Fractional Calculus

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Abstract

This paper generalises the limit definitions of calculus to define 'differintegrals' of order $\alpha \in \mathbb{C}$, calculates some differintegrals of elementary functions, and introduces the notion of a fractional differential equation. An application to quantum theory is explored, and we conclude with some operator algebra. Functions in this paper will only have one variable.

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1 Differintegrals of integer order

The n^{th} derivative of a function makes intuitive sense when $n \in \mathbb{N}$. What about

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha} \tag{1}$$

where $\alpha \in \mathbb{Q}$? What does it mean to half-differentiate a function? The half-derivative operator $\mathrm{D}^{1/2} = \frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}$ is the operator which, when applied twice to a differentiable function f, differentiates f one time. If we let D be the derivative operator, then

$$\mathrm{D}^{1/2}\mathrm{D}^{1/2}f(t) = \frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}\frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}f(t) = \frac{\mathrm{d}f}{\mathrm{d}t} = \mathrm{D}f(t) \tag{2}$$

The use of the derivative operator $\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}$ as a mathematical object was explored in its infancy by Arbogast in [1]. Operator methods for differential equations, which we will introduce in section 5, were investigated throughout the nineteenth century and fully developed by Heaviside in [2]. To see how useful operators are, consider the following three ways of writing the same linear ordinary differential equation.

$$a_n y^{(n)} + \ldots + a_2 y'' + a_1 y' + a_0 y = f \tag{3}$$

$$a_n(t)\frac{\mathrm{d}^n y}{\mathrm{d}t^n}(t) + \ldots + a_2(t)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}(t) + a_1(t)\frac{\mathrm{d}y}{\mathrm{d}t}(t) + a_0(t)y(t) = f(t) \tag{4}$$

$$a_n \mathrm{D}^n y + \ldots + a_2 \mathrm{D}^2 y + a_1 \mathrm{D} y + a_0 y = f \tag{5}$$

Equation (5) can be written as $(a_n\mathrm{D}^n + \ldots + a_2\mathrm{D}^2 + a_1\mathrm{D} + a_0)y = f$, where the operators are factorised. This allows algebraic techniques to be used when solving differential equations. The derivative of $y(t)$ of order $n \in \mathbb{N}$, with respect to t, is

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} = \mathrm{D}_t^n y \tag{6}$$

or simply $\mathrm{D}^n y$ when the variable being used is clear. Moreover, $\mathrm{D}^1 y = \mathrm{D}y$ and $\mathrm{D}^0 y = y$. For partial derivatives, we may use the notation $\partial_t^n y$ or $\partial^n y$. An integral may be thought of as a derivative of order -1. We will unify derivatives and integrals into a single class of operators called **differintegrals**. When we extend the definition to include differintegrals of order $\alpha \in \mathbb{C}$, we will need to define the gamma function and the beta function as in [3] and [4] respectively.

Definition 1.1. For all $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, the **gamma function** is defined by the convergent integral

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, \mathrm{d}t \tag{7}$$

By analytic continuation, this may be extended to all $\alpha \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

Definition 1.2. For all $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$, the **beta function** is defined by the convergent integral

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \, \mathrm{d}t \tag{8}$$

The gamma function, whose domain is almost all of $\mathbb{C}$, is related to the factorial function by $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. This relationship can be seen by the two equations $n! = n(n-1)!$ and $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, the latter of which arises when (7) is integrated by parts. For now we will simply state the following formula from [5].

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \tag{9}$$

Definition 1.3. Let ${}_a\mathrm{I}_t$, or simply I_t or just I, represent the integral from a to t

$$ {}_a\mathrm{I}_t f(t) = \int_a^t f(\tau)\ \mathrm{d}\tau \tag{10} $$

where $f(t)$ is an integrable function and τ is a dummy variable. For a repeated integral, let ${}_a\mathrm{I}_t^n$, or simply I_t^n or just I^n, represent n repeated integrals from a to t

$$ {}_a\mathrm{I}_t^n f(t) = \int_a^t \int_a^{\tau_n} \cdots \int_a^{\tau_2} f(\tau_1)\ \mathrm{d}\tau_1 \mathrm{d}\tau_2 \cdots \mathrm{d}\tau_n \tag{11} $$

where $f(t)$ is an n-times integrable function and each τ_i is a dummy variable.

This repeated integral can be simplified with the following result due to Cauchy [6].

Lemma 1.4 (Cauchy's Repeated Integral Formula)**.** *The n^{th} repeated integral of an n-times integrable function $f(t)$ where $n \in \mathbb{Z}^+$ from a to t is given by*

$$ {}_a\mathrm{I}_t^n f(t) = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau)\ \mathrm{d}\tau \tag{12} $$

Proof. We use induction on n. Consider the base case where $n = 1$.

$$ \frac{1}{(1-1)!} \int_a^t (t-\tau)^{1-1} f(\tau)\ \mathrm{d}\tau = \int_a^t f(\tau)\ \mathrm{d}\tau = {}_a\mathrm{I}_t^1 f(t) $$

Assume that (12) is true for $n = k \in \mathbb{Z}^+$. Now consider the case where $n = k + 1$.

$$ \begin{aligned} {}_a\mathrm{I}_t^{k+1} f(t) &= {}_a\mathrm{I}_t \left[\frac{1}{(k-1)!} \int_a^\sigma (\sigma-\tau)^{k-1} f(\tau)\ \mathrm{d}\tau \right] \\ &= \frac{1}{(k-1)!} \int_a^t \left[\int_a^\sigma (\sigma-\tau)^{k-1} f(\tau)\ \mathrm{d}\tau \right] \mathrm{d}\sigma \\ &= \frac{1}{(k-1)!} \int_a^t \left[\int_\tau^t (\sigma-\tau)^{k-1}\ \mathrm{d}\sigma \right] f(\tau)\ \mathrm{d}\tau \\ &= \frac{1}{(k-1)!} \int_a^t \left[\frac{1}{k}(\sigma-\tau)^k \right]_{\sigma=\tau}^{\sigma=t} f(\tau)\ \mathrm{d}\tau \\ &= \frac{1}{(k-1)! \cdot k} \int_a^t \left[(t-\tau)^k - (\tau-\tau)^k \right] f(\tau)\ \mathrm{d}\tau \\ &= \frac{1}{k!} \int_a^t (t-\tau)^k f(\tau)\ \mathrm{d}\tau \end{aligned} $$

The order of integration was swapped in the third equality which caused the limits of integration to change. Our result therefore holds for all $n \in \mathbb{Z}^+$. □

The factorial $(n-1)!$ will later be replaced with $\Gamma(\alpha)$ in section 2 when we generalise I_t^n to I_t^α where $\alpha \in \mathbb{C}$. For now we will look at the differintegral operator D with integer order. When n is positive, D^n represents the n^{th} derivative, and when n is negative it represents the n^{th} repeated integral.

