Discrete motives for moonshine

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Abstract

From the holographic perspective in quantum gravity, topological field theories like Chern-Simons are more than toy models for computation. An algebraic construction of the CFT associated to Witten's j-invariant for $2+1$ dimensional gravity aims to compute coefficients of modular forms from the combinatorics of quantum logic, dictated by axioms in higher dimensional categories, with heavy use of the golden ratio. This paper is self contained, including introductory material on lattices, and aims to show how the Monster group and its infinite module arise when the automorphisms of the Leech lattice are extended by special point sets in higher dimensions, notably the 72 dimensional lattice of Nebe.

Contents

"I missed the opportunity of discovering a deeper connection between modular forms and Lie algebras, just because the number theorist Dyson and the physicist Dyson were not speaking to each other." Freeman Dyson (1972)

1 Introduction

The j-invariant is a function on the moduli space of tori specified by a fundamental domain for the modular group in the upper half plane. Each torus comes with a marked point, given by the product structure on the space. A boundary of this puncture might draw out a trefoil knot on a (traced) three stranded ribbon diagram with half twists, so that the knot is the boundary of the ribbons. For us, ribbon diagrams represent particle states in categories for quantum computation and condensed matter physics.

Suitable categories, like that of the Fibonacci anyon, label diagram elements with algebraic data. A braiding brings a deformation parameter into the algebra, and we begin with this rather than a loop parameter z from an affine algebra. In this view, infinite dimensional modules are built from the ground up. Even $\mathbb{C}$ itself is not fundamental, as we are permitted to replace it with dense subsets based on special number fields, including $\mathbb{Q}(\phi)$, for ϕ the golden ratio. Homotopically, $\mathbb{C}/0$ is just as good as the unit circle. Now a vertex operator algebra (VOA) [1][2] should be seen as an operadic creature, governed by the axioms of quantum computation rather than complicated arguments from string theory.

String theory emerged from the dual resonance models of the 1970s, but these ideas also led eventually to the operad formalism for scattering amplitudes, where the duality between s and t channels, with one leg on the diagram fixed, is like the associator map between 3-leaved trees. What we would like, then, is a new set of axioms for VOAs, which streamline the number theoretic information and clarify the physical content of moonshine theorems. This paper does not go so far, but hopefully clarifies the Monster group and j-invariant from this point of view, which we call motivic quantum gravity.

There is no doubt that the j-invariant $j(q)$ carries interesting number theoretic properties, as well as its module structure for the Monster group. Monstrous moonshine [3][4] employs a vertex operator algebra for bosonic strings, but here the string dimensions denote the number of strands on a ribbon diagram, prior to the emergence of spacetime, which is not in itself particularly interesting. In particular, the 27 dimensions of bosonic M theory give the dimension of a state space for three qutrits, which we will describe. The axiomatic framework promises to shed light on the basic structure of modular forms, along with the operads underlying VOAs.

Borcherd's formula states that

$$\frac{j(q)-j(p)}{\frac{1}{q}-\frac{1}{p}} = \prod_{n,m\geq 1}^{\infty}(1-q^n p^m)^{c(nm)} = \prod_{N\geq 1}\prod_{d|N}(1-q^d p^{N/d})^{c(N)} \qquad (1)$$

for d a divisor of N, where $c(i)$ is an integer coefficient of j. Recently in [5], it was used to study a duality between the inverse temperature in the usual argument q of j and a chemical potential associated to the number of copies k of j defining the CFT at $c = 24k$. This construction begins with Witten's j-invariant [6] for $k = 1$, giving the Bekenstein-Hawking entropy for BTZ black holes. When $c = 24$, we think immediately of the Leech lattice and its associated modular form. Their partition function is $\sum_{k \geq 0} p^{k+1} Z_k(q)$, where $Z_k(q) = \text{Tr} q^{\Delta - k}$. Here the coefficient $c(n)$ is the number of states with $\Delta = n + 1$ in Witten's CFT.

The thermal AdS regime in [5] is characterised by Bose-Einstein condensate ground states, and a BEC screening of central charge. The q and p variables are interchanged by a $\mathbb{Z}_2$ symmetry, implying a correspondence between temperature and minimal AdS_3 masses. We expect such behaviour in quantum gravity, which has a Fourier supersymmetry [7] between massive neutrinos and CMB photons, under which the present day CMB temperature corresponds precisely to a neutrino mass.

Borcherd's formula is closely related [8] to partition functions of the form $\sum_{n \geq 0} j_n(q) p^n$, where p is our second variable and $j_n(q)$ is derived from $j(q)$ by action of the Hecke operator $nT_{n,0}$ of weight zero, with $j_1(q)$ our usual j function. For $f(q) = \sum a(n) q^n$, this is the operator

$$T_{n,0} f(q) \equiv \sum_{i \geq 0} (a(ni) + a(i/n)/n) q^n, \tag{2}$$

assuming that $a(i/n)$ is only non zero for integral arguments.

Dualities and trialities are always information theoretic [9][10], and Bose-Einstein condensation is fundamental to the localisation of mass in the cosmological neutrino vacuum, for which the neutrino IR scale and its dual Planck scale underpin the Higgs mechanism [11]. The Fourier supersymmetry between Standard Model fermions and bosons introduces the 24 dimensions of the Leech lattice as hidden structure for the Z boson, roughly speaking. This is related to the non local states of the neutrino. As is well known, square roots of rest masses come in eigenvalue triplets with simple parameters, and we hope soon to complete the rest mass derivations with the scale ratios, starting with the large $\sqrt{Z/\nu_R}$ ratio, coming from lattice combinatorics in high dimensions.

There are no classical string manifolds and no supersymmetric partners. In fact, the quantum topos perspective aims to reformulate $\mathbb{R}$ and $\mathbb{C}$ completely, matching the combinatorics of categorical polytopes to generalised discrete root systems using the canonical rings that define our CFTs.

We leave further discussion of the Monster CFTs to the last two sections, starting with the combinatorial structure of lattices and the j-invariant, motivated by motivic quantum gravity. The next few sections are quite elementary, for readers unfamiliar with modular mathematics, but we begin with some pertinent remarks on set theory and cohomology, which are crucial to the category theoretic philosophy and its implications for the underlying motivic axioms. The section on braids includes definitions of important finite group elements. In section 9 we introduce a 72-dimensional lattice, taking the discussion beyond the Leech lattice. As a result, the Golay code has higher dimensional analogs, related to spinor spaces in the generalised algebra approach.

2 Sets and icosians

In order to build a CFT for a holographic theory, we want to throw a great deal of higher dimensional information into two dimensions. For instance, the roots of any Lie algebra are projected onto the so called magic star [12], which extends in exceptional periodicity [13] beyond the exceptional Lie algebras using a broken Jacobi rule for T-algebras. Broken rules for algebras make perfect sense in operads, which come with an infinite tower of rules for operad composition. The L_∞ operad [14] replaces Lie algebras when we are focused on homotopy, which secretly we are.

We are also interested in quasilattices, sometimes generated by projections to lower dimensions, particularly the two dimensional Penrose quasilattice [15]. This employs the golden ratio $\phi = (1+\sqrt{5})/2$ and $\rho = \sqrt{\phi+2}$ to map $\mathbb{Z}^8/2$ densely into $\mathbb{C}$ under the real maps

$$a + b\phi + c\rho + d\phi\rho, \tag{3}$$

for integral a, b, c, d.

The category theory is also essential for another reason: physical measurements are statements in quantum logic, where a dimension of a Hilbert space replaces the classical cardinality of a set. Thus quantum mechanics forces us immediately into an infinite dimensional setting. To make a measurement, we must also account for the classical data, whose logic is governed by the category of sets. It is therefore perfectly natural to map the subset lattice for an n point set in the category of sets onto a cube in n dimensions. Such cubes are fundamental as targets of the power set functor on the category of sets, dictating Boolean logic in the topos [16]. They are viewed here as diagrams for cohomology [17], along with other canonical polytopes like the associahedra.

The spinor dimensions in the $\mathbf{e_8}$ chain of exceptional periodicity [13] go up by factors of 16 (as in $8, 128, 2048, 32768$). Each 2^{4n-1} counts the number of vertices on a cube in dimension $4n-1$. In particular, 2^3 will give the (negative) charges of leptons and quarks, just as it defines a basis for $\mathbb{O}$, and 2^7 traditionally carries magnetic data. If the arrows from the source vertex represent a basis for the space, the cube also contains all other subsets of the basis set. For example, label the 8 vertices of the three dimensional cube

$$1, e_1, e_2, e_3, e_1e_2, e_2e_3, e_3e_1, e_1e_2e_3. \tag{4}$$

Here 1 denotes the empty set or zero point. The XOR product on subsets (either A or B but not both) is addition in the Boolean ring with intersection as product [18]. This recovers the structure of the Fano plane in ${\mathbb{F}_2}^3$, and hence the units in $\mathbb{O}$ [19][20]. Intersection is defined using the arrows and faces of the cube. Our subsets are also denoted by the $\mathbb{F}_2$ sign strings, so that e_1 is $+--$. These are anyon charges for the leptons and quarks.

The square, which labels two qubit states, is a module diagram for the Klein 4-group [17]. These diagrams are used to compute the cohomology $H^*(G,k)$ over a field of characteristic p, where $p=2$ in this case. The square labels are the e_1 and e_2 given above, where we work with $k[e_1,e_2]/(e_1^2,e_2^2)$. Cohomology for group algebras is natural for us, since we have Hopf algebras with suitable categorical axioms.

Given this algebraic richness, the recovery of a dense set in $\mathbb{C}$ using only the ring $\mathbb{Q}(\sqrt{\phi+2})$ [15] permits us to dispose of the usual continua in favour of more topos friendly constructions. The reals are equal in cardinality to the power set of $\mathbb{N}$, which is just a second application of the power set monad to our initial category, picking out dimensions $n=2^k$. In exceptional periodicity [13] this includes the spatial dimensions 8 and 32, where four copies of $\mathbb{Z}^8/2$ are needed for $SL_2(\mathbb{C})$, the cover of the Lorentz group.

The infinite dimensional cube defines reals, within the surreals, that are not finite dyadic: an infinite string of minus signs is the infinitesimal, and the infinite plus string is the surreal infinity. Thus the surreals, or something like them, assign a natural normalisation of 2^{-n} to n qubits.

When each e_i coordinate of (4) is extended to a discrete line $e_i, e_i^2, \cdots$, as if the line is a path space, then coordinates represent the prime factors of $N=\prod_{i=1}^r p_i^{k_i}$ for $N\in\mathbb{N}$, in a general rectangular array of points. All divisors d of N sit on the points below N, which is the target of the rectangular block. For example, $N=30=2\cdot 3\cdot 5$ is modeled on the basic 3-cube, and each square free N gives a parity cube in dimension r. In this picture, $\mathbb{Z}$ is an infinite dimensional cubic cone, with a discrete axis for each prime. This is natural, because the Cartesian product $N\times M$ has either NM points or $N+M$ dimensions as a vector space. Taking all subsets of a basis enumerates all possible sets of linearly independent vectors within the basis.

