Hypergeometric Function – Identities - Pi

Edgar Valdebenito

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abstract
We give some identities for Pi

1. Introducción

En esta nota mostramos algunas fórmulas que involucran a la constante Pi y la función hipergeométrica $F(a,b;c;x) = {}_2F_1(a,b;c;x)$.

1.1. Definición. La función hipergeométrica se define por la serie de Gauss:

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} z^n =$$

$$= 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 + \ldots = \tag{1}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\, n!} z^n \qquad ,|z|<1$$

En (1) $\Gamma(x)$ es la función Gamma y $(a)_n$ es el símbolo de pochhammer:

$$(a)_n = a(a+1)(a+2)\ldots(a+n-1) \quad ,n \geq 1 \tag{2}$$

$$(a)_0 = 1 \quad ,a \neq 0 \tag{3}$$

1.2. Representación integral de $F(a,b;c;x)$:

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} \, dt \quad ,\mathrm{Re}\, c > \mathrm{Re}\, b > 0 \tag{4}$$

1.3. Valores particulares:

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad ,\mathrm{Re}\, c > \mathrm{Re}(a+b) \tag{5}$$

$$F\left(1,1;\frac{3}{2};\frac{1}{2}\right)=\frac{\pi}{2} \tag{6}$$

2. Pi - Fórmulas

Entry 1. Si $m\in\mathbb{N}=\{1,2,3,\ldots\}$ entonces

$$\frac{\pi}{3\sqrt{3}}=m\sum_{k=0}^{m-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{m^2+km+k^2}\right)^{n+1}F\left(n+1,2n+1;2n+2;-\frac{m+2k}{m^2+km+k^2}\right) \tag{7}$$

Entry 2. Si $m\in\mathbb{N}=\{1,2,3,\ldots\}, q(m,k)=m^2+km+k^2+m+2k,\ k=0,\ldots,m-1$, entonces

$$\frac{\pi}{3\sqrt{3}}=m\sum_{k=0}^{m-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{q(m,k)}\right)^{n+1}F\left(n+1,1;2n+2;\frac{m+2k}{q(m,k)}\right) \tag{8}$$

Entry 3. Si $m\in\mathbb{N}=\{1,2,3,\ldots\}, p(m,k)=m^2+km+k^2, q(m,k)=m^2+km+k^2+m+2k$, $k=0,\ldots,m-1$, entonces

$$\frac{\pi}{3\sqrt{3}}=m\sum_{k=0}^{m-1}\left(\frac{q(m,k)}{p(m,k)}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{p(m,k)}{\left(q(m,k)\right)^2}\right)^{n+1}F\left(2n+1,n+1;2n+2;\frac{m+2k}{q(m,k)}\right) \tag{9}$$

Entry 4. Si $m\in\mathbb{N}=\{1,2,3,\ldots\}, p(m,k)=m^2+km+k^2, q(m,k)=m^2+km+k^2+m+2k$, $k=0,\ldots,m-1$, entonces

$$\frac{\pi}{3\sqrt{3}}=m\sum_{k=0}^{m-1}\left(\frac{1}{p(m,k)}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{q(m,k)}\right)^{n}F\left(n+1,1;2n+2;-\frac{m+2k}{p(m,k)}\right) \tag{10}$$

3. Ejemplos

Entry 5. $m=1$:

$$\frac{\pi}{3\sqrt{3}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}F\left(n+1,2n+1;2n+2;-1\right) \tag{11}$$

$$\frac{\pi}{3\sqrt{3}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{2}\right)^{n+1}F\left(n+1,1;2n+2;\frac{1}{2}\right) \tag{12}$$

$$\frac{\pi}{3\sqrt{3}}=2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{4}\right)^{n+1}F\left(2n+1,n+1;2n+2;\frac{1}{2}\right) \tag{13}$$

$$\frac{\pi}{3\sqrt{3}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{2}\right)^n F\left(n+1,1;2n+2;-1\right) \tag{14}$$

Entry 6. $m=2$:

$$\frac{\pi}{3\sqrt{3}}=$$

$$=2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left\{\left(\frac{1}{4}\right)^{n+1}F\left(n+1,2n+1;2n+2;-\frac{1}{2}\right)+\left(\frac{1}{7}\right)^{n+1}F\left(n+1,2n+1;2n+2;-\frac{4}{7}\right)\right\} \tag{15}$$

$$\frac{\pi}{3\sqrt{3}}=2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left\{\left(\frac{1}{6}\right)^{n+1}F\left(n+1,1;2n+2;\frac{1}{3}\right)+\left(\frac{1}{11}\right)^{n+1}F\left(n+1,1;2n+2;\frac{4}{11}\right)\right\} \tag{16}$$

$$\frac{\pi}{3\sqrt{3}}=$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left\{3\left(\frac{1}{9}\right)^{n+1}F\left(2n+1,n+1;2n+2;\frac{1}{3}\right)+\frac{22}{7}\left(\frac{7}{121}\right)^{n+1}F\left(2n+1,n+1;2n+2;\frac{4}{11}\right)\right\} \tag{17}$$

$$\frac{\pi}{3\sqrt{3}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left\{\frac{1}{2}\left(\frac{1}{6}\right)^{n}F\left(n+1,1;2n+2;-\frac{1}{2}\right)+\frac{2}{7}\left(\frac{1}{11}\right)^{n}F\left(n+1,1;2n+2;-\frac{4}{7}\right)\right\} \tag{18}$$

Entry 7. $m=3$:

$$\begin{aligned}\frac{\pi}{3\sqrt{3}}=&\,3\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{9}\right)^{n+1}F\left(n+1,2n+1;2n+2;-\frac{1}{3}\right)+\\&+3\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{13}\right)^{n+1}F\left(n+1,2n+1;2n+2;-\frac{5}{13}\right)+\\&+3\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{19}\right)^{n+1}F\left(n+1,2n+1;2n+2;-\frac{7}{19}\right)\end{aligned} \tag{19}$$

$$\frac{\pi}{3\sqrt{3}} = 3\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{12}\right)^{n+1} F\left(n+1,1;2n+2;\frac{1}{4}\right)+ \\ +3\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{18}\right)^{n+1} F\left(n+1,1;2n+2;\frac{5}{18}\right)+ \\ +3\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{26}\right)^{n+1} F\left(n+1,1;2n+2;\frac{7}{26}\right) \tag{20}$$

$$\frac{\pi}{3\sqrt{3}} = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{16}\right)^{n+1} F\left(2n+1,n+1;2n+2;\frac{1}{4}\right)+ \\ +\frac{54}{13}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{13}{324}\right)^{n+1} F\left(2n+1,n+1;2n+2;\frac{5}{18}\right)+ \\ +\frac{78}{19}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{19}{676}\right)^{n+1} F\left(2n+1,n+1;2n+2;\frac{7}{26}\right) \tag{21}$$

$$\frac{\pi}{3\sqrt{3}} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{12}\right)^{n} F\left(n+1,1;2n+2;-\frac{1}{3}\right)+ \\ +\frac{3}{13}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{18}\right)^{n} F\left(n+1,1;2n+2;-\frac{5}{13}\right)+ \\ +\frac{3}{19}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{26}\right)^{n} F\left(n+1,1;2n+2;-\frac{7}{19}\right) \tag{22}$$

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