The Josephus Numbers

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Abstract We give explicit formulas to compute the Josephus-numbers $J(n)$ where n is positive integer . Furthermore we present a new fast algorithm to calculate $J(n)$. We also offer prosperities , and we generalized it for all positive real number non-existent, Finally we give .the proof of properties.

Keywords*: Integer number , Mersenne Numbers , Primes number., Numerical sequences , Real numbers*

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1. Introduction

The Josephus problem in its original form goes back to the Roman historian Flavius Josephus during the Jewish-Roman war in the first century (see [1],[2]).

The legend reports that Flavius and 40 companions escaped and were trapped in a cave by the roman army. Fearing capture they decided to kill themselves. They decided to stand in a circle and kill every third man until only two were left until there was only one survivor who would kill himself.

By choosing the position 19 in the circle, Josephus saved their lives at the end of the process.

According to the story (told only by Josephus himself) had some skills in Math's and wanted none of this suicide non-sense, he quickly calculated where he should place himself in the circle so that they would be the last survivors.

Instead of eliminating every third person, we eliminate every second person, and we are only interested in the final survivor, not the final two survivors. (How Josephus became Roman historian is another interesting story.)

The story makes an interesting mathematical problem: how do you choose the correct position in the circle in order to stay alive the elimination process?.

Let us fix some notations:

$[x]$: Integer part of x .

Define the ordered set $Z_n = \{a_1, a_2, a_3, \dots, a_n\}$ We number the n positions in the circle by $1,2,3,\dots,n$ and start counting at number 1 .

We remove cyclically, from left to right, each m-th element of Z_n , then every m th element is removed we define new ordered set $Z_n^{(m)}$.

The questions arise: Which element of Z_n is the nth to be removed?

We will use $J(n)$ to represent the seat number of the person who survives (the survival number).

Example1.1:

Suppose there are 12 people at the table. The elimination sequence starts $2,4,6,8,10,12\dots$ at which point we are back to 1 , with only persons $1,3,5,7,9,11$ remaining. We continue the process, eliminating $1,5,9$, leaving 9 as the sole survivor. Thus $J(12)=9$.

We can test this by trying another value. Let's compute $J(1), J(2), J(3), \dots, J(20)$ with the following table:

Table 1. *An account of Josephus Numbers for* $n = \{1,2,3,\dots,20\}$

n	$J(n)$
1	1
2	1
3	3
4	1
5	3
6	8
7	7
8	1
9	3
10	5
11	7
12	9
13	11
14	13
15	15
16	1
17	3
18	5
19	7
20	9

From table(1), we can look at a few more powers of 2 : $J(2^0)=J(2^1)=J(2^2)=J(2^3)=J(2^4)=1$, player 1 survives. For this we deduce $J(2^s)=1$ for all $s \geq 0$.

But what about situations where n is not a power of 2 ?

We can write n uniquely as $n = 2^s + k$, where k is the number by which n exceeds the largest power of 2 .

That is $2^s \leq n$ and we can get $s = [\log(n)/\log(2)]$ and we can easily compute the ranks of k i,e $k = n - 2^s$.

We can look that each group s is to start at 1 when the game starts, after k people are eliminated, 2^s players remain, and the very next player is the winner, see the following table:

Table 2. *An account the values* $(n, s, 2^s, k, J(n))$

n	s	2^s	k	$J(n)$
1	0	1	0	1
2	1	2	0	1
3	1	2	1	3
4	2	4	0	1
5	2	4	1	3
6	2	4	2	5
7	2	4	3	7
8	3	8	0	1
9	3	8	1	3
10	3	8	2	5
11	3	8	3	7
12	3	8	4	9

13	3	8	5	11
14	3	8	6	13
15	3	8	7	15
16	4	16	0	1
17	4	16	1	3
18	4	16	2	5
19	4	16	3	7
20	4	16	4	9

Frome table (2) note that for each natural numbers trapped between two successive force of number 2 has the same integer part $s=\left[\log(n)/\log(2)\right]$, that's mean:

1) For integer number s we have $2^s \le n < 2^{s+1}$, rewards $n=\left\{2^{s+1}-1, 2^{s+1}-2, \ldots, 2^s\right\}$.

2) For $0 \le n+r < 2^s$ we have $\left[\log(n+r)/\log(2)\right]=s$, and for $n+r=2^s$ we have $\left[\log(n+r)/\log(2)\right]=s+1$

Now we can figure out the relationship between the k number and $j(n)$.

Therefore: $j(n)=2k+1$. It's just that easy.

Definition1.2: The Josephus Numbers is a positive integer writes:

$$2)\ J(n)=2n-2^{s+1}+1 \cdots\cdots\cdots (1)$$

where $s=\left[\log(n)/\log(2)\right]$ for all natural number n .