Sign strings of length n arise as signature classes for permutations in S_{n+1}. For example, (2314) in S_4 belongs to the class $+-+$, with a plus denoting an increase in numerals as we read the permutation left to right. Eight vertices on the 3-cube are therefore derived from the 24 vertices of the S_4 permutohedron, a polytope in dimension 3. The group algebra kS_{n+1} descends to the Solomon Hopf algebra on the vertices of the cube, starting with the elements $\sum_i \pi_i$ for $\pi_i\in S_{n+1}$ ranging over the signature class. The vertices of the permutohedron, which also tiles three dimensional Euclidean space, are denoted by the divisors of the number $N={p_1}^n{p_2}^{n-1}{p_3}^{n-2}\cdots p_n$. For S_4, we get the 24 points

$$\begin{aligned} 1, p_1, p_2, p_3, {p_1}^2, {p_2}^2, {p_1}^3, \\ p_2p_3, p_1p_3, p_1p_2, p_3{p_2}^2, p_3{p_1}^2, p_2{p_1}^2, p_1{p_2}^2, p_2{p_3}^2, \\ p_2{p_3}^3, {p_2}^2{p_3}^3, {p_2}^2{p_3}^2, p_1p_2p_3, \\ p_1{p_2}^2p_3, p_1{p_2}^2{p_3}^2, p_1{p_2}^2{p_3}^3, p_1p_2{p_3}^2, p_1p_2{p_3}^3, \end{aligned} \tag{5}$$

viewed as permutations of $(-3,-1,1,3)/2$ in $\mathbb{Z}^4/2$, or more often, $(1,2,3,4)$.

This permutohedron is mapped to the $(1,0,0,1)$ coordinates of the 24-cell as follows. A $(1,2,3,4)$ vector sends the 3 and 4 to 1, and the 1 and 2 to 0. Since each parity square face on the S_4 polytope has 3 and 4 in the same positions, the square adds the signs to the resulting vector. Extending the 24-cell coordinates into 8 dimensions using four extra zeroes, we get as usual the 112 bosonic roots of $\mathbf{e_8}$. Opposite pairs of squares on S_4 give three separate copies of the 3-cube basis for $\mathbb{O}$, so that 9 copies of S_4 provide sets of 24 roots on the 7 internal points of the magic star in the plane. One further copy of S_4 catalogs (i) 6 points for the $\mathbf{a_2}$ hexagon and (ii) three $J_3(\mathbb{O})$ diagonal elements on each tip of the star.

Replacing sets and permutations by vector spaces and endomorphisms, for quantum logic, we naturally consider the vector space analog of transpositions, namely reflections. This is why generalised root lattices appear naturally in quantum gravity.

The Leech lattice [21] is easily defined in terms of the 120 norm 1 icosians of (45). As with the $\mathbb{O}^3$ Leech lattice, the icosian lattice is a subset of vectors $(x, y, z) \in \mathbb{H}^3$ with x, y and z all icosians. First consider the 24 icosians

$$\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k). \tag{6}$$

Since the basis $+\{1, i, j, k\}$ forms a parity square in the plane, as just described for $e_1 = i$ and $e_2 = j$, we would like to group these 24 icosians into three pairs of squares, such that the numbers with three minus signs belong to one square. These sets are not orthogonal (in sets of three) in $\mathbb{H}$, but they are in $\mathbb{H}^3$ if we spread the squares out into different copies of $\mathbb{H}$. Then we project from 12 dimensions down to 3, obtaining three pairs of square faces on the permutohedron S_4. For example, the square

$$-1 + i + j + k, \;\; 1 + i - j + k, \;\; 1 - i + j + k, \;\; 1 + i + j - k \tag{7}$$

starts with ++ for the signs on i and j, and then flips i or j on its diagonal.

Let A be the set of 24 icosians in (6) rescaled by 1/2, which we now use to define the 240 roots of $\mathbf{e_8}$, following [22]. Let B be the set $(1 + i)A$. The 240 roots are the ten sets of vectors in 8 dimensions, given as pairs of quaternions (q_1, q_2),

$$\begin{gathered} (B, 0), \;\; (0, B), \;\; (A, A), \;\; (A, -A), \;\; (A, iA), \\ (A, jA), \;\; (A, -jA), \;\; (A, kA), \;\; (A, -kA), \;\; (A, -iA). \end{gathered} \tag{8}$$

The nine non zero types of entry in these vectors define nine copies of the permutohedron above. Alternatively, each of the ten vector types defines an 8-dimensional analog of S_4.

There is a natural way to take 5 copies of a permutohedron in dimension 3 to build a 120 vertex polytope known as the permutoassociahedron [23], obtained by replacing each vertex on S_4 with a pentagon. This polytope is crucial to the axioms for ribbon categories. A braiding and fusion rule in a ribbon category is directly analagous to the breaking of commutativity and associativity in a loop. A 4-valent double permutoassociahedron is a 240 vertex polytope in dimension 4.

These ten sets define the S^3 fibres of a discrete Hopf fibration [22] for S^7, where the discrete base S_4 is given by the 10 quaternions q_1/q_2, including ∞ for

$(B,0)$. These points map to a cubic basis of type $(\pm 1,0,0,0,0)$ in 5 dimensions, indicating that we can split the ten permutohedra into two sets of five, as desired.

The other 96 icosians are of the form

$$\frac{1}{2}(\pm 0 \pm i \pm \phi j \pm \phi^{-1} k) \tag{9}$$

up to signs and even permutations on $\{1,i,j,k\}$. That is, 8 copies of A_4 in $\pm$ pairs.

3 The Fibonacci reflection and braids

The simplest expression for the j-invariant in (22) is invariant under the S_3 permutations of three roots for a cubic, where group multiplication is function substitution under the correspondence

$$z = (1),\ \ 1-\frac{1}{z} = (312),\ \ \frac{1}{1-z} = (231), \tag{10}$$
$$\frac{1}{z} = (31),\ \ \frac{z}{z-1} = (32),\ \ 1-z = (21).$$

This is almost, but not quite, represented by matrices in the modular group $\Gamma = PSL_2(\mathbb{Z})$. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{11}$$

be the two generators of Γ and let

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{12}$$

be a reflection across the axis in the plane. Then the correct representation is

$$(1) = I,\ \ (312) = TS,\ \ (231) = ST^{-1}, \tag{13}$$
$$(31) = -SZ,\ \ (32) = -ST^{-1}SZ,\ \ (21) = -TZ.$$

We have to multiply by Z on the right to obtain the usual $PSL_2(\mathbb{F}_2)$, for which $T^2 = I$. Now observe that the powers $(TSZ)^n$ generate the Fibonacci numbers F_k. At mod m, the set of F_k is a cycle of length $L(m)$, depending on primes associated to k. In particular, we have F_4 mod $3 \equiv 0$, so that $T^8 = I$ and $(TSZ)^4 = I$, giving a representation of S_4. For $PSL_2(\mathbb{F}_7)$ we need F_k mod 13. Up to mod 12, all cycles fit into a cycle of length 240.

Recall that the limit of F_{k+1}/F_k is the golden ratio $\phi = (1+\sqrt{5})/2$ [24]. In order to understand the relationships between different structures on the Leech lattice, we need to look at the rotation between normed division algebra braids and Fibonacci anyon representations [25]. The cyclic braid group B_3^c on three strands [25][26] is given by quaternion units i, j, k in

$$\sigma_1 = \frac{1}{\sqrt{2}}(1+i),\ \ \sigma_2 = \frac{1}{\sqrt{2}}(1+j),\ \ \sigma_3 = \frac{1}{\sqrt{2}}(1+k), \tag{14}$$

such that $\sigma_i^8 = 1$. This representation corresponds to a phase $\pi/4$ in a circle of representations. The complex B_2 generator σ_1 is used to describe ribbon twists in an extension of B_3^c to ribbon diagrams for particle states. Similarly, B_7^c has a representation [26]

$$\sigma_1 = \frac{1}{\sqrt{2}}(1 + e_2 e_1), \;\; \sigma_2 = \frac{1}{\sqrt{2}}(1 + e_3 e_2), \;\; \sigma_3 = \frac{1}{\sqrt{2}}(1 + e_4 e_3), \tag{15}$$

$$\sigma_4 = \frac{1}{\sqrt{2}}(1 + e_5 e_4), \;\; \sigma_5 = \frac{1}{\sqrt{2}}(1 + e_6 e_5), \;\; \sigma_6 = \frac{1}{\sqrt{2}}(1 + e_7 e_6),$$

$$\sigma_7 = \frac{1}{\sqrt{2}}(1 + e_1 e_7),$$

where the e_i Clifford algebra elements all satisfy $e_i^2 = -1$ and $e_i e_j = -e_j e_i$. This B_7^c will appear in associative algebras based on $\mathbb{O}$.

The B_3 representation based on j and k, viewing $\mathbb{H}$ as 2×2 matrices, is rotated to a matrix Fibonacci representation in $SU(2)$ [25] using golden phases,

$$M = e^{7\pi j/10}, \quad P = j\phi^{-1} + k\sqrt{\phi}^{-1}, \quad N = PMP^{-1}, \tag{16}$$

satisfying the braid relation $MNM = NMN$. This rotation is 9°, where $(\phi\sqrt{\phi + 2})^{-1} = \tan 18^\circ$ and $\phi = 2\cos(2\pi/10)$. Another special angle of 38.17° defines the Great Pyramid triangle, whose Pythagorean triple is $(\phi, \sqrt{\phi}, 1)$. Its Fibonacci approximations give $(\sqrt{a^2 + b^2}, a, b)$ for $a^2 = F_{n+1}$ and $b^2 = F_n$, but there are only three F_n that are squares, namely $F_1 = F_2 = 1$ and $F_{12} = 144$. The Fibonacci analog of Pythagorean triples is the rule $F_{n+1}^2 - F_n^2 = F_{n+2}F_{n-1}$. The braid relation of (16) is rewritten in the form $MX = XM$ for $X = PMP^{-1}MP$.

We define Fibonacci matrix triples. Note that the numbers F_n/F_{n+1} belong to Farey sets. When n is even, $F_n/F_{n+1} < F_{n-1}/F_n$, and vice versa for odd n. The Farey triple of modular group matrices for even n is then

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \quad \begin{pmatrix} F_{n+1} & F_{n+2} \\ F_n & F_{n+1} \end{pmatrix}, \quad \begin{pmatrix} F_{n+2} & F_n \\ F_{n+1} & F_{n-1} \end{pmatrix}, \tag{17}$$

with the first matrix equal to $(TSZ)^n$. The columns of this first matrix define the initial left and right Farey fractions, while the other matrices insert the mediant either on the left or right. Allowing $F_{-1} = -1$ puts both T and S into the middle matrix, and Z in the first. Check that the second and third matrix sum under Farey sums, giving yet another determinant 1 matrix.