For positive n and m, we have the basic 'index law' $\mathrm{D}^n\,\mathrm{D}^m = \mathrm{D}^{n+m}$ from basic calculus. In order to find an analogous index law for repeated integrals it is necessary to define the n^{th} **derivative** ${}_a\mathrm{D}_t^n$ **evaluated from basepoint** $t = a$. This is different to the usual notion of a derivative being evaluated from the basepoint $t = 0$, meaning that the graph has not been translated along the t-axis. In operator notation we will write D_t^n instead of ${}_0\mathrm{D}_t^n$ to avoid clutter.

Definition 1.5. The n^{th} derivative of $f(t)$ evaluated from basepoint a is

$$ {}_a\mathrm{D}_t^n f(t) = \frac{\mathrm{d}^n f}{\mathrm{d}(t-a)^n} \tag{13}$$

Later on, we will use (13) to compare ${}_a\mathrm{I}_t^n$ and ${}_a\mathrm{D}_t^n$. The evaluation of a derivative from basepoint a will resemble the evaluation of an integral with a lower limit of integration a. The following lemma and proof is found in [7].

Lemma 1.6. *Let $n \in \mathbb{Z}^+$ and let $f(t)$ be an n-times differentiable function. Then the derivative of $f(t)$ of order n, with respect to t, is given by*

$$\mathrm{D}_t^n f(t) = \lim_{h\to 0}\left[\frac{1}{h^n}\sum_{i=0}^{n}(-1)^i\binom{n}{i} f(t-ih)\right] \tag{14}$$

Proof. We use induction on n. When $n = 1$, we see that

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \mathrm{D}_n^1 f(t) = \lim_{h\to 0}\left[\frac{f(t)-f(t-h)}{h}\right] \tag{15}$$

which is the limit definition for the first derivative of $f(t)$. We assume that (14) is true for n. For $n+1$, since h and h' converge to zero simultaneously, we have

$$\begin{aligned}
\mathrm{D}_t^{n+1} f(t) &= \mathrm{D}_t\mathrm{D}_t^n f(t) = \lim_{h'\to 0}\left[\frac{\mathrm{D}_t^n f(t) - \mathrm{D}_t^n f(t-h')}{h'}\right] \\
&= \lim_{h'\to 0}\left[\frac{\lim\limits_{h\to 0}\left(\sum\limits_{i=0}^{n}\frac{(-1)^i}{h^n}\binom{n}{i} f(t-ih)\right) - \lim\limits_{h\to 0}\left(\sum\limits_{i=0}^{n}\frac{(-1)^i}{h^n}\binom{n}{i} f(t-ih-h')\right)}{h'}\right] \\
&= \lim_{h\to 0}\left[\frac{1}{h^{n+1}}\left(\sum_{i=0}^{n}(-1)^i\binom{n}{i} f(t-ih) - \sum_{i=0}^{n}(-1)^i\binom{n}{i} f(t-(i+1)h)\right)\right] \\
&= \lim_{h\to 0}\left[\frac{1}{h^{n+1}}\sum_{i=0}^{n+1}(-1)^i\left[\binom{n}{i} + \binom{n}{i-1}\right] f(t-ih)\right] \\
&= \lim_{h\to 0}\left[\frac{1}{h^{n+1}}\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i} f(t-ih)\right]
\end{aligned}$$

since $\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$ for all $n, i \in \mathbb{Z}$ where $0 \le i \le n$. Therefore (14) holds and the lemma is true for all $n \in \mathbb{Z}^+$. $\square$

We say that $f(t)$ is **n-times differintegrable** when the differintegral of order n exists for the function $f(t)$. If $n = 4$, this means that the fourth *derivative* of $f(t)$ exists. If $n = -19$, this means that the nineteenth *repeated integral* of $f(t)$ exists.

The fact that differentiation and integration are inverse operators suggests, in operator notation, that ${}_a\mathrm{D}_t^{-1} = {}_a\mathrm{I}_t$ and ${}_a\mathrm{I}_t^{-1} = {}_a\mathrm{D}_t$. The following theorem, proven in [8], expands our notion of D_t^n and provides a limit definition for the differintegral ${}_a\mathrm{D}_t^n$ for any integer order $n \in \mathbb{Z}$, with respect to t, from basepoint $a \in \mathbb{R}$.

Theorem 1.7. *Let $f(t)$ be an n-times differintegrable function where $n \in \mathbb{Z}$. Then the differintegral of $f(t)$ of order n, with respect to t, from basepoint a, is defined by*

$$ {}_a\mathrm{D}_t^n f(t) = \lim_{N\to\infty} \left[\frac{1}{\Gamma(-n)} \left(\frac{t-a}{N}\right)^{-n} \sum_{i=0}^{n} \frac{\Gamma(i-n)}{\Gamma(i+1)} f\left(t - i\left(\frac{t-a}{N}\right)\right)\right] \tag{16} $$

Proof. Equation (14) resembles equation (16). To generalise (14) to differintegrals of order $n \in \mathbb{Z}$, consider the repeated integral ${}_a\mathrm{I}_t^n$ with lower limit a. Suppose that $a < t$ and partition the interval $[a, t]$ into N smaller intervals of equal width. Define

$$ h = \frac{t-a}{N} $$

to be the width of one of the N smaller intervals. Consider the updated expression

$$ {}_a\mathrm{D}_t^n f(t) = \lim_{N\to\infty} \left[\left(\frac{t-a}{N}\right)^{-n} \sum_{i=0}^{N-1} (-1)^i \binom{n}{i} f\left(t - i\left(\frac{t-a}{N}\right)\right)\right] \tag{17} $$

for the n^{th} derivative, which holds since $\binom{n}{i} = 0$ for all $i > n$. Now consider the integral of $f(t)$ from a to t, obtained by summing up areas $\delta A = f(t) \cdot h$, given by

$$ \begin{aligned} {}_a\mathrm{I}_t f(t) &= \int_a^t f(\tau)\, \mathrm{d}\tau \\ &= \lim_{h\to 0} \big[h\,(f(t) + f(t-h) + f(t-2h) + \cdots + f(a+h))\big] \\ &= \lim_{N\to\infty} \left[\left(\frac{t-a}{N}\right) \sum_{i=0}^{N-1} f\left(t - i\left(\frac{t-a}{N}\right)\right)\right] \end{aligned} \tag{18} $$