Furthermore we present a new fast algorithm to calculate $j(n)$ which is based into two steps as following:

Step1: Let n be the number of people in the group. Compute,

$$s=\left[\log(n)/\log(2)\right]$$

Step 2: Compute,

$$J(n)=2n-2^{s+1}+1$$

The algorithm is used to compute The Josephus Numbers by an program R ; version i386 3.3.3.

The R is an integrated suite of software facilities for data manipulation, calculation and graphical display. R is very much a vehicle for newly developing methods of interactive data analysis. It has developed rapidly, and has been extended by a large collection of packages, many people use R as a statistics system. There are about 25 packages supplied with R (called \standard" and \recommended" packages) and many more are available through the CRAN family of Internet sites (via http://CRAN.R-project.org) and elsewhere.

Example1.3:

Let us calculate $J(n)$ for $n=41$, $n=1243$, $n=1111111111$. Using the algorithm given earlier, we find the following results:

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> josephus.R(41)
[1] 19
[1] "josephus is successful"
""""""""""""""""""""""""""""""""""""""""""""""""""""""""
> josephus.R(1243)
[1] 439
[1] "josephus is successful"
""""""""""""""""""""""""""""""""""""""""""""""""""""""""
> josephus.R(1111111111)
[1] 74738575
[1] "josephus is successful"
""""""""""""""""""""""""""""""""""""""""""""""""""""""""
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Meaning that: $J(41)=39, J(1243)=439,\ J(1111111111)=74738575,$

2.Josephus numbers and some identities

According with the formula (1) for the Josephus Numbers we get for this numbers the following interesting identity.

Proposition2.1: For positive integer number s and all natural number n .we have :

1) $J(2^s)=1$

2) $J(2^{s+1}-1)=2^{s+1}-1$

3) $J(n)=2n-M_{s+1} \cdots\cdots\cdots (2)$

where $M_{s+1}=2^{s+1}-1$ is Mersenne numbers.

Note that the Proposition(2.1) indicate that the Josephus numbers is different between $2n$ and Mersenne number M_{s+1} where $s=\left[\log(n)/\log(2)\right]$. and also we conclude the different between Josephus number and Mersenne number is an even number as described in (2). We known that Mersenne number became an prime numbers for any prime $(s+1)$ and by note(1) derived from table(2) allows us to say that there was at least a Josephus number prime , but the result has to be circulated. And from Proposition(2.1) inequality(2) we get for

$2^s \le n < 2^{s+1}$,and $s=\left[\log(n)/\log(2)\right]$

$$\begin{cases} J(2^{s+1}-1)=2^{s+1}-1 \\ J(2^{s+1}-(n+1))=J(2^{s+1}-n)-2 \end{cases}$$

and for $n \ge 1$, $s=\left[\log(n)/\log(2)\right]$ we defined a multi-sequences $(J_{n,j})$ for $j=\{0,1,\ldots,s\}$ by,

$$\begin{cases} J_{n,j}(2^{j+1}-1)=2^{j+1}-1 \\ J_{n,j}(2^{j+1}-(n+1))=J(2^{j+1}-n)-2 \end{cases}.$$

Proposition2.2: For the sequence (J_n), we have:

$$\sum_{i=1}^{n} J(i)=(n+1)(n-2^{s+1})+\frac{2}{3}(2^{2s+1}+1)+n \cdots\cdots (3)$$

For example with $n=20$ we have $s=4$ and

$$\sum_{i=1}^{20} J(i)=(20+1)(20-2^5)+\frac{2}{3}(2^9+1)+20=110$$

is the same result with the sum of values of the fifth column of table(2).

Proposition2.3: *For* positive integer numbers r and for $n=2^s+k$ where k is positive integer, we have:

$$j(n+r)=2(n+r)-2^{s+a+1}-1 \ldots (4)$$

for $a=\left[\log(n+r)/\log(2)-s\right]$

Note that for a=0, in Proposition (2.3) we get the following results:

1) $J(n+r)-j(n)=2r$ for $r+k<2^s$

2) $J(n+r)-j(n)=-2k$ for $r+k=2^s$.

3) $J(n+1)-j(n)=2$ for $1+k<2^s$.

Proposition2.4: For t is real value we have

$$1)\ J(n^t)=2n^t-2^{t\times s+1}+1 \cdots\cdots (5)$$

where $t \in \mathbb{R}$. And for $a=\left[\log(t\times n)/\log(2)-s\right]$ with $t>0$ we get

$$2)\ J(t\times n)=2t\times n-2^{(s+a)+1}\cdots(6)$$

For $b=\left[log(n+t)/\log(2)-s\right]$. and $n+t>0$,

$$3)\ J(n+t)=2(n+t)-2^{s+b+1}+1\cdots(7)$$

With Proposition(2.4) we can expand the Josephus problem to the real values, so it does not remain restricted to natural numbers only. This allows us to define it as a function.

***Definition2.5*: The Josephus function