Extraspecial 2-groups are important to the Monster, and there is an entanglement representation [25] related to braids. To be specific, consider a 2^n dimensional state space $V^{\otimes n}$ for n qubits. Let $\mathbf{i}$ be the complex unit. The quaternions i, j and k are represented by the matrices

$$i = \mathbf{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \mathbf{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{18}$$

Let $E = j \otimes j$, which satisfies $E^2 = -I$. In $V^{\otimes n}$ we position E in two adjacent positions and put the identity I in the other places. So for $V^{\otimes 3}$ we define

$$E_1 = E \otimes I, \quad E_2 = I \otimes E. \tag{19}$$

In general, the E_i satisfy $E_i E_{i+1} = -E_{i+1} E_i$ and for $|i - j| > 1$ we have $E_i E_j = E_j E_i$. The 2-group relations follow from $E_1 E_2 E_1 E_2 = -I$. Using quaternions, $(iE_1)(jE_2) = (jE_2)(iE_1)$.

For the dihedral group, define a generator

$$R = \frac{1}{\sqrt{2}} e^{\pi \mathbf{i}/4} \begin{pmatrix} 1 & \mathbf{i} \\ \mathbf{i} & 1 \end{pmatrix} \tag{20}$$

such that $R^4 = I$. Then $\mathbf{i}i = R^2$ and $-I$ represents a flip. The columns of R are the eigenvectors of the Pauli matrix $\mathbf{i}j$. This qubit basis is mutually unbiased [27][28][29] with respect to the eigenvectors for the other Pauli matrices, which give I along with the Hadamard matrix (56). Let $\omega = \exp(2\pi\mathbf{i}/3)$. For p a prime, in dimension p there are $2p$ matrices given by $p+1$ mutually unbiased bases and their Fourier transforms. When $p = 3$ these are the five matrices

$$F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix}, \quad R_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & 1 \\ 1 & 1 & \omega \\ \omega & 1 & 1 \end{pmatrix}, \quad \overline{R_3} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & \overline{\omega} \\ \overline{\omega} & 1 & 1 \\ 1 & \overline{\omega} & 1 \end{pmatrix}, \tag{21}$$

$$F(R_3) = -\mathbf{i} \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad F(\overline{R_3}) = \mathbf{i} \begin{pmatrix} \overline{\omega} & 0 & 0 \\ 0 & \overline{\omega} & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

along with the identity I_3. The circulant R_3 generates the cyclic group C_{12}. In section 7 we see how F_3 and the underlying C_3 form a basis for the 27 dimensional exceptional Jordan algebra.

4 The j-invariant and golden ratio

Before discussing the Leech lattice, we introduce the j-invariant for the modular group Γ. A divisor function of k-th powers is denoted $\sigma_k(n)$. Let $z = a + ib$ and

$$J(z) = 1728 \cdot \frac{4}{27} \cdot \frac{(z^2 - z + 1)^3}{z^2 (z-1)^2}, \tag{22}$$

so that $J(i) = 1728$. It's famous Fourier expansion is

$$j(q) = J(q) - 744 = q^{-1} + 196884q + 21493760q^2 + \cdots \tag{23}$$

Let's imagine we are interested in real values of the j-invariant. Using the numerator and denominator of (22), we define a 2×2 matrix using the coeffients of

$$J(z) = \frac{A + iB}{C + iD}. \tag{24}$$

Such a splitting of terms covers two cases of interest: (i) A, B, C, D all real, and (ii) A, C rational and B, D pure imaginary irrational. The reality condition is

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0, \tag{25}$$

giving a degree 9 or 10 polynomial P in a and b. At $a = 1$ we obtain

$$P(b) = b^3(b^2 - 1)(b^2 - ib + 1)(b^2 + ib + 1), \tag{26}$$

which defines the critical values

$$0, \pm 1, \pm \phi, \pm \phi^{-1}. \tag{27}$$

For small values of b, we pick up the standard critical values

$$0, \pm 1, \infty, \frac{1}{2}, 2, e^{\pi i/3}, e^{-\pi i/3} \tag{28}$$

of the ribbon graph, as roots of

$$P(a) = a(a-1)(a+1)(a-2)(2a-1)(a^2 - a + 1)^2. \tag{29}$$

Note the invariance under $a \mapsto 1 - a$, and

$$J(\pm\phi) = J(\pm\phi^{-1}) = 2048 = 2^{11}. \tag{30}$$

This golden ratio value appears in the combinatorics of the first shell of the Leech lattice Λ, where $\sigma_{11}(2) = 2049$ appears in the second term of the lattice form

$$f_\Lambda = \sum_{i=0}^{\infty} \frac{65520}{691}(\sigma_{11}(i) - \tau(i)) = 1 + 196560q^2 + 16773120q^3 + \cdots \tag{31}$$

which includes Ramanujan's τ coefficients for the modular discriminant

$$\Delta(q) = \sum_{n=1}^{\infty} \tau(n) q^n = q - 24q^2 + 252q^3 - \cdots \tag{32}$$

The nicest Leech lattice integers appear with the Hecke operator T_2, which acts on the dimension 2 space of weight 12 forms for Γ, giving

$$T_2(f_\Lambda) = 2049 + 196560q + \cdots, \quad T_2(\Delta) = 0 - 24q + \cdots \tag{33}$$

For quadratic fields over $\mathbb{Q}$, the only good integral values of $j(z)$ are the twelve critical values listed in (27) and (28), as follows. Let $z = a\sqrt{n} + b$ for $a, b \in \mathbb{Q}$. When the numerator and denominator of j are rational, we look for a numerator that is a multiple of the denominator, in the form

$$(64x^2 - 32x + 4)(x + s) \tag{34}$$

with $x = a^2 n$. The constant term forces $s = 27/4$ and we find the solution $x = 5/4$. Defining $y = 4a^2 n$ and seeing that j is proportional to

$$\frac{(y+3)^3}{(y-1)^2}, \tag{35}$$

it must be that the prime factors of $y-1$ are obtained in the factors of the numerator, restricting us to $p = 2$. This forces the solution $a = \pm 1/2$, $n = 5$, giving $z = \phi$ or ϕ^{-1}. Consider now

$$f(a\sqrt{n} + b) \equiv a^2 n + b^2 + (2ab - a)\sqrt{n} - b \tag{36}$$

in the numerator. There must exist $c \in \mathbb{Q}$ such that $f^3 + 3cf + c = 0$, but then f^2 is also rational, which is only the case for n a square. Siegel's theorem [30] states that, besides $\mathbb{Q}$, the only (totally real) algebraic number field for which every ordinal is a sum of at most three squares is $\mathbb{Q}(\sqrt{5})$, which contains the golden ratio integers.

Let $C(n)$ be the Fourier coefficients of $j(q) + 24$, using (23). A recursion formula [31] for $C(n)$ is

$$C(n) = -\sum_{i=-1}^{n-1} C(i)\tau(n+1-i) + \frac{65520}{691}(\sigma_{11}(n+1) - \tau(n+1)). \tag{37}$$

In particular,

$$C(1) = 196884 = 24^2 - 252 + 196560 \tag{38}$$

reminds us of the Monster. The $C(n)$ coefficients exhibit Ramanujan type congruences [32] such as

$$C(5^i k) \equiv 0 \mod 5^{i+1}, \quad C(7^i k) \equiv 0 \mod 7^i, \quad C(11k) \equiv 0 \mod 11, \tag{39}$$

where i is any positive ordinal. Here $5, 7, 11$ are equal to $n + 3$ for $n = 2, 4, 8$, the dimensions of the division algebras. We will see that these three primes correspond to $6 \cdot 8 \cdot 12 = 24^2$. When talking about $j(q)$, we will denote the coefficients by little $c(n)$.

5 Cubes and the Leech lattice

We look at the Leech lattice Λ for the octonions $\mathbb{O}$, for the icosians in $\mathbb{H}$, and for $\mathbb{C}$. Later we will also look at a $\mathbb{Z}[\alpha]$ structure where, once and for all, we fix $\alpha = (1 + \sqrt{-7})/2$. Table 1 gives the octonion multiplication table following [33], which is close to subset notation.

The Leech lattice is described in terms of integral octonions in $\mathbb{O}^3$ by Wilson [34][35]. Start with the 8 dimensional root lattice L_8 of $\mathbf{e_8}$, generated by a set of 240 unit octonions. These are the 112 octonions of the form $\pm e_i \pm e_j$ for any distinct units e_i and e_j of $\mathbb{O}$, and the 128 octonions of the form $(\pm 1 \pm i \pm j \pm \cdots \pm l)/2$ with an odd number of minus signs. We write $L = L_8$. We could start

Table 1: multiplication of units in $\mathbb{O}$

	i	j	k	il	jl	kl	l
i	-1	k	$-j$	$-l$	kl	$-jl$	il
j		-1	i	$-kl$	$-l$	il	jl
k			-1	jl	$-il$	$-l$	kl
il				-1	k	$-j$	$-i$
jl					-1	i	$-j$
kl						-1	$-k$
l							-1

instead with a right lattice R, or the lattice $2B = LR$, and this will be discussed in section 8. The Eisenstein form E_4 is the norm function for L_8, counting the vectors of length $2n$. Let $s = (-1 + i + j + \cdots + kl + l)/2$. Then the Leech lattice Λ [34] is the set of all triplets $(u, v, w) \in \mathbb{O}^3$ such that

1. $u,\ v,\ w \in L$
2. $u + v,\ v + w,\ w + u \in L\overline{s}$
3. $u + v + w \in Ls$.

Note that $L\overline{s} \cap Ls = 2L$. For future reference, given a root X in L, Λ includes the norm 4 vectors

$$(X\overline{s}, X\overline{s}, 0), \quad (2X, 0, 0), \quad (Xs, X, X). \tag{40}$$

Consider the Leech vector

$$(1 + i)(i + jl)s = -1 - i - j - k - il - jl + kl - l \tag{41}$$

of the form $2X$, where $X \in L$. If we multiply X on the right by any of the 8 units we obtain other spinors in L, and hence 24 vectors of shape $(2X, 0, 0)$ in Λ. The number of short vectors in Λ is $196560 = 24 \cdot 8190$, where $819 = 3 \cdot 273$. The number

$$273 = 1 + 16 + 16^2 \tag{42}$$

accounts [34] for the sign choices for the charts of a projective plane $\mathbb{OP}^2$.

When Λ is embedded in the exceptional Jordan algebra $J_3(\mathbb{O})$ [36], triality acts on the off diagonal elements of

$$\begin{pmatrix} a & X & \overline{Y} \\ \overline{X} & b & Z \\ Y & \overline{Z} & c \end{pmatrix}, \tag{43}$$

for a, b, c real. Triality is given by a triple of maps: left, right and two sided multiplication (L, R, B) by elements of $\mathbb{O}$. We look further at triality in section 8.