Similarly, the double integral is given by

$$ \begin{aligned} {}_a\mathrm{I}_t^2 f(t) &= \int_a^t \int_a^{\tau_2} f(\tau_1)\, \mathrm{d}\tau_1 \mathrm{d}\tau_2 \\ &= \lim_{h\to 0} \big[h^2\,(f(t) + 2f(t-h) + 3f(t-2h) + \cdots + Nf(a+h))\big] \\ &= \lim_{N\to\infty} \left[\left(\frac{t-a}{N}\right)^2 \sum_{i=0}^{N-1} (i+1) f\left(t - i\left(\frac{t-a}{N}\right)\right)\right] \end{aligned} \tag{19} $$

and the triple integral is given by

$$
\begin{aligned}
{}_a\mathrm{I}_t^3 f(t) &= \int_a^t \int_a^{\tau_3} \int_a^{\tau_2} f(\tau_1)\ \mathrm{d}\tau_1 \mathrm{d}\tau_2 \mathrm{d}\tau_3 \\
&= \lim_{N\to\infty} \left[\left(\frac{t-a}{N}\right)^3 \sum_{i=0}^{N-1} \frac{(i+1)(i+2)}{2} f\left(t - i\left(\frac{t-a}{N}\right)\right)\right] \qquad (20)
\end{aligned}
$$

Equations (18), (19) and (20) are the first three special cases of

$$
\begin{aligned}
{}_a\mathrm{I}_t^n f(t) &= \int_a^t \int_a^{\tau_n} \cdots \int_a^{\tau_2} f(\tau_1)\ \mathrm{d}\tau_1 \mathrm{d}\tau_2 \cdots \mathrm{d}\tau_n \\
&= \lim_{N\to\infty} \left[\left(\frac{t-a}{N}\right)^n \sum_{i=0}^{N-1} \binom{i+n-1}{i} f\left(t - i\left(\frac{t-a}{N}\right)\right)\right] \qquad (21)
\end{aligned}
$$

which may be proven inductively (left to the reader as an exercise). We will present without proof the following collection of formulas, which we will continue to use.

$$
(-1)^i \binom{n}{i} = \frac{\Gamma(i-n)}{\Gamma(-n)\Gamma(i+1)} = \binom{i+n-1}{i} = \frac{(i+n-1)!}{i!(n-1)!} \qquad (22)
$$

By substituting various expressions from (22) into (17) and (21), we obtain the limit definition that we expected. By defining ${}_a\mathrm{D}_t^{-n} = {}_a\mathrm{I}_t^n$ for $n > 0$, we conclude that

$$
\begin{aligned}
{}_a\mathrm{D}_t^n f(t) &= \lim_{N\to\infty} \left[\left(\frac{t-a}{N}\right)^{-n} \sum_{i=0}^{N-1} \frac{\Gamma(i-n)}{\Gamma(-n)\Gamma(i+1)} f\left(t - i\left(\frac{t-a}{N}\right)\right)\right] \\
&= \lim_{N\to\infty} \left[\frac{1}{\Gamma(-n)} \left(\frac{t-a}{N}\right)^{-n} \sum_{i=0}^{N-1} \frac{\Gamma(i-n)}{\Gamma(i+1)} f\left(t - i\left(\frac{t-a}{N}\right)\right)\right]
\end{aligned}
$$

for any positive or negative integer n, as required. □

In the next section we extend this definition from order $n \in \mathbb{Z}$ to order $\alpha \in \mathbb{C}$.

2 Differintegrals of complex order

Definition 2.1 (The Grünwald-Letnikov Differintegral)**.** Let $\alpha \in \mathbb{C}$ and let $f(t)$ be a α-times differintegrable function. Then the differintegral of $f(t)$ of order α, with respect to t, from basepoint a, is given by

$$
{}_a\mathrm{D}_t^\alpha = \lim_{N\to\infty} \left[\frac{1}{\Gamma(-\alpha)} \left(\frac{t-a}{N}\right)^{-\alpha} \sum_{i=0}^{n} \frac{\Gamma(i-\alpha)}{\Gamma(i+1)} f\left(t - i\left(\frac{t-a}{N}\right)\right)\right] \qquad (23)
$$

To see this, as in [8], replace $n \in \mathbb{Z}$ in (16) by arbitrary order $\alpha \in \mathbb{C}$. There is a worry that (16) and (23) may be undefined for order $\alpha \in \{0, 1, 2, ...\}$ since $\Gamma(-z)$ diverges when $z \in \{0, 1, 2, ...\}$. The Grünwald-Letnikov differintegral is actually defined for all complex numbers. The quotient of $\Gamma(i-\alpha)$ and $\Gamma(-\alpha)$, seen in (23), converges even though they both diverge. The proof of the next lemma is found in [8, p. 48].

Lemma 2.2. *Let f be an n-times, an α-times, and also an $(n+\alpha)$-times differentiable function. Then we have $\mathrm{D}^n\mathrm{D}^\alpha f = \mathrm{D}^{n+\alpha} f$ for all $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$.*

The Grünwald-Letnikov differintegral is not always useful. We will try to generalise repeated integral formula from Lemma 1.4 instead of starting with (16).

Definition 2.3 (The Riemann-Liouville Integral)**.** Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$ and let $f(t)$ be an α-times integrable function. The integral of $f(t)$ of order α, with respect to t, from lower limit a, is given by

$$ {}_a\mathrm{I}_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\ \mathrm{d}\tau \tag{24}$$

The Riemann-Liouville integral can be constructed by extending (12) to include order $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$. Both operators become the identity operator when $\alpha = 0$. We have defined the RL integral for order α with positive real component. This definition may be extended to any complex order α using the ceiling function.

Definition 2.4 (Ceiling Function)**.** The *ceiling* of α, denoted $\lceil\alpha\rceil$, is the lowest integer such that $\mathrm{Re}(\alpha) \le \lceil\alpha\rceil$.

As an example, the ceiling of $4.1 + 2.3i$ is 5 and the ceiling of $-9.5 - i\sqrt{23}$ is -9.

