side the solution $x(t) = \mathrm{col}\{x_1(t), ..., x_{2n}(t)\}$ of system (2.10) is bounded on the semiaxis $[0, +\infty)$ is equivalent to the exponential estimate (2.13) of the Cauchy matrix $C(t,s)$.*

Definition 2.2. *The system (2.9) is called positive if all the entries of the Cauchy matrix $C(t,s) = \{c_{i,j}(t,s)\}_{i,j=1,...,n}$* of (2.10) *in even lines are nonnegative in the triangle $0 \leq s \leq t < \infty$.*

3. Main results

Denote $|q_{ii}|^* = \mathrm{esssup}_{t\geq 0} |q_{ii}(t)|$, $|q_{ii}|_* = \mathrm{essinf}_{t\geq 0} |q_{ii}(t)|$.

We obtain the following assertion.

Theorem 3.1. *Assume that*

$$p_{ii}(t) < 0, \ q_{ii}(t) < 0, \ |q_{ii}|^* \theta_{ii}^* \leq \frac{1}{e}, \ \theta_{ii}(t) \leq \tau_i(t) \leq \tau_i^* < \infty, \ i = 1,...,n, \tag{3.1}$$

$$4|p_{ii}(t)| \leq |q_{ii}|_*^2, \ i = 1,...,n, \tag{3.2}$$

and there exist a positive vector $z = \mathrm{col}\{z_1, ..., z_n\}$ such that

$$p_{ii}(t) z_i + \sum_{j=1, \ j\neq i}^{n} |p_{ij}(t)| z_j \leq -\varepsilon < 0, \ i = 1,...,n, \tag{3.3}$$

then system (2.9) is exponentially stable.

Lemma 3.1. *Let the condition (3.1) and (3.2) be fulfilled, then the Cauchy functions of all scalar diagonal equations (2.1) for $i = 1,...,n$, are positive in the triangle $0 \leq s \leq t < +\infty$.*

Proof. The proof follows from Theorem 16.12 [9].

Proof of Theorem 3.1. Using Lemma 3.1, we prove the positivity of the system

$$x_i''(t) = q_i(t) x_i'(t - \tau_i(t)) + p_{ii}(t) x_i(t - \theta_{ii}(t)) +$$

$$\sum_{j=1, \ j\neq i}^{n} |p_{ij}(t)| x_j(t - \theta_{ij}(t)), \quad i = 1,...,n, \quad t \in [0, +\infty), \tag{3.4}$$

repeating the proof of Theorem 3.1 in [5] and the exponential stability of system (2.9), repeating the proof of Theorem 3.2 in [5].

Proof of Theorem 1.1. In order to prove sufficiency we note that conditions *A)* and *B)* are equivalent for the Metzler matrix P (see Proposition 1.1). The condition *B)* coincides with the condition (3.3) of Theorem 3.1. Then all the conditions of Theorem 3.1 are fulfilled and, according to Theorem 3.1, we obtain the exponential stability of system (1.6).

To prove necessity, let us consider the initial value problem

$$x_i^{''}(t) = q_i x_i'(t-\tau_i(t)) + \sum_{j=1}^{n} p_{ij} x_j(t-\theta_{ij}(t)) + f_i(t), \quad i=1,...,n, \quad t \in [0,+\infty), \tag{3.5}$$

$$x_i(0) = z_i, \ x_i'(0) = 0, \tag{3.6}$$

where $f_i(t) \equiv 1$ for $t \geq \Theta$, $i = 1, ..., n$,

$\Theta = \max\left\{\max_{1\leq i\leq n} \operatorname{esssup}_{t\geq 0}\tau_i(t), \max_{1\leq i,j\leq n} \operatorname{esssup}_{t\geq 0}\theta_{ij}(t)\right\}$.

The constant vector $z = \operatorname{col}\{z_1, ..., z_n\}$ has to satisfy this initial value problem. The representation of solutions (2.12) leads to the equalities

$$z_i = \int_0^t \sum_{j=1}^{2n} c_{2i,j}(t,s) f_i(s) ds + \sum_{j=1}^{2n} c_{2i,j}(t,0) z_i = \int_0^\Theta \sum_{j=1}^{2n} c_{2i,j}(t,s) f_i(s) ds + \int_\Theta^t \sum_{j=1}^{2n} c_{2i,j}(t,s) ds + \sum_{j=1}^{2n} c_{2i,j}(t,0) z_i, \ i = 1,...,n. \tag{3.7}$$

The exponential estimate (2.13) of the Cauchy matrix of system (2.10) implies that

$$\int_0^\Theta \sum_{j=1}^{2n} c_{2i,j}(t,s) f_i(s) ds \to 0, \ c_{2i,j}(t,0) \to 0 \text{ for } t \to +\infty, \quad i = 1,...,n. \tag{3.8}$$

The condition $c_{2i,2i}(s,s) = 1$ leads to existence of the interval $[s, s+\delta]$, where $c_{2i,2i}(t,s) > 0$. This and nonnegativity of all $c_{2i,j}(t,s)$ lead to the conclusion that all components of the constant vector z are positive. We have proven that the exponential estimate (2.13) implies assertion *B)* for system (1.6). Equivalence of *A)* and *B)* (see Proposition 1.1) completes the proof.

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