Inside the 16 dimensional $\mathbb{O}^2$ lies an S^{15}, which can be associated to a discrete Hopf fibration $S^{15} \to S^8$ [22][37]. The number of short vectors also satisfies

$$196560 = 196608 - 48, \tag{44}$$

where $196608/2 = 3 \cdot 2^{15} = 24 \cdot 2^{12}$. That is, identifying opposite points in the sphere of short vectors in Λ takes us to the projective $\mathbb{OP}^2$, which has three charts based on $\mathbb{O}^2$. Each $\mathbb{O}^2$ contains a cube with 2^{15} vertices, and we subtract 8 basis points for each copy of $\mathbb{O}$ so that the 2-forms are minimal [37].

Compare this to the standard 24 vectors in the real Λ, which are of the form $(3, 1, 1, \cdots, 1)$, considered to be in $\mathbb{C}^{12}$, so that there are only 2^{12} sign choices. To remove 24 vectors, fix one positive sign on a 3 and make all remaining signs negative.

Instead of the usual $\mathbb{OP}^2$ decomposition $\mathbb{O}^2 + \mathbb{O} + 1$, we may wish to add a *regulator* triangle of three points near infinity on the affine plane. This is analogous to adding two points on a line in a triangle model for $\mathbb{RP}^1$, turning the triangle into the famous pentagon [38][39]. Adding three points to a discrete cohomological $\mathbb{RP}^2$ gets us the 14 vertex associahedron.

In the complex Λ [40] over $\mathbb{Z}[\omega]$, for ω the primitive cubed root, it is useful to use the vector $(\theta, \theta, \theta, \theta, \theta, \theta, 0, 0, 0, 0, 0, 0)$, where $\theta = \omega - \overline{\omega}$. Its integral norm is 18, which is adjusted down to 4 by a factor of 2/9, a parameter that will often crop up, most notably in the $J_3(\mathbb{C})$ Koide mass matrices for leptons.

As noted above, the quaternion Leech lattice [35][41] uses the 120 norm 1 icosians of the form

$$\frac{1}{2}(\pm 1 \pm i \pm j \pm k), \quad \frac{1}{2}(\pm i \pm \phi^{-1} j \pm \phi k), \quad \pm 1, \quad \pm i, \quad \pm j, \quad \pm k, \tag{45}$$

representing $SL_2(5)$. In analogy to the $\mathbb{O}^3$ construction, consider vectors (x, y, z) in $\mathbb{H}^3$, such that x, y, z are icosians. Let $h = (-\sqrt{5} + i + j + k)/2$ and $\overline{h}$ its quaternion conjugate. The number h defines a left ideal in the icosians, as do the four other numbers given by an odd number of minus signs on (i, j, k) in h, which includes $\overline{h}$. These five ideals define the 600 norm 2 icosians. Let L_h and $L_{\overline{h}}$ denote the two ideals, given by left multiplication in the full icosian ring. Then Λ is the lattice defined by

1. $x \equiv y \equiv z \mod L_h$
2. $x + y + z \equiv 0 \mod L_{\overline{h}}$.

Conway [42] looks at Lorentzian lattices in $\mathbb{R}^{d,1}$ for $d \equiv 1 \mod 8$. In particular, at $d = 25$ there exists an infinite group of automorphisms for the Leech lattice, including the translations, generated by the reflections of fundamental roots, where a root $u \in \Lambda$ satisfies $u \cdot u = 2$ and $u \cdot w = -1$ for the norm zero 26-vector

$$w = (0, 1, 2, \cdots, 23, 24, 70). \tag{46}$$

In today's Atlas notation, the finite group $\text{Aut}(\Lambda)$ is called $2 \cdot \text{Co}_1$, where the simple Conway group Co_1 has order $25 \cdot 27 \cdot 196560 \cdot 2^8 \cdot |M_{24}|$, for M_{24} the Mathieu group.

Recall the braid group rotation (16). We want to use it to understand how the $\mathbb{O}^3$ structure on Λ relates to the $\mathbb{H}^3(\phi)$ icosian structure. Similarly, in 12 dimensions, we will relate the $\mathbb{Z}[\omega]$ structure to the $\mathbb{Z}[\alpha]$ one, but this requires a peek at the 72 dimensional lattice.

With the golden ratio and octonions appearing everywhere, we realise that 5 point bases are just as important as 3 point ones. After all, $SU(3)$ for color uses a 3-space and a 5-space. The permutohedron S_6 has 720 points in dimension 5. Taking the Cartesian product of a 5-space with $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, we get the triplet of dimensions

$$5+2=7, \quad 5+4=9, \quad 5+8=13, \tag{47}$$

so that a tensor product of quantum components has dimension $7 \cdot 9 \cdot 13 = 819 = 3 \cdot 273$. This 819 counts the so called integral Jordan roots [43] in $J_3(\mathbb{O})$, which define a simplex for $\mathbb{OP}^2$ [44] distinct from the optimal simplices based on mutually unbiased bases. The idea is to think of the crux pitch in the Monster in terms of such canonical discrete spheres, leaving most of the group elements to the simple MUBs.

6 Constructing a Jordan algebra

For $\mathbb{RP}^n$ the important polytopes are the associahedra [38][45], which are naturally embedded in an n-dit cube of dimension n inside an $(n+1)$-dit space [46] as follows. Look at the diagonal simplex defined by paths of length n on a cubic lattice. Parking function words, which are noncommutative paths such that the order of the i-th letter is greater than or equal to i, fit onto a subset of vertices in the triangular simplex, which defines commutative monomials in path letters.

In particular, the pentagon sits inside the 10 point tetractys simplex for 27 paths on a 64 point 3-cube, while the associator edge carries 3 paths on the diagonal of a 9 point square. The target vertex on an associahedron carries a copy of S_n, with S_4 appearing on the 14 vertex polytope.

Consider the 27 length 3 paths in the letters X, Y and Z. Our trit letters might represent powers of the three matrices

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad W = \omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{48}$$

in a product $U^X V^Y W^Z$, where ω is the primitive cubed root of unity. The spatial matrix V generates C_3, while the momentum operator is obtained as its quantum Fourier transform. These 27 matrices define a discrete phase space, and are used to define the exceptional Jordan algebra $J_3(\mathbb{O})$ using $\mathbb{F}_3 \times {\mathbb{F}_3}^3$, as shown in [47].

These four trits arise in the following simplex. To properly separate the 27 components of the Jordan algebra, the tetractys needs to be replaced by a simplex carrying 81 paths of length 4. These paths live in a cube with $5^3 = 125$

vertices (which counts parking functions). A parity 3-cube for a basis of $\mathbb{O}$ appears when one selects one out of four letters. For instance, choosing $ZXXX$ out of four possible permutations marks the first letter for deletion, leaving the word XXX, so that a parity cube is now labelled

$$ZXXX, \{XXXZ, XXZX, XZXX\}, \{ZXZZ, ZZXZ, ZZZX\}, XZZZ. \quad (49)$$

On this 81 path simplex, the unused corners $XXXX$, $YYYY$ and $ZZZZ$ are free to define the diagonal of a 3×3 matrix, so that the boundary of the simplex (without edge centre points) gives the 27 dimensions of $J_3(\mathbb{O})$.

Observe that the central 54 paths, which are not included in our 27, reduce under the letter deletion operation either to existing paths on the parity cubes or to the six XYZ paths of S_3. In this way the spinor splitting $27 = 16 + 11$, which ignores the shadow 54, reduces on the tetractys tile to $12 + 9$, so that the tetractys centre S_3 is sourced from the shadow paths. Selecting 6 out of 27 tetractys paths is one way to obtain that factor of $2/9$, which occurs as a lepton Koide phase for rest mass triplets [11][7] in $\mathbb{C}S_3$. Below we will look at a 72 dimensional lattice, which uses three copies of the Leech lattice. If the leptons only see one copy, then this shadow $2/9$ is related to the $2/9$ that appears in the normalisation of the complex lattice, as noted in section 6.

An example of a 3-cube inside S_4 on the paths $ZZZY$ and $YYYZ$ is

$$\begin{array}{ll} YYY : 1324 \quad YYZ : 2314, 3142, 1423 \\ ZZY : 2413, 4132, 3241 \quad ZZZ : 4231. \end{array} \quad (50)$$

Thus 9 copies of S_4 fill 216 roots on the magic star for $\mathbf{e_8}$, leaving 18 diagonal entries plus 6 points for $\mathbf{a_2}$. On this tenth S_4, permutations are spread around the triangles of the star. For example, relative to an outer $\mathbf{a_2}$ hexagon that fixes a 1 in the first coordinate, one vertex on the star carries a 432 subcycle, for the other three possible positions of 1, giving the $J_3(\mathbb{O})$ diagonal $(4132, 4312, 4321)$. A triple of Jordan algebras is then the 1-circulant set $\{432, 243, 324\}$, which is the usual basis for Koide mass matrices.

In the magic $\mathbf{a_2}$ plane [12][13], a 27 dimensional $J_3(\mathbb{O})$ is assigned to each point on the star, as a piece of the 240 roots of $\mathbf{e_8}$. This plane is tiled by tetractys simplices, each carrying three pentagons, and a discrete blowup in the plane replaces a point on the star with a simplex. In the 64 vertex cube, which has a total of 1680 paths from source to target, there are actually two diagonals that hold a tetractys, their triangle boundaries pointing in opposite directions, so that the projection of these two simplices along the diagonal gives the magic star. Combining associahedra and permutohedra, we obtain the 120 vertex permutoassociahedron [23] in dimension 3, counting half the roots in $\mathbf{e_8}$. In 4 dimensions, a permutoassociahedron has $1680 = 5! \cdot 14$ vertices.

7 Triality and integral forms

"Really it is an instance of a much more general construction which can be used for almost all simple groups; if, as usually happens, there is an involution with

three conjugates whose product is the identity, then in most instances, there is a triality automorphism ...” R. A. Wilson (2009)

In section 6 we defined the Leech lattice Λ using one of the lattices L, R and B. To understand the distinction between these lattices, we require the so called *integral octonions* $I = I(\mathbb{O})$ [48]. Let

$$q = a_0 + a_1 i + a_2 j + a_3 k + a_4 il + a_5 jl + a_6 kl + a_7 l \tag{51}$$

be in $\mathbb{O}$, with norm $N(a) = \overline{a}a$. A set closed under addition and multiplication, with a 1, is *integral* if (i) $2a_0$ and $N(a)$ are in $\mathbb{Z}$, (ii) it is not contained in a larger such set. Thus the Gaussian $\mathbb{Z}[i]$ are integral in $\mathbb{C}$, and for $\mathbb{H}$ it is the 24 units of (6). In $\mathbb{O}$, an element $e = q_1 + q_2 l$ is defined in terms of two quaternions q_1 and q_2 using the unit l of Table 1. Now let $t = (i + j + k + l)/2$. Coxeter [48] then defines I in terms of the 8 elements

$$1, i, j, k, t, it, jt, kt, \tag{52}$$

which close under multiplication. The 240 units include the 16 basis units, with a sign, 112 numbers of the form $(\pm 1 \pm j \pm l \pm jl)/2$ and 112 of the form $(\pm j \pm k \pm jl \pm kl)/2$ or $(\pm i \pm j \pm k \pm l)/2$.