Theorem 2.5 (The Riemann-Liouville Differintegral)**.** *Let $\alpha \in \mathbb{C}$. The differintegral of $f(t)$ of order α, with respect to t, from lower limit a, is given by*

$$ {}_a\mathrm{D}_t^\alpha f(t) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)}\ {}_a\mathrm{D}_t^{\lceil\alpha\rceil} \int_a^t (t-\tau)^{\lceil\alpha\rceil-\alpha-1} f(\tau)\ \mathrm{d}\tau \tag{25}$$

Proof. We know that ${}_a\mathrm{D}_t^\alpha f(t) = {}_a\mathrm{D}_t^{\lceil\alpha\rceil}\,{}_a\mathrm{D}_t^{\alpha-\lceil\alpha\rceil} f(t)$ from Lemma 2.2, so we have

$$\begin{aligned} {}_a\mathrm{D}_t^\alpha f(t) &= {}_a\mathrm{D}_t^{\lceil\alpha\rceil}\left[{}_a\mathrm{D}_t^{\alpha-\lceil\alpha\rceil} f(t)\right] \\ &= {}_a\mathrm{D}_t^{\lceil\alpha\rceil}\left[{}_a\mathrm{I}_t^{\lceil\alpha\rceil-\alpha} f(t)\right] \\ &= {}_a\mathrm{D}_t^{\lceil\alpha\rceil}\left[\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \int_a^t (t-\tau)^{(\lceil\alpha\rceil-\alpha)-1} f(\tau)\ \mathrm{d}\tau\right] \\ &= \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\ {}_a\mathrm{D}_t^{\lceil\alpha\rceil} \int_a^t (t-\tau)^{\lceil\alpha\rceil-\alpha-1} f(\tau)\ \mathrm{d}\tau \end{aligned}$$

since $\mathrm{Re}(\lceil\alpha\rceil - \alpha) > 0$. This gives our result. □

This result found in [11] provides a method to calculate the differintegral of any order $\alpha \in \mathbb{C}$ by taking a Riemann-Liouville integral of order $\lceil\alpha\rceil - \alpha$ and then a derivative of order $\lceil\alpha\rceil$. If we wanted ${}_a\mathrm{D}_t^\alpha f(t)$, where $\alpha = 3.7 + 2.9i$, we would take a Riemann-Liouville integral of order $\lceil\alpha\rceil - \alpha = 0.3 - 2.9i$ and then $\lceil\alpha\rceil = 4$ derivatives.

Lemma 2.6. *Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$ and let $f(t)$ and $g(t)$ be α-times integrable functions. Then Riemann-Liouville integrals are linear. That is, for $\lambda, \mu \in \mathbb{C}$,*

$$ {}_a\mathrm{I}_t^\alpha\left(\lambda f(t) + \mu g(t)\right) = \lambda\, {}_a\mathrm{I}_t^\alpha f(t) + \mu\, {}_a\mathrm{I}_t^\alpha g(t) \tag{26}$$

Proof. By applying (24) we can deduce that

$$\begin{aligned}
{}_a\mathrm{I}_t^\alpha \left(\lambda f(t) + \mu g(t)\right) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \left(\lambda f(t) + \mu g(t)\right) \mathrm{d}\tau \\
&= \frac{1}{\Gamma(\alpha)} \int_a^t \left[\lambda f(t)(t-\tau)^{\alpha-1} + \mu g(t)(t-\tau)^{\alpha-1}\right] \mathrm{d}\tau \\
&= \frac{\lambda}{\Gamma(\alpha)} \int_a^t f(t)(t-\tau)^{\alpha-1}\, \mathrm{d}\tau + \frac{\mu}{\Gamma(\alpha)} \int_a^t g(t)(t-\tau)^{\alpha-1}\, \mathrm{d}\tau \\
&= \lambda\, {}_a\mathrm{I}_t^\alpha f(t) + \mu\, {}_a\mathrm{I}_t^\alpha g(t).
\end{aligned}$$

Therefore Riemann-Liouville integrals of order $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$ are linear. □

Theorem 2.7. *Let $\alpha \in \mathbb{C}$ and let $f(t)$ and $g(t)$ be α-times differintegrable functions. Then Riemann-Liouville differintegrals are linear. That is, for $\lambda, \mu \in \mathbb{C}$,*

$$ {}_a\mathrm{D}_t^\alpha \left(\lambda f(t) + \mu g(t)\right) = \lambda\, {}_a\mathrm{D}_t^\alpha f(t) + \mu\, {}_a\mathrm{D}_t^\alpha g(t) \tag{27}$$

Proof. We use Theorem 2.5 to get

$$\begin{aligned}
&{}_a\mathrm{D}_t^\alpha \left(\lambda f(t) + \mu g(t)\right) = {}_a\mathrm{D}_t^{\lceil\alpha\rceil}\, {}_a\mathrm{I}_t^{\lceil\alpha\rceil-\alpha} \left(\lambda f(t) + \mu g(t)\right) \\
&= {}_a\mathrm{D}_t^{\lceil\alpha\rceil} \left[\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \int_a^t (t-\tau)^{(\lceil\alpha\rceil-\alpha)-1} \left(\lambda f(t) + \mu g(t)\right)\, \mathrm{d}\tau\right] \\
&= {}_a\mathrm{D}_t^{\lceil\alpha\rceil} \left[\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \int_a^t \left(\lambda f(t)(t-\tau)^{\lceil\alpha\rceil-\alpha-1} + \mu g(t)(t-\tau)^{\lceil\alpha\rceil-\alpha-1}\right)\, \mathrm{d}\tau\right] \\
&= \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\, {}_a\mathrm{D}_t^{\lceil\alpha\rceil} \left[\lambda \int_a^t f(t)(t-\tau)^{\lceil\alpha\rceil-\alpha-1}\, \mathrm{d}\tau + \mu \int_a^t g(t)(t-\tau)^{\lceil\alpha\rceil-\alpha-1}\, \mathrm{d}\tau\right] \\
&= \frac{\lambda\, {}_a\mathrm{D}_t^{\lceil\alpha\rceil}}{\Gamma(\lceil\alpha\rceil-\alpha)} \int_a^t f(t)(t-\tau)^{\lceil\alpha\rceil-\alpha-1}\, \mathrm{d}\tau + \frac{\mu\, {}_a\mathrm{D}_t^{\lceil\alpha\rceil}}{\Gamma(\lceil\alpha\rceil-\alpha)} \int_a^t g(t)(t-\tau)^{\lceil\alpha\rceil-\alpha-1}\, \mathrm{d}\tau \\
&= \lambda\, {}_a\mathrm{D}_t^{\lceil\alpha\rceil}\, {}_a\mathrm{I}_t^{\lceil\alpha\rceil-\alpha} f(t) + \mu\, {}_a\mathrm{D}_t^{\lceil\alpha\rceil}\, {}_a\mathrm{I}_t^{\lceil\alpha\rceil-\alpha} g(t) \\
&= \lambda\, {}_a\mathrm{D}_t^\alpha f(t) + \mu\, {}_a\mathrm{D}_t^\alpha g(t)
\end{aligned}$$