Given I, the lattices L, R and $2B$ are defined by [35]

$$L = (1 + l)I, \quad R = I(1 + l), \quad 2B = (1 + l)I(1 + l). \tag{53}$$

Here we see clearly the actions on each lattice, which may be embedded in the X, Y and Z components of $J_3(\mathbb{O})$. These are related to Peirce decompositions [49] for a noncommutative ring, splitting idempotents. The choice of one special octonion (in this case l) is used [11][50][51] to separate leptons from quarks in the $\mathbb{C} \otimes \mathbb{O}$ ideal algebra for Standard Model particles.

In section 6 we saw the number 819 as a factor of 196560. It appears now in the integral form of $J_3(\mathbb{O})$. Observe that the primes $7 = 4 + 2 + 1$ and $13 = 9 + 3 + 1$ count the points in projective planes for $\mathbb{F}_2$ and $\mathbb{F}_3$. The so called monomial subgroup G [35] of order $3 \cdot 2^{12}$ is generated by the maps

$$\begin{pmatrix} x & A & \overline{B} \\ \overline{A} & y & C \\ B & \overline{C} & z \end{pmatrix} \mapsto \begin{pmatrix} x & eA & \overline{Be} \\ \overline{eA} & y & eCe \\ Be & \overline{eCe} & z \end{pmatrix}, \tag{54}$$

and the permutations $(x, y, z; A, B, C) \mapsto (y, z, x; B, C, A)$ and $(x, z, y; \overline{A}, \overline{C}, \overline{B})$, for any unit e. Now let $s = (1 + i + j + \cdots + l)/2$ be the vector above. The group G acts on the elements

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 2 & 1 & \overline{s} \\ 1 & 1 & \overline{s} \\ s & s & 1 \end{pmatrix}, \tag{55}$$

to give $819 = 768 + 48 + 3$ elements. The twisted finite simple group ${}^3D_4(2)$ is generated by Jordan reflections in these 819 roots, and has order $819 \cdot 63 \cdot 2^{12}$. The full automorphism group is the semidirect product ${}^3D_4(2) : C_3$ [35].

In constructing $F_4(2)$ and other finite groups, we often encounter the two dimensional discrete Fourier transform

$$F_2 \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} . \tag{56}$$

The matrix $F_2 \otimes I_2$ maps short roots to long roots. The D_4 triality automorphism T is essentially given by $F_2 \otimes F_2$ in the form

$$T \equiv -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{57}$$

We see that $T^3 = 1$, as in the lattice triality $L \to R \to B$. The columns of T are the eigenvectors of the γ_5 matrix in the Dirac representation, making T one of the five mutually unbiased bases in dimension 4. Two copies of T are used in conjunction with ϕ to project the L_8 lattice down to a four dimensional quasi-lattice [52][53], in the $\mathbf{e_8}$ approaches to gravity that were pioneered by Tony Smith [54].

There are 7 choices of domain for the integral octonions, and another 7 if we include a coassociative set. With 14 copies of $\mathbf{e_8}$, there are $10080 = 14 \cdot 720$ roots in a triple $\mathbb{O}^3$ basis for the Leech lattice. In the next few sections we will see that 10080 is also interesting as the number whose divisors characterise the primes dividing $|\mathbb{M}|$.

Codes and simplices associated to $\mathbb{OP}^2$ are studied in [44], including a design of 819 points. For $\mathbb{FP}^n$, over any division algebra, many of these codes come from mutually unbiased bases [27][28][29]. The 819 points form a distinct structure, with $819 = \sum_{i=1}^{13} i^2$ suggesting a Lorentzian vector $(0, 1, 2, \cdots, 13, 9, 30)$ in 15+1 dimensions.

Since its inception, the Monster has been studied in terms of the Parker loop [55]. We must discuss its axioms in the context of codes [62][63], because beyond the Golay code is a code on 40 elements, which we will need in the construction of the 72-dimensional lattice.

8 A 72 dimensional lattice

Monster moonshine begins with a vertex operator algebra based on Λ, and its trivalent vertices suggest looking at three copies of Λ in higher dimensional lattices. In the magic star [12][13], one copy of the Leech lattice in $J_3(\mathbb{O})$ extends to three copies around a triangle in the star. Now there exists an even unimodular lattice Φ in dimension 72 [56] whose minimal vectors have norm 8, making it an extremal lattice. It is constructed using three copies of Λ, along with a 6 dimensional lattice Θ known as the Barnes lattice.

Let $\alpha = (1 + \sqrt{-7})/2$. Θ is a subset of vectors $v = (x, y, z)$ with components in $\mathbb{Z}[\alpha]$. Define a Hermitian form on Θ by

$$f : \Theta \times \Theta \to \mathbb{Z}[\alpha], \qquad (v, w) \mapsto \frac{1}{2}\sum_{i=1}^{3} v_i \overline{w}_i. \tag{58}$$

Θ is usually the span of the vectors $(1, 1, \alpha)$, $(0, \overline{\alpha}, \overline{\alpha})$ and $(0, 0, 2)$. Its automorphism group is $C_2 \cdot PSL_2(7)$, of order $336 = 24 \cdot 14$. We find a matrix A and its conjugate $B = I - A$, so that $(\alpha, \overline{\alpha})$ maps to (A, B), defining the three rows of

$$Q = \begin{pmatrix} I & I & A \\ 0 & B & B \\ 0 & 0 & 2I \end{pmatrix}. \tag{59}$$

The matrix A is 24-dimensional over $\mathbb{Z}$ and should satisfy

$$AGA^t = 2G, \qquad GA^tG^{-1} = B, \tag{60}$$

where G is a Gram matrix for Λ. There are 9 solutions for A modulo the automorphisms of Λ [56], but one natural choice for an extremal Φ. Now Φ is the sublattice of $(\Lambda, \Lambda, \Lambda)$ defined by Q, so that over $\mathbb{Z}[\alpha]$ it is the lattice $\Theta \otimes \Lambda$. Letting Tr denote the trace on $\mathbb{Z}[\alpha]/\mathbb{Z}$, a $\mathbb{Z}[\alpha]$ structure for Λ is $\frac{1}{7}\mathrm{Tr} \cdot f$. Then Λ contributes an $SL_2(25)$ of order 15600 to the automorphisms in 36 dimensions. The number of norm 8 vectors in Φ equals

$$31635 \cdot 196560 = 2025 \cdot 9139 \cdot |C_2 \cdot PSL_2(7)|. \tag{61}$$

Note that there is a prime factor of 37 here, which does not divide the order of $\mathbb{M}$, but the mod 37 Fibonacci numbers have a length 19 cycle. The automorphisms of Φ include the semidirect product $PSL_2(7) \cdot SL_2(25) : C_2$, of order $6400 \cdot 819$, which is four times $1209600 + 100800$, where 1209600 is the order of $2 \cdot J_2$, the symmetry of the icosian Leech lattice.

Observe that $|SL_2(25)|/(9 \cdot 240)$ equals $7 + 2/9$, where we take 9 copies of the $\mathbf{e_8}$ roots in Φ. Recall that $1080 = 9 \cdot 120$ is the order of the triple cover $3 \cdot A_6$. Now $240 = 1080 \cdot (2/9)$ reminds us of the nine copies of 24 on octonion points in the magic star. The remaining $1080 \cdot 7$ will appear in section 10, suggesting a strong link between Φ and the Monster.

The matrix (59) suggests the 72×72 circulant Koide matrix

$$K = \begin{pmatrix} I & A & B \\ B & I & A \\ A & B & I \end{pmatrix}, \tag{62}$$

where (A, I, B) is a projective splitting of idempotents with $BA = 2I$, suggesting a natural rescaling of Θ by $1/\sqrt{2}$. Now consider a map $\mathbb{Z} \to \mathbb{Z}$ mod 3 taking

$$\alpha^2 - \alpha + 2 = 0 \quad \to \quad \phi^2 - \phi - 1 = 0, \tag{63}$$

where ϕ is the golden ratio. This says that mod 3 arithmetic is closely related to the appearance of ϕ in lattice coordinates and mass phenomenology.

The Barne's matrix (59) also gives three out of four orbits for minimal vectors in the icosian Λ. Recall that $h = (-\sqrt{5} + i + j + k)/2$ generates one of the five ideals of the 600-cell. Three orbits are characterised by the vectors $(2, 0, 0)$, $(0, h, h)$ and $(\overline{h}, 1, 1)$, which looks exactly like the map $\overline{\alpha} \mapsto h$, but we need to look at something different first. These vectors generate Λ when thought of as a (left) module over the icosians [35].

The fourth orbit comes from $(1, \phi u, -\phi^{-1}\overline{u})$, where $u = (-1 + i + j + k)/2$, so that the map $u \mapsto \omega$ is analagous to $s \mapsto \alpha$. Indeed, we have $u^2 = \overline{u}$ and $u^2 + u + 1 = 0$. Then $h = u - \phi^{-1}$, so the natural reduction is $h \mapsto \omega - \phi^{-1}$, which satisfies a quadratic. Note that ω, ϕ and their conjugates solve all four quadratics $x^2 \pm x \pm 1 = 0$. This summarises the $\mathbb{Z}[\alpha]$, $\mathbb{Z}[\omega]$ and $\mathbb{Z}[\phi]$ structures on Λ in dimension 12.

Now under $\overline{\alpha} \mapsto h$ for mod 3 arithmetic, we would require $\omega \mapsto 0$, which is conveniently done with $J(\omega) = 0$ for $J(z) = j(z) + 744$.

Let us recall the automorphisms for Λ. Ignoring the S_3 permutations, Wilson's $\mathbb{O}^3$ version of the automorphism group $2 \cdot \mathrm{Co}_1$ for Λ is generated by the 3×3 matrices [36]

$$-\frac{1}{2}\begin{pmatrix} 0 & \overline{\alpha} & \overline{\alpha} \\ \alpha & 1 & -1 \\ \alpha & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, \quad \frac{1}{\sqrt{2}}(1-i)I_3, \quad \frac{1}{\sqrt{2}}(1+e)I_3, \tag{64}$$

where I_3 is the identity, e is one of the six units other than 1 and i, and α appears above in the structure of Φ. Recall A and B in (59). The vector $(0, B, B)$ maps to $(0, \overline{\alpha}, \overline{\alpha})$ in the first row of the first matrix. But $(A, 1, 1)$ is replaced by $(A, 1, -1)$, which is possible in Θ with $(A, 1, 1) - (0, 0, 2)$. Thus the first matrix defines Θ for Φ.

Most of the Monster is obtained by adjoining the Conway group to the extraspecial group 2_+^{1+24}, defined using quantum information in the next section.

Our musings about modular arithmetic are motivated by the prime powers p^r of quantum Hilbert spaces, with p^r acting as a coarse graining on a cubic lattice with a discrete dimension labelled by sets of p points. Going from the (root) lattices to the exponentiated group, whatever analog of a group we might use, is a process of quantization, because a root is an element of a *set* while the same root later contributes a dimension. We saw how the 8 points of a 3-cube denote the 8 dimensions of $\mathbb{O}$. With one, two or three qutrits we are in dimensions $3, 9, 27$, which combine with three qubits to give dimensions $24, 72, 216$.

How many qudits do we need for the fundamental degrees of freedom of gravity? In dimension 72, we can put 9 copies of $\mathbb{Z}^8/2$ into a 3×3 matrix to recover complex number entries using the symplectic map. The mod 27 cycle of Fibonacci numbers has length 72, which includes the length 24 cycles that start at mod 6. These lengths are expected to govern dimensions of modules in quantum gravity.

Beyond 72 is its triple $216 = 8 \cdot 81$, which is where we would construct $J_3(\mathbb{O} \otimes \mathbb{O})$. This is just sufficient to account for all copies of $\mathbb{O}$ in the magic plane. Then we have the beautiful fact that

$$\frac{1}{24} - \frac{1}{27} = \frac{1}{216} \tag{65}$$

so that $216 = 9 \cdot 24 = 8 \cdot 27$, and $1728 = 72 \cdot 24 = 64 \cdot 27$, where the 64 covers three Dirac spinors and the 27 adds three qutrits. Note that 1728 covers 27 copies of $\mathbb{O} \otimes \mathbb{O}$ on Leech lattice points in the magic plane.

Finally we ask: does three copies of Φ get us close to the *group* $\mathbf{e_8}$, whose roots define $240 = 216 + 24$ dimensions. The idea is that the third tripling should somehow take us back to where we started. Recall that 24 roots are selected in the magic plane outside of the $\mathbb{O}$ components. Thus ten copies of Λ reduce to a vector $(\Phi, \Phi, \Phi, \Lambda)$ in dimension 240, and the Λ directions give the special 24 roots. The usual coordinates for the $\mathbf{a_2}$ hexagon in three dimensions are the permutations of XYZ, which were included as ribbon charges in the 24 qutrit words for the Z boson [11], along with the remaining 18 diagonal elements. Now we see that the 27 dimensions of bosonic M theory have little to do with classical spaces.

9 Prime triples and code loops

Qubit state spaces label vertices on a parity cube, while qutrits require cubes with midpoints on each edge, using up the coordinates $\{0, \pm 1\}$. Coarse graining a three qubit cube requires only $3 = 2^2 - 1$ points along an edge, to create eight little 3-cubes. Thus primes of the form $2^n - 1$ are the only coarse graining primes. 3 is the only prime squeezed between two prime powers. Each point on a prime power edge stands for a digit $\{0, 1, \cdots, p-1\}$ in $\mathbb{F}_p$. Digits are quantised by circulant mutually unbiased bases [27][28][29], leaving out the quantum Fourier transform. Putting a power r on every divisor n in a finite cubic lattice defines a divisor function $\sigma_r(n)$ on the target of the cubic block, so that an Eisenstein series is naturally a discrete analog of a monomial function x^r.

Three dimensions are physical. It makes sense for us to choose primes as labels for each direction in discrete space, so that three dimensions denote a prime triple (p_1, p_2, p_3). When we look at the orders of finite simple groups, we immediately think of triplets of their primes. Consider the primes dividing the order of $\mathbb{M}$, the Monster. A triple (p_1, p_2, p_3) is mapped to its polygon triple $(p_1 + 1, p_2 + 2, p_3 + 3)$, because factorization for $N \in \mathbb{N}$ occurs in chordings of a polygon with $N + 1$ sides, and there are $p + 1$ mutually unbiased bases in dimension p, including the $p \times p$ Fourier transform. Starting from the big numbers in

$$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71, \tag{66}$$

we observe that

$$72 \cdot 60 \cdot 48 = 196560 + 10800, \tag{67}$$

$$42 \cdot 32 \cdot 30 = 4 \cdot 10080,$$

$$24 \cdot 20 \cdot 18 = 8 \cdot 1080,$$
$$60 \cdot 48 \cdot 42 = 120960 = 196560 - 7 \cdot 10800,$$
$$48 \cdot 42 \cdot 32 = 64512 = 2^{16} - 2^{10},$$
$$60 \cdot 30 \cdot 24 = 4 \cdot 10800,$$

where $10080 + 720 = 10800$ occurs in the squares of the modular discriminant and related Eisenstein series. Observe that

$$72 \cdot 60 \cdot 48 = (273 + 15) \cdot 720 = 196560 + 8 \cdot 720 + 7!, \tag{68}$$

where $7!$ is the number of vertices on S_7, and $8 + 7$ is the splitting $128 + 112$ for L_8. In the last section we saw that $7 \cdot 1080$ appears in the structure of the 72 dimensional lattice, and $8 \cdot 10800 = 5 \cdot 17280$, where 17280 counts holes in the L_8 lattice. We have

$$72 \cdot 60 \cdot 48 = 3 \cdot 2^{16} + 3 \cdot 14 \cdot 2^{8}, \tag{69}$$

where the second term is $10477 + 275$ and 10477 is prime. Thinking of $\mathbb{OP}^2$, 196883 also equals $3 \cdot 2^{16} + 275$, where 275 is the dimension of the 256 spinor T-algebra at level 2, of shape

$$\begin{pmatrix} 1 & 16 & 128 \\ 16 & 1 & 128 \\ 128 & 128 & 1 \end{pmatrix}. \tag{70}$$

Higher level algebras should be studied for the lattice Φ, and we are also interested in an algebra of dimension 75, with dimension 24 in all off-diagonal spots.

The prime product $71 \cdot 59 \cdot 47 = 196883$ is the dimension of the Griess module [57][35], and $196884 = 300 + 98280 + 98304$ the dimension of the algebra, traditionally described as follows. Given two symmetric 24×24 matrices X and Y, the Griess product is $X * Y = 2(XY + YX)$, in a 300 dimensional space of matrices. In Λ, there are $98280 = 196560/2$ positive vectors v which we can use as basis vectors in dimension 98280. The remaining 98304 dimensions have basis vectors $f \otimes b$ for f a fermion spinor in dimension 4096 and b a boson in dimension 24, as occurs in the level 3 T-algebra [13] of shape

$$\begin{pmatrix} 1 & 24 & 2048 \\ 24 & 1 & 2048 \\ 2048 & 2048 & 1 \end{pmatrix}. \tag{71}$$

The action of X on b is

$$X * (f \otimes b) = f \otimes bX + \frac{1}{8}(\mathrm{Tr}X)(f \otimes b). \tag{72}$$

The Mathieu group M_{24} in $\mathbb{M}$ acts on 2048 elements of the Golay code, and on

$$4096 = 1 + 24 + 276 + 2024 + 1771, \tag{73}$$

where $1771 = 7 \cdot 11 \cdot 23$ has polytope prime dimension 48^2. The dimension 276 carries one of the rank 3 permutation representations for M_{24}, and $276 = 4 \cdot 3 \cdot 23$, with polytope dimension 480. The 2024 is required for particle mixing matrices [58], which have to be magic, requiring a space of dimension $2024 = 2048 - 24$. And $2024 = 8 \cdot 11 \cdot 23$. The Golay octads give $759 = 3 \cdot 11 \cdot 23$. Observe that these Golay prime power triples only use primes of the form $d - 1$ for $d|24$, and 24 is the largest integer for which all the $d - 1$ are prime. These divisors label Niemeier lattices in umbral moonshine [59]. In the cocode, the number of zeroes in a word belongs to the set $\{1 \cdot 23, 2 \cdot 11, 3 \cdot 7, 4 \cdot 5\}$.

The primes 3, 7, 11 and 23 all satisfy $-p \equiv 1 \bmod 4$, so that $(1 + \sqrt{-p})/2$ gives an integral ring. The norm [60] of $a + b\phi$ in $\mathbb{Q}(\sqrt{5})$ is $a^2 - b^2 + ab$, indicating that the trace of ϕ equals 1 and the norm is -1, and we have $5 \equiv 1 \bmod 4$. In the integers for $\mathbb{Q}(\sqrt{-p})$, the trace is 1 whenever $-p \equiv 1 \bmod 4$, allowing the 1/2. This extends to the usual trace and norm definitions for noncommutative and nonassociative algebras, where a trace of 1 is a projector condition for quantum states. In other words, quantum measurements naturally select integers from a quadratic field, which is real or imaginary depending on $p \bmod 4$, starting with $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{5})$, drawing hexagons and pentagons for our axioms. Since such primes often denote the diagonal length on a cube, we include $\sqrt{-71}$ for Φ. There is no unit trace for the Gaussian integers, and for negative discriminants $\mathbb{Q}(\sqrt{-p})$ is contained in $\mathbb{Q}(\omega_p, i)$ for ω_p the p-th root. In our ribbon diagrams, this distinguishes a B_2 twist from ribbon braids in B_n.

Adding up the number of words in the code and cocode, omitting the zero vectors, we obtain

$$2 \cdot 4095 = 8190, \tag{74}$$

where $8190 \cdot 24 = 196560$ in Λ. Congruence classes of norm 8 vectors in Λ come in sets of 48, called *crosses*, such that the stabiliser of a cross in $2 \cdot \mathrm{Co}_1 = \mathrm{Aut}(\Lambda)$ is the semidirect product $2^{12} : M_{24}$. There are $48 \cdot 45^3 \cdot 7 \cdot 13$ norm 8 vectors.

We are interested in n dimensional codes in $\mathbb{F}^{2n}$, known as $[2n, n]$ codes. The Golay code over $\mathbb{F}_2$ is a $[24, 12]$ code and easily defined using the hexacode over $\mathbb{F}_4$, which has a standard generator matrix given by the 3×3 Fourier transform F_3 so that $\mathbb{F}_4$ is given by $\{0, 1, \omega, \overline{\omega}\}$.

Observe the interesting ring homomorphisms that enter here. Let $R = \mathbb{Z}/3\mathbb{Z}$. If we start with $\mathbb{Z}[\alpha]$, taking mod 3 introduces $R[\phi]$, the 9 element ring using the values (27). In $R[\phi]$ with ϕ indeterminate, we recover formal Lucas numbers $L_n = \phi^n + (-1/\phi)^n$, but the Fibonacci numbers satisfy

$$5F_n = 2L_{n+1} - L_n, \tag{75}$$

requiring a fractional ideal in $\mathbb{Z}[\phi]$. A projective representation of A_5 over $R[\phi]$ is given by the Hecke group generators [61]

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & -1 \\ 1 & \phi \end{pmatrix}, \tag{76}$$

with $H^5 = I$ and $(SH)^3 = I \bmod 3$. In the full algebra, a principal ideal is generated by $5 - 2\phi$ and its quotient gives $PSL_2(11)$.

Recall that a binary codeword is a subset of a given set, and intersection of sets is a product in the Boolean ring. Griess suggested [62] defining Moufang loops in terms of doubly even codes, thinking of the Parker loop for $\mathbb{M}$, which doubles the Golay code using signs. The operations on a code loop are [63] (i) power $P(u) = \frac{1}{4}|u| \bmod 2$, (ii) commutator $C(u,v) = \frac{1}{2}|u \cap v| \bmod 2$ and (iii) associator $A(u,v,w) = |u \cap v \cap w| \bmod 2$, for u,v,w in a code $\mathcal{C}$. In the Parker loop, this gives $[u,v] = (-1)^{|u \cap v|/2}$ and so on. The Parker loop carries a form of triality. For u,v,w in the loop (rather than the code) and a any even subset of the 24 element set, define the three flip maps [55]

$$x_a(u,v,w) = ((-1)^{|u\cap a|}u^{-1}, (-1)^{|w\cap a|}w^{-1}, (-1)^{v\cap a}v^{-1}), \tag{77}$$

$$y_a(u,v,w) = ((-1)^{|w\cap a|}w^{-1}, (-1)^{|v\cap a|}v^{-1}, (-1)^{|u\cap a|}u^{-1}),$$
$$z_a(u,v,w) = ((-1)^{|v\cap a|}v^{-1}, (-1)^{|u\cap a|}u^{-1}, (-1)^{|w\cap a|}w^{-1}).$$

These satisfy $x_a y_a = y_a z_a = z_a x_a$, which send (u,v,w) to (v,w,u). The triality maps are

$$X_a(u,v,w) = (\overline{a}u\overline{a}, av, wa), \quad Y_a(u,v,w) = (ua, \overline{a}v\overline{a}, aw), \tag{78}$$

$$Z_a(u,v,w) = (au, va, \overline{a}w\overline{a}).$$

Conway used these maps, along with M_{24}, to generate the Griess maximal subgroup of $\mathbb{M}$.

The Monster requires the extraspecial 2-group 2_+^{1+24}, a cover of which is generated by the X_a and x_a. The quantum extraspecial groups use the generators of (19). Here we have $E_1, E_2, \cdots E_{11}$ on $V^{\otimes 12}$ and also the dihedral generator (20). Let

$$R_2 = -I \otimes R \otimes R^2 = I \otimes R \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{79}$$

and $R_1 = -R^2 \otimes R \otimes I$. These satisy the tetrahedron equation

$$D \equiv R_1 R_2 R_1 R_2 = R_2 R_1 R_2 R_1 \tag{80}$$

and $D^4 = I$, which is fitting for the symmetry of a square. This shows that moonshine goes beyond ordinary braids to the composition of higher dimensional arrows and knotted surfaces.

In the next section we see that we should go beyond the Golay code to a subspace of 2^{40}, in order to construct Φ. Encouraged by the relation between the numbers $d-1$ for $d|24$ in the primes of M_{24} moonshine, we observe that the primes in $|\mathbb{M}|$ are exactly of the form $d-1$ for $d \leq 72$ and $d|10080$.

10 Spinors for a Monster

It seems clear that an information theoretic approach to gravity brings clarity to the physical meaning of CFSG and moonshine coincidences. In quantum gravity, the observer measures, and the observer's frame of mind depends on cosmological parameters. BTZ black holes are directly relevant to gravity in a holographic approach, where we restrict our attention to true two dimensional CFTs. The j-invariant appears in Witten's Monster CFT [6] at $c = 24$.

Combining a Dirac 4-dit and qutrit in three dimensions gives a 1728 vertex cube, and the normalisation for the j-invariant. The number 1728 also counts the 27 copies of $\mathbb{O} \otimes \mathbb{O}$ that might be assigned to the 27 off-diagonal entries on 3×3 matrices on the magic star [13].

Let us write the first few terms of (31) as

$$\begin{aligned} 196560 &= 240 \cdot 819 \\ 16773120 &= 5 \cdot 819 \cdot 2^{12} \\ 398034000 &= 25 \cdot 81 \cdot 240 \cdot 819 \\ 4629381120 &= 5 \cdot 276 \cdot 819 \cdot 2^{12} \\ 34417656000 &= 25 \cdot 7004 \cdot 240 \cdot 819 \\ 187489935360 &= 5 \cdot 11178 \cdot 819 \cdot 2^{12} \\ 814879774800 &= 5 \cdot 829141 \cdot 240 \cdot 819 \\ 2975551488000 &= 5 \cdot 177400 \cdot 819 \cdot 2^{12}, \end{aligned} \tag{81}$$

where 276, 7004, 11178, 829141 and 177400 all have three factors. The first coefficient of the j-invariant (23) is

$$c(1) = 196884 = 196560 + 324 = 3 \cdot 2^{16} + 276, \tag{82}$$

recalling the 15-spheres for $\mathbb{OP}^2$. Frenkel's study of the Kac-Moody algebra associated to the Lorentzian Leech lattice [64] gives a bound of 324 on multiplicities of roots of norm -2, where 324 is $p_{24}(2)$, the number of partitions in 24 colors of 2. It includes $276 = \binom{24}{2}$ when the colors are distinct.

The second coefficient of j is

$$c(2) = 21493760 = 5 \cdot 819 \cdot 2^{11} + 25 \cdot 2^{19}, \tag{83}$$

noting the appearance of the norm 6 number 16773120/2 from f_Λ, along with the 2^{19} spinor in the T-algebra of shape

$$\begin{pmatrix} 1 & 40 & 2^{19} \\ 40 & 1 & 2^{19} \\ 2^{19} & 2^{19} & 1 \end{pmatrix} \tag{84}$$

at level 5, which is two levels beyond the Leech lattice. The dimension of this algebra is $11 \cdot 13 \cdot 7333$, introducing new primes. The 2^{19} will be associated to a code on a 40 element set, just as the Golay code uses 24 elements.

At level 6 the spinor dimension becomes $2^{32} = 65536^2$, and for the 72 dimensional lattice Φ we go up to level 9, with $2^{72} = (2^{24})^3$. The number 2^{24} counts the elements in the power set on a 24 element set. Now observe that

$$2^{24} - 16773120 = 2^{12} = 4096, \tag{85}$$

recovering the spinor dimension at level 3. Then

$$(2^{24})^2 = (2^{24} + 2^{13}) \cdot 4095^2 + 4096 \cdot (3 \cdot 4095 + 1), \tag{86}$$

making heavy use of the $4095 = 196560/48 = 5 \cdot 819$, which counted words in the Golay code. Taking three copies of Λ for Φ, we see that

$$3 \cdot 196560 = 3^{14} - 3^2 - 2^{22} + 2^{10} \tag{87}$$

maintains a power difference of 12 on threes and twos. Differences in powers of 2 of the form

$$D_n(\delta) = 2^{n+\delta} - 2^n \tag{88}$$

obviously have a fixed set of prime factors. For example, at $\delta = 20$ we have the primes $2, 3, 5, 11, 31, 41$. All primes for $\delta \in \{4, 12, 20\}$ (associated to levels 1, 3 and 5) divide $|\mathbb{M}|$. The T-algebra bosonic components at these levels sum to

$$72 = 8 + 24 + 40, \tag{89}$$

justifying (83). The 24 element Golay set generalises to self-dual codes on sets of size $8(2n + 1)$ [1]. Dimension 8 has the Hamming code, dimension 24 the Golay code, and dimension 40 a new code. Each such code defines an even, unimodular lattice.

We are also interested in ternary codes, such as those used for $\mathbf{e_8}$. Under the symplectic Penrose map (3) into $\mathbb{C}$, nine copies of L_8 make $SL_3(\mathbb{C})$. Four copies make $SL_2(\mathbb{C})$. The primes 2 and 3 underlie a great deal indeed, just as Francis Brown discovered with multiple zeta values, which extend the integral arguments of the Riemann zeta function to noncommutative partitions. Taking binary and ternary rooted trees restricts the valency of nodes to 4, as in ϕ^4 quantum field theory.

In the next section we show that the arguments 2 and 3 correspond to segments of a Fibonacci chain. In the full quantum theory, we focus on matrix algebras in dimensions 2 and 3, or higher. The dimensions F_n give Fibonacci braid group representations for B_n, and the group $SU(F_n - 1)$ [65], with $F_4 = 3$ giving $SU(3)$. Quark structure uses this B_4, while electric charge is fully accounted for by B_3.

11 A vertex operator algebra

Binary rooted planar trees label the vertices of the associahedron [38][39], and trees with an ordering on nodes label permutations in S_n, where $n + 1$ is the number of leaves on the binary tree. In a VOA [1], trees become string diagrams, or Riemann spheres with punctures. The real points of the moduli space of Riemann spheres are tiled by the associahedra. What about the complex structure? An affine algebra has basis elements like $z^n \otimes E_i$, for z^n in the Laurent series on the loop and E_i a typical Lie algebra generator. But like in everything else, we want to work with higher dimensional categories and their polytopes, rather than circles per se. We cannot begin with the Virasoro algebra unless we work with the full division algebras.

At the simplest level a VOA combines a kind of universal algebra with a Laurent series in a circle parameter, but for us the natural complex parameter is a deformation parameter for the braiding. Just as there were two angles for Fibonacci braids and quaternions, we see two nice angles in dimension 8 [22]. Given a dual Coxeter number $h^\vee$, an appropriate deformation parameter for the representation category is $\exp(\pi i/(h^\vee + 1))$.

But we want to focus on 5-th roots of unity. The ribbon category of the Fibonacci anyon [66][67] is universal [68] for quantum computation [65], and the fusion map for two anyons gives an associator arrow on the pentagon.

Let $F(abcd)^y_x$ be a fusion coefficient for an internal edge y on the input tree and internal edge x in the set of allowed trees, with d labelling the root of a three leaved tree. Our anyon objects are 1 and τ, such that $\tau \circ \tau = 1 + \tau$. Following [66], the interesting coefficients satisfy the pentagon relation

$$\begin{aligned} F(\tau\tau c\tau)^d_a F(a\tau\tau\tau)^c_b = {} & F(\tau\tau\tau d)^c_1 F(\tau 1 \tau\tau)^d_b F(\tau\tau\tau b)^1_a \\ & + F(\tau\tau\tau\tau)^c_\tau F(\tau\tau\tau\tau)^d_b F(\tau\tau\tau b)^\tau_a. \end{aligned} \tag{90}$$

When $(abcd)$ contains a 1, the coefficients are 0 or 1. At $(abcd) = (\tau 11\tau)$, we obtain $F(\tau\tau\tau\tau)^1_1 = (F(\tau\tau\tau\tau)^1_\tau)^2$. From $(abcd) = (1\tau 1\tau)$ it follows that $F(\tau\tau\tau\tau)^1_1 = -F(\tau\tau\tau\tau)^1_1$. Let $F(\tau\tau\tau\tau)^1_1 = -A$ and $F(\tau\tau\tau\tau)^1_\tau = i\sqrt{A}$. Then $(abcd) = (\tau\tau\tau\tau)$ gives $A^2 - A - 1 = 0$ with solution $A = -1/\phi$. In summary, the all-τ coefficients are

$$\begin{pmatrix} F^1_1 & F^\tau_1 \\ F^1_\tau & F^\tau_\tau \end{pmatrix} = \begin{pmatrix} \frac{1}{\phi} & \frac{i}{\sqrt{\phi}} \\ \frac{i}{\sqrt{\phi}} & \frac{-1}{\phi} \end{pmatrix}. \tag{91}$$

These appear in the B_3 representation

$$\sigma_1 = \begin{pmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \frac{e^{4\pi i/5}}{\phi} & \frac{e^{-3\pi i/5}}{\sqrt{\phi}} \\ \frac{e^{-3\pi i/5}}{\sqrt{\phi}} & \frac{-1}{\phi} \end{pmatrix}, \tag{92}$$

with phases from the hexagon rule.

Consider the number of fusion diagrams on d leaves when all inputs are set to τ and the bracketing is nested to the left. That is, separate out the associator trees from lists of possible labels on internal edges. This effectively labels a corolla with d leaves with the word in 1 and τ attached to the tree. Or, instead of the corolla, we could draw a path of d edges to match the number of letters in a word, making the word into a kind of Fibonacci sequence.

We write words in 1 and τ by following the internal edges from a leaf down to the root. Since all words start with τ, we often omit this letter, leaving words of length $d-1$. For three leaves the words are $\tau 1$, 1τ and $\tau\tau$, counted by the Fibonacci number F_{d+1}. Such words mark the vertices of a parity cube whose dimension equals the number of τ letters, so that a + marks the placement of a

1 in the word. The number F_{d+1} is graded across cubes of different dimension,

$$F_{d+1} = \sum_{n=0}^{f(d/2)} \binom{d-n}{n}, \tag{93}$$

where $f(i)$ is the integer part.

Let us now recall the connection [69][70] between knots and multiple zeta values [71]. From our perspective, the appearance of the golden ratio *limit* in a fusion map is reminiscent of the Drinfeld associator, with its infinite series of multiple zeta values. In the iterated integral form, a zeta argument is a word in two letters such that one letter only occurs as a singlet, much like the 1 in our fusion words. A multiple zeta value (MZV) is the unsigned case of the signed Euler sum

$$\zeta(n_1, n_2, n_3, \cdots, n_l; \sigma_1, \cdots, \sigma_l) = \sum_{k_i > k_{i+1} > 0} \frac{\sigma_1^{k_1} \sigma_2^{k_2} \cdots \sigma_l^{k_l}}{k_1^{n_1} k_2^{n_2} \cdots k_l^{n_l}} \tag{94}$$

of *depth* l and *weight* $n = \sum_i n_i$, with $\sigma_i \in \pm 1$. Recall that the Mobius function $\mu(n)$ on $\mathbb{N}$ is zero on non square free n and $(-1)^r$ for r prime factors. The square free $n \in \mathbb{N}$ are the targets of parity cubes. An MZV is irreducible if not expressed as a $\mathbb{Q}$ linear combination of other MZVs of the same weight. The number E_n of irreducible signed Euler sums of weight n is [69][70]

$$E_n = \frac{1}{n} \sum_{D|n} \mu(n/D) L_D = \frac{1}{n} \sum_{D|n} \mu(n/D)(F_{D-1} + F_{D-3}), \tag{95}$$

where L_D is the Lucas number. The number M_n of irreducible MZVs of weight n is the number of knots with n positive crossings (and no negative crossings). It's value replaces L_D by P_D, the Perrin number, satisfying the recursion

$$P_D = P_{D-2} + P_{D-3} \tag{96}$$

for $P_1 = 0$, $P_1 = 2$ and $P_3 = 3$.

An argument $(n_1, \cdots, n_l)$ of an MZV, such that only n_l may equal 1, is expressed as a word in two letters A and B, such that all words start with A and end with B, and B only occurs as a singleton. First reduce the argument to the ordinals $(n_1 - 1, n_2 - 1, \cdots, n_l - 1)$. The corresponding word is $A^{n_1-1} B A^{n_2-1} B \cdots A^{n_l-1} B$. Each copy of A is assigned the form dz/z and each B the form $dz/(1-z)$ in the iterated integral expression for the MZV. For example

$$\zeta(3,1) = \int_0^1 \int_0^1 \int_0^1 \int_{0, z_4 > \cdots > z_1}^1 \frac{dz_1}{z_1} \frac{dz_2}{z_2} \frac{dz_3}{1-z_3} \frac{dz_4}{1-z_4}. \tag{97}$$

A *Fibonacci word* belongs to the sequence B, A, AB, ABA, $\cdots$, where each word is the concatenation of the previous two. A general Fibonacci chain takes the form $ABAABA \cdots$, where only A or AA occurs in between the single instances of B. These are then the MZVs with arguments taking the values only

2 and 3, here representing the breeding of Adult and Baby rabbits [24]. Recall that Brown's theorem shows that all MZVs are expressed as combinations of those with arguments 2 and 3.

More general arguments correspond to fusion words with arbitrary strings of τ, such that 1 still occurs as a singleton. Consider putting the τ at the start of every fusion word. Then add a formal 1 at the end of every allowed word, to obtain precisely the set of MZV words. This extra 1 adds a bigon piece to the root edge of the polygon that is being chorded by a dual tree. The length of the internal word is essentially the weight,

$$\sum n_i - 2 = n - 2 = d - 1, \tag{98}$$

where d is the number of leaves on the fusion tree. Thus the weight $d+1$ is associated to braids in the category on d strands, but fewer than d strands may be used to draw a knot.

An example of a positive knot with n crossings and $n-1$ strands is the trefoil knot σ_1^3 in B_2. It corresponds to $\zeta(3)$, from the internal word τ on two leaves. The word 1 on two leaves gives $\zeta(2,1)$. Other torus knots of type $(2k+1,2)$ define the zeta values $\zeta(2k+1)$ [69].

Since we have separated out the associators, fusion words label a corolla tree with d leaves, which is a building block for symmetric trees in renormalization Hopf algebras. By restricting to the F_d words that end in τ, we ensure that the grafting of little corollas onto other trees is always possible. This gives the F_{D-1} term in (95). F_{D-3} counts the number of internal words ending in $\tau 1$ and starting with τ. Thus L_D only excludes words that begin and end with 1, the so called vacuum words. Values of M_n correspond to full words with even clusters of τ letters, corresponding to odd arguments for MZVs, as proved in [72].

Recall the shuffle algebra for MZVs. The shuffle unit is the empty letter. The recursion law on A and B words is

$$\begin{aligned} l_1 l_2 \cdots l_u \cup k_1 k_2 \cdots k_v = & \, l_1(l_2 \cdots l_u \cup k_1 \cdots k_v) \\ & + k_1(l_1 \cdots l_u \cup k_2 \cdots k_v), \end{aligned} \tag{99}$$

so that the minimum zeta shuffle is

$$\zeta(2) \cup \zeta(2) = AB \cup AB = 2ABAB + 4AABB = 2\zeta(2,2) + 4\zeta(3,1). \tag{100}$$

Since this is a weight 4 rule, the trivalent vertex for $\zeta(3)$ comes from non MZV words. In particular, $\tau \cup \tau 1$ gives $2\tau\tau 1 + \tau 1 \tau$, which is $2\zeta(3)$ plus the word $\tau 1 \tau$. The fusion vertex $\tau \circ \tau$ corresponds to $\tau\tau 1 + \tau 11$, giving also $2\zeta(3)$. Thus a trivalent fusion graph resembles the dual of the Tutte graph [73] for the trefoil knot. For our fusion letter 1 we have $1 \cup 1 = 2 \cdot 11$ and $1^{\cup n} = 2^n \cdot 1 \cdots 1$. Note that only 1/4 of the vertices on a parity cube are in the MZV algebra.

A Fibonacci chain in one dimension may be obtained from a cubic lattice in two dimensions by the projection of a slice taken at an angle with tangent $1/\phi$. A finite such sequence that repeats corresponds to a rational approximation F_{n+1}/F_n to ϕ. In any dimension D, a nice quasilattice comes from a $2D$ cubic lattice and a projection of a slice based on angles related to ϕ. In other words, the Fibonacci chain is the one dimensional template for quasiperiodic orders in higher dimensions. In two dimensions, we have Penrose tilings. In four dimensions, one can obtain the Elser-Sloane quasicrystal from the $\mathbf{e_8}$ lattice using triality, as noted above. In three dimensions, an icosahedral cell comes from the 6-cube. Halving dimensions is familiar from the Langlands program. Our Chern-Simons theories bound four dimensional theories, just as their knot strands bound surfaces.

It is well known how to define CFTs using categorical ribbons. With trees and braids, including categories beyond the Fibonacci anyon, we have the ingredients for alternative VOAs. The central axiom should be, as usual [1], a broken Jacobi rule. Recall that for a Lie algebra, the commutators in the Jacobi rule come from the boundary of an r operator in the classical Yang-Baxter equation [74]

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{101}$$

which quantises to the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{102}$$

where $R \in \mathcal{A} \otimes \mathcal{A}$ is the R-matrix for the quasitriangular Hopf algebra [75]. A braid group representation for B_n assigns R to $V_i \otimes V_{i+1}$ in a string of n copies of V. This is generalised to an R acting on $V^{\otimes 3}$ by Kitaev and Wang [76], using special objects in fusion categories. Our Fibonacci object τ almost satisfies this criterion, except that V has a quantum dimension of ϕ, which solves the familiar quadratic

$$d_\tau^2 = d_\tau + d_1 \tag{103}$$

following from the fusion rule. In other words, we can think of the Fibonacci representations for B_n as giving an R-matrix on some fractal representation space, with a classical limit defining a Jacobi rule for the ambient Lie algebra. The dimension ϕ permits a neat estimate of the fine structure constant as a Hopf link invariant. From this perspective, broken Jacobi rules for T-algebras [13] are to be expected.

12 Mass and $E_6(p^r)$

In conclusion, as a taste for what follows, given the importance of the Lie group E_6 in the traditional Higgs mechanism, we note that $259200 = (8+7) \cdot 17280$. Here 259200 is the order of a simple group whose automorphisms are the Weyl group for E_6. It equals $360 \cdot 6!$, which is traditionally the number of seconds in three days, or years in ten Earth precession cycles. Time is dimensionless when measured as a ratio.

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