This proves that Riemann-Liouville differintegrals are linear operators. □

The previous two results are found in [11]. Riemann-Liouville differintegrals are generally not commutative. Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$. With the ceiling function it becomes obvious that ${}_a\mathrm{D}_t^\alpha$ is the left-inverse of ${}_a\mathrm{I}_t^\alpha$ since

$$ {}_a\mathrm{D}_t^\alpha\, {}_a\mathrm{I}_t^\alpha f(t) = {}_a\mathrm{D}_t^{\lceil\alpha\rceil} \left[{}_a\mathrm{I}_t^{\lceil\alpha\rceil-\alpha}\, {}_a\mathrm{I}_t^\alpha f(t)\right] = {}_a\mathrm{D}_t^{\lceil\alpha\rceil} \left[{}_a\mathrm{I}_t^{\lceil\alpha\rceil} f(t)\right] = f(t) \tag{28}$$

On the other hand ${}_a\mathrm{I}_t^\alpha$ is not the left-inverse of ${}_a\mathrm{D}_t^\alpha$. This is because information is lost when differentiating and ${}_a\mathrm{I}_t^\alpha$ is not able to recover it, as shown in [11, p. 21]:

$$ {}_a\mathrm{I}_t^\alpha\, {}_a\mathrm{D}_t^\alpha f(t) = f(t) - \sum_{j=1}^{\lceil\alpha\rceil} \frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)} \left[{}_a\mathrm{D}_t^{\alpha-j} f(t)\right]_{t=a} \tag{29}$$

Composition of complex differintegrals is explored further in [8, pp. 82-87]. Putting differintegrals aside, it turns out that Riemann-Liouville integrals do commute.

Theorem 2.8. *Let $f(t)$ be an $(\alpha+\beta)$-integrable function with $\alpha, \beta \in \mathbb{C}$ such that* $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$. *Then the following Riemann-Liouville integrals commute.*

$$ {}_a\mathrm{I}_t^{\alpha}\, {}_a\mathrm{I}_t^{\beta} f(t) = {}_a\mathrm{I}_t^{\beta}\, {}_a\mathrm{I}_t^{\alpha} f(t) \tag{30} $$

Proof. The left hand side of 30 gives

$$\begin{aligned}
{}_a\mathrm{I}_t^{\alpha}\, {}_a\mathrm{I}_t^{\beta} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \left(\frac{1}{\Gamma(\beta)} \int_a^{\tau} (\tau-\sigma)^{\beta-1} f(\sigma)\ \mathrm{d}\sigma \right) \mathrm{d}\tau \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_a^{\tau} (t-\tau)^{\alpha-1} (\tau-\sigma)^{\beta-1} f(\sigma)\ \mathrm{d}\sigma\ \mathrm{d}\tau \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_{\sigma}^{t} (t-\tau)^{\alpha-1} (\tau-\sigma)^{\beta-1} f(\sigma)\ \mathrm{d}\tau\ \mathrm{d}\sigma \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(\sigma) \left(\int_{\sigma}^{t} (t-\tau)^{\alpha-1} (\tau-\sigma)^{\beta-1}\ \mathrm{d}\tau \right) \mathrm{d}\sigma \\
&= \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-\sigma)^{\alpha+\beta-1} f(\sigma)\ \mathrm{d}\sigma \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \right) \int_a^t (t-\sigma)^{\alpha+\beta-1} f(\sigma)\ \mathrm{d}\sigma \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-\sigma)^{(\alpha+\beta)-1} f(\sigma)\ \mathrm{d}\sigma \\
&= {}_a\mathrm{I}_t^{\alpha+\beta} f(t)
\end{aligned}$$

Addition commutes in the positive real half of $\mathbb{C}$, so we have

$$ {}_a\mathrm{I}_t^{\alpha}\ {}_a\mathrm{I}_t^{\beta} f(t) = {}_a\mathrm{I}_t^{\alpha+\beta} f(t) = {}_a\mathrm{I}_t^{\beta+\alpha} f(t) = {}_a\mathrm{I}_t^{\beta}\ {}_a\mathrm{I}_t^{\alpha} f(t) \tag{31} $$

which proves our result. □

Theorem 2.9. *Let $\alpha \in \mathbb{C}$ and let $f(t)$ and $g(t)$ be α-differintegrable functions. Then the product rule for Riemann-Liouville differintegrals is given by*

$$ {}_a\mathrm{D}_t^{\alpha} \left(f(t)g(t) \right) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}\ {}_a\mathrm{D}_t^{\alpha-k} f(t)\ {}_a\mathrm{D}_t^{k} g(t) \tag{32} $$

The previous result, whose proof is due to T. J. Osler [13], provides an explicit formula for the complex differintegral of a product of functions - a generalisation of both

$$ \mathrm{D}(uv) = u\,\mathrm{D}v + v\,\mathrm{D}u \tag{33} $$

and the generalised Leibniz rule

$$ \mathrm{D}^n(uv) = \sum_{i=0}^{n} \binom{n}{k} \mathrm{D}^i u\, \mathrm{D}^{n-i} v \tag{34} $$

for positive integer order n to include arbitrary order $\alpha \in \mathbb{C}$.

3 Fractional differential equations

In this section we will explore the differintegrals of some elementary functions and then we will introduce linear fractional differential equations with basic solutions. Let C be a constant. For arbitrary α, we find that the arbitrary differintegral of C is

$$\begin{aligned}
{}_a\mathrm{D}_t^\alpha(C) &= C\ {}_a\mathrm{D}_t^\alpha(1) \\
&= C \lim_{N\to\infty}\left[\frac{1}{\Gamma(-\alpha)}\left(\frac{t-a}{N}\right)^{-\alpha}\sum_{i=0}^{N-1}\frac{\Gamma(i-\alpha)}{\Gamma(i+1)}\right] \\
&= \frac{C}{(t-a)^\alpha}\lim_{N\to\infty}\left[N^\alpha\frac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)\Gamma(N)}\right] \\
&= \frac{C}{(t-a)^\alpha\,\Gamma(1-\alpha)}
\end{aligned} \tag{35}$$

where we have taken the following statements to be true without proof [8, p. 20].

$$\sum_{i=0}^{N-1}\frac{\Gamma(i-\alpha)}{\Gamma(-\alpha)\Gamma(i+1)} = \frac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)\Gamma(N)} \qquad\qquad \lim_{N\to\infty}\left[N^\alpha\frac{\Gamma(N-\alpha)}{\Gamma(N)}\right] = 1$$

Subsequently we have ${}_a\mathrm{D}_t^\alpha(0) = 0$ for all α since the denominator of (35) is finite when $t > \alpha$. Now let $p > -1$ and let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) < 0$. From [11] we have

$$\begin{aligned}
{}_a\mathrm{D}_t^\alpha(t-a)^p &= {}_a\mathrm{I}_t^{-\alpha}(t-a)^p \\
&= \frac{1}{\Gamma(-\alpha)}\int_a^t \frac{(\tau-a)^p}{(t-\tau)^{\alpha+1}}\,\mathrm{d}\tau \\
&= \frac{1}{\Gamma(-\alpha)}\int_a^t \frac{\sigma^p}{(t-\sigma-a)^{\alpha+1}}\,\mathrm{d}\sigma \\
&= \frac{(t-a)^{p-\alpha}}{\Gamma(-\alpha)}\int_0^1 \frac{\rho^p}{(1-\rho)^{\alpha+1}}\,\mathrm{d}\rho \\
&= \frac{(t-a)^{p-\alpha}}{\Gamma(-\alpha)}\,B(-\alpha, p+1) \\
&= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}\,(t-a)^{p-\alpha}
\end{aligned}$$

where we have used the substitutions $\sigma = \tau - a$ and $\rho = \frac{\sigma}{t-a}$. Now, for $\mathrm{Re}(\alpha) \geq 0$,

$$\begin{aligned}
{}_a\mathrm{D}_t^\alpha(t-a)^p &= {}_a\mathrm{D}_t^{\lceil\alpha\rceil}\,{}_a\mathrm{I}_t^{\lceil\alpha\rceil-\alpha}(t-a)^p \\
&= \frac{\Gamma(p+1)}{\Gamma(p+\lceil\alpha\rceil-\alpha+1)}\,{}_a\mathrm{D}_t^{\lceil\alpha\rceil}(t-a)^{p+\lceil\alpha\rceil-\alpha} \\
&= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}\,(t-a)^{p-\alpha}
\end{aligned}$$

The results for $\mathrm{Re}(\alpha) < 0$ and $\mathrm{Re}(\alpha) \geq 0$ are identical. When $\alpha = 0$ we obtain

$$\frac{\Gamma(p+1)}{\Gamma(p+1)}\,(t-a)^p = (t-a)^p$$

which is the original function, as one would expect.

We have found the differintegrals of the constant function C and the polynomial function $(t-a)^p$, but now we will find the differintegral of the exponential function. Let C be a constant. For arbitrary α, using power series [8, p. 94], we obtain

$$\begin{aligned}
{}_a\mathrm{D}_t^\alpha(e^{C-t}) &= {}_a\mathrm{D}_t^\alpha(e^{C-a}e^{-(t-a)}) \\
&= {}_a\mathrm{D}_t^\alpha\left(e^{C-a}\sum_{i=0}^{\infty}(-1)^i\frac{(t-a)^i}{\Gamma(i+1)}\right) \\
&= e^{C-a}\left(\sum_{i=0}^{\infty}(-1)^i\,\frac{{}_a\mathrm{D}_t^\alpha(t-a)^i}{\Gamma(i+1)}\right) \\
&= e^{C-a}\left(\sum_{i=0}^{\infty}(-1)^i\,\frac{(t-a)^i}{\Gamma(i+1)}\frac{\Gamma(i+1)}{\Gamma(i-\alpha+1)}\,(t-a)^{i-\alpha}\right) \\
&= \frac{e^{C-a}}{(t-a)^\alpha}\left(\sum_{i=0}^{\infty}(-1)^i\,\frac{(t-a)^i}{\Gamma(i-\alpha+1)}\right) \\
&= \frac{\gamma^*(-\alpha,a-t)}{(t-a)^\alpha}e^{C-t}
\end{aligned}$$

where we have used the **incomplete gamma function** defined in [8, p. 20] by

$$\gamma^*(\alpha,\beta) = e^{-\beta}\sum_{i=0}^{\infty}\frac{\beta^i}{\Gamma(i+\alpha+1)} \tag{36}$$

This reflects what we would have expected since the exponential function is a factor of its differintegral.

Liouville used the existence of a solution to $\mathrm{D}^n y = 0$ for $n \in \mathbb{Z}$ and investigated possible solutions to to $\mathrm{D}^\alpha y = 0$ for $\alpha \in \mathbb{C}$, as shown in [12]. A function $f(t)$ is a **fixed function of order** α if it satisfies $\mathrm{D}_t^\alpha f(t) = 0$. Denote the set of all such fixed functions by $\mathcal{F}^\alpha$, and denote the set of all α-differintegrable functions by $\mathcal{C}^\alpha$. We define a **linear fractional differential equation of order** α, where $\alpha \in \mathbb{R}$ and $0 < \alpha \leq 1$, as an equation of the form

$$\mathrm{D}_t^\alpha y(t) = a(t)y(t) + b(t) \tag{37}$$

where $y(t) \in \mathcal{C}^\alpha$ and both a and b are complex valued functions [14, p. 330]. If $b(t) = 0$ for all t, then we say that $\mathrm{D}_t^\alpha y(t) = a(t)y(t)$ is also **homogeneous**.

Lemma 3.1. *The solutions to the above linear homogeneous fractional differential equation of order α are of the form*

$$y(t) = k(t)e^{A(t)} \tag{38}$$

where $k(t) \in \mathcal{F}^\alpha$ and $\mathrm{D}_t^\alpha A(t) = a(t)$.

Proof. Taking the differintegral of $\ln(y(t))$, we obtain

$$\mathrm{D}_t^\alpha\left(\ln(y(t))\right) = \frac{1}{y(t)}\mathrm{D}_t^\alpha(y(t)) = \frac{a(t)y(t)}{y(t)} = a(t)$$

Now let $A(t)$ be the function such that $\mathrm{D}_t^\alpha A(t) = a(t)$. Then $\ln(y(t)) = a(t) + c(t)$ for some function $c(t) \in \mathcal{F}^\alpha$. Taking exponents of both sides, it follows quickly that $y(t) = e^{A(t)+c(t)} = e^{c(t)}e^{A(t)} = k(t)e^{A(t)}$ □

Theorem 3.2. *If each $y_i(t) \in \mathcal{C}^\alpha$ is a solution to $\mathrm{D}_t^\alpha y(t) = a(t)y(t) + b_i(t)$ for each $i \in I$, where I is some countable indexing set, then $y(t) = \sum_{i\in I} y_i(t)$ is a solution to*

$$\mathrm{D}_t^\alpha y(t) = a(t)y(t) + b(t) \tag{39}$$

where $b(t) = \sum_{i\in I} b_i(t)$.

Proof. Substituting $y(t) = \sum_{i\in I} y_i(t)$ into (39), we obtain

$$\begin{aligned}
\mathrm{D}_t^\alpha y(t) &= \mathrm{D}_t^\alpha \sum_{i\in I} y_i(t) \\
&= \sum_{i\in I} \mathrm{D}_t^\alpha y_i(t) \\
&= \sum_{i\in I} \left(a(t)y_i(t) + b_i(t)\right) \\
&= a(t)\sum_{i\in I} y_i(t) + \sum_{i\in I} b_i(t) \\
&= a(t)y(t) + b(t)
\end{aligned}$$

which shows that $y(t)$ is indeed a solution. □

In some cases, an FDE may be manipulated to become an ODE with the intention of making it easier to solve. Consider the linear homogeneous FDE [8] given by

$$\mathrm{D}^{1/2}y + y = 0$$

Applying $\mathrm{D}_t^{1/2}$ to each term, and considering composition rules [11], one finds that

$$\mathrm{D}_t\, y - ct^{-3/2} + \mathrm{D}_t^{1/2}y = 0$$

where c is constant. When we replace $\mathrm{D}_t^{1/2}y$ with $-y$, we obtain

$$\mathrm{D}_t y - y = ct^{-3/2}$$

which is a first order ODE. Further working [15] yields the solution

$$y(t) = Ce^t\,\mathrm{erfc}(t^{1/2}) - \frac{C}{\sqrt{\pi t}}$$

where C is constant. The full FDE-to-ODE transform method is beyond the scope of this paper, but the interested reader is referred to [8, pp. 157–158].

4 Fractional quantum mechanics

Fractional quantum mechanics is a branch of physics with many recent developments. The **time-dependent Schrödinger equation** for a single non-relativistic particle of mass m describes how the quantum state[1] $\psi(\mathbf{r},t)$ of a three dimensional system changes with time, where $\mathbf{r}=(x,y,z)$, as found in [17]. We have

$$i\hbar\partial_t\psi = \frac{-\hbar^2}{2m}\nabla^2\psi + V\psi \tag{40}$$

where $\hbar$ is the reduced Planck constant, ∂_t is the partial derivative with respect to t, $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplacian operator and $V(\mathbf{r},t)$ is the potential energy of the system. The wavefunction $\psi(\mathbf{r},t)$ determines the probability $|\psi(\mathbf{r},t)|^2$ of a particle appearing at $\mathbf{r}$ at time t. The **Hamiltonian** of the system [18] is the operator

$$\mathrm{H} = \frac{-\hbar^2}{2m}\nabla^2 + V \tag{41}$$

corresponding to the total energy of the system. On the right-hand-side are the kinetic and potential energy operators respectively. We can then rewrite (40) as

$$i\hbar\partial_t\psi = \mathrm{H}\psi \tag{42}$$

The **time-independent Schrödinger equation** for a single non-relativistic particle of mass m describes the stationary states of the system, that is, the states which do not change with time. In three dimensions, the equation is written as

$$E\phi = \frac{-\hbar^2}{2m}\nabla^2\phi + V\phi \tag{43}$$

where the constant E represents the total energy of the system and the wavefunction $\psi(\mathbf{r},t)$ has been factored into $\phi(\mathbf{r})\rho(t)$. It makes sense to factorise in this way since we are exploring quantum states which do not change as t changes. With (41) we get

$$E\phi = \mathrm{H}\phi \tag{44}$$

Now, the **fractional time-dependent Schrödinger equation** [16] for a single non-relativistic particle, where $1 < \alpha \leq 2$, is given by

$$i\hbar\partial_t\psi = M_\alpha(-\hbar^2\Delta)^{\alpha/2}\psi + V\psi \tag{45}$$

where $(-\hbar^2\Delta)^{\alpha/2}$ is the three dimensional quantum Riesz derivative[2] and M_α is a scale constant. We must note that both $M_\alpha = \frac{1}{2m}$ and $(-\hbar^2\Delta)^{\alpha/2} = \nabla^2$ if we let $\alpha = 2$. We may think of the Riesz derivative [19] as the generalisation of the Laplacian, and we may incorporate it into the Hamiltonian operator to define the **α-Hamiltonian**

$$\mathrm{H}_\alpha = M_\alpha(-\hbar^2\nabla^2)^{\alpha/2} + V \tag{46}$$

[1]The state of a quantum system is described by a vector $|\psi\rangle \in \mathcal{H}$ for some Hilbert space $\mathcal{H}$.

[2]The full definition of the three dimensional Riesz derivative is beyond the scope of this paper.

which allows us to rewrite the fractional time-dependent Schrödinger equation as

$$i\hbar\partial_t\psi = \mathrm{H}_\alpha\psi \tag{47}$$

A stationary state is a quantum state in which all observable properties of the state are constant. Factoring $\psi(\mathbf{r},t)$ into $\phi(\mathbf{r})\rho(t)$ leads to $\rho(t) = e^{-iEt/\hbar}$ after some basic manipulation. The wavefunction then becomes $\psi(\mathbf{r},t) = \phi(\mathbf{r})e^{-iEt/\hbar}$ which means that the probability $|\psi(\mathbf{r},t)|^2$ becomes $|\phi(\mathbf{r})|^2$. For a single non-relativistic particle, the **fractional time-independent Schrödinger equation** is given by

$$E\phi = M_\alpha(-\hbar^2\Delta)^{\alpha/2}\phi + V\phi \tag{48}$$

and may be rewritten as $E\phi = \mathrm{H}_\alpha\phi$. The FDE given by equation (48) describes the stationary states of the three dimensional quantum system. The probability associated to a wavefunction is, as before, equal to $|\phi(\mathbf{r})|^2$.

The **Bohr model** of the hydrogen atom [20] describes a small electron of charge $-1e$ orbiting a nucleus of positive charge $+1e$. Taking the origin to be at the nucleus, the potential energy for a hydrogen atom is given by Coulomb's law

$$V(\mathbf{r}) = -\frac{e}{|\mathbf{r}|} \tag{49}$$

where Coulomb's constant is omitted since it is determined by the choice of units. Substituting 49 into (48) yields the fractional time-independent eigenvalue equation

$$M_\alpha(-\hbar^2\Delta)^{\alpha/2}\phi(\mathbf{r}) - \frac{e}{|\mathbf{r}|}\phi(\mathbf{r}) = E\phi(\mathbf{r}) \tag{50}$$

The general case has been solved by Laskin [21]. The first Bohr postulate [22] states that electrons in an atom exists in a stationary state; they orbit the nucleus without emitting or absorbing electromagnetic radiation so their energies remain constant. For an electron at each energy level n, where $n = 1$ is the lowest energy level, Laskin finds the general formula for the radius a_n and the energy E_n of the electron to be

$$a_n = n^{\frac{\alpha}{\alpha-1}}\left(\frac{\alpha M_\alpha \hbar^\alpha}{e^2}\right)^{\frac{1}{\alpha-1}} \qquad E_n = (1-\alpha)n^{\frac{\alpha}{\alpha-1}}\left(\frac{e^2}{\alpha M_\alpha^{1/\alpha}\hbar}\right)^{\frac{\alpha}{\alpha-1}} \tag{51}$$

The second Bohr postulate states that a transition between energy levels happens when an electron either absorbs or emits electromagnetic radiation if its energy level is increasing or decreasing, respectively. The energy of the radiation absorbed when transitioning from energy level n_i to a different energy level n_j is given by

$$\Delta E = (1-\alpha)E_0\left[n_j^{\frac{\alpha}{1-\alpha}} - n_i^{\frac{\alpha}{1-\alpha}}\right] \tag{52}$$

Theorem 4.1. *The Bohr model is a special case of the fractional Bohr model.*

Proof. Setting $\alpha = 2$, one obtains the radii and energies

$$a_n = \frac{n^2\hbar^2}{e^2M} \qquad E_n = -\frac{e^4m}{2\hbar^2n^2}$$

and the transition energy is given by

$$\Delta E = (1-\alpha)E_0\left[n_j^{-2} - n_i^{-2}\right]$$

where $E_0 = \frac{e^4m}{2\hbar^2}$. Unsurprisingly, these are the results that Niels Bohr once stated. □

5 Derivative operators

In this final section we will look at another way of thinking about derivative operators. Consider the linear homogeneous second-order ordinary differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 5\frac{\mathrm{d}y}{\mathrm{d}t} + 6y = 0 \tag{53}$$

which can be written as

$$\mathrm{D}^2 y - 5\mathrm{D}y + 6y = 0$$

where $\mathrm{D} = \mathrm{D}_t = \frac{\mathrm{d}}{\mathrm{d}t}$. Factorising the operators, we get

$$(\mathrm{D}^2 - 5\mathrm{D} + 6)y = 0$$

which can be further factorised to give

$$(D-2)(D-3)y = 0$$

This means that either $(\mathrm{D}-2)y = 0$ or $(\mathrm{D}-3)y = 0$, so we need to solve

$$\frac{\mathrm{d}y}{\mathrm{d}t} - 2y = 0 \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$$

The solutions to each equation are

$$y = Ae^{2x} \qquad\qquad y = Be^{3x}$$

which, by the **principle of superposition**, gives the general solution

$$y = Ae^{2x} + Be^{3x}$$

These are the solutions one would obtain by solving (53) some other way. Continuing to treat derivatives as algebraic objects, consider the exponential of a derivative

$$e^{\mathrm{D}} = 1 + \mathrm{D} + \frac{\mathrm{D}^2}{2!} + \frac{\mathrm{D}^3}{3!} + \cdots \tag{54}$$

$$= \sum_{k=0}^{\infty} \frac{\mathrm{D}^k}{k!} \tag{55}$$

Recall that the Maclaurin expansion of a function $f(t)$ is

$$f(t) = f(0) + tf'(0) + \frac{t^2}{2!}f''(0) + \frac{t^3}{3!}f'''(0) + \cdots \tag{56}$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[\frac{\mathrm{d}^k f}{\mathrm{d}t^k}\right]_{t=0} \tag{57}$$

Using (54) and while letting $\mathrm{D} = \mathrm{D}_\tau$, we obtain

$$\begin{aligned} f(t) &= f(0) + tf'(0) + \frac{t^2}{2!}f''(0) + \frac{t^3}{3!}f'''(0) + \cdots \\ &= \left[f(\tau) + t\,\mathrm{D}f(\tau) + \frac{t^2}{2!}\,\mathrm{D}^2 f(\tau) + \frac{t^3}{3!}\,\mathrm{D}^3 f(\tau) + \cdots \right]_{\tau=0} \\ &= \left[\left(1 + (t\,\mathrm{D}) + \frac{(t\mathrm{D})^2}{2!} + \frac{(t\mathrm{D})^3}{3!} + \cdots \right) f(\tau) \right]_{\tau=0} \\ &= \left[e^{t\mathrm{D}} f(\tau) \right]_{\tau=0} \end{aligned} \tag{58}$$

It is clear, then, that the Taylor series

$$\begin{aligned} f(t) &= f(a) + (t-a)f'(a) + \frac{(t-a)^2}{2!}f''(a) + \frac{(t-a)^3}{3!}f'''(a) + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(t-a)^k}{k!} \left[\frac{\mathrm{d}f}{\mathrm{d}t} \right]_{t=a} \end{aligned} \tag{59}$$

may be written as

$$f(t) = \left[e^{(t-a)\mathrm{D}} f(\tau) \right]_{\tau=a}$$

Lastly, let $f : \mathbb{R}^n \to \mathbb{R}$ with $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{t} = (t_1, \ldots, t_n), \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and consider the gradient operator $\nabla = (\partial_1, \ldots, \partial_n) = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. By playing around with summations, we obtain an expression for the Taylor series of $f : \mathbb{R}^n \to \mathbb{R}$.

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{a}) + \sum_{k=1}^{n} (x_k - a_k) \frac{\partial f(\mathbf{a})}{\partial x_k} + \frac{1}{2!} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} (x_{k_1} - a_{k_1})(x_{k_2} - a_{k_2}) \frac{\partial^2 f(\mathbf{a})}{\partial x_{k_1} \partial x_{k_2}} + \cdots \\ &= \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} \frac{(x_1 - a_1)^{k_1} \cdots (x_n - a_n)^{k_n}}{k_1! \, \cdots \, k_n!} \left[\frac{\partial^{k_1 + \cdots + k_n} f(\mathbf{x})}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right]_{\mathbf{x}=\mathbf{a}} \\ &= \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} \left[\left(\prod_{i=1}^{i=n} \frac{(x_i - a_i)^{k_i}}{k_i!} \, \partial_i^{k_i} \right) f(\mathbf{x}) \right]_{\mathbf{x}=\mathbf{a}} \end{aligned}$$

It is left to the tenacious reader to verify that we can, in fact, write

$$f(\mathbf{t}) = \left[e^{(\mathbf{t}-\mathbf{a})\cdot\nabla} f(\mathbf{x}) \right]_{\mathbf{x}=\mathbf{a}} \tag{60}$$

Conclusion

On the surface, fractional calculus appears to be devoid of real applications. However, quantum theory is just one of many fields which make use of it. The mathematical methods which we have introduced here are useful in the study of fluid dynamics, electrochemistry and space-time fractional diffusion, among other topics. For further reading, the book by Oldham and Spanier [8] is a good starting point, as well as the more recent book by Herrmann [23] for which a good grasp of physics will be useful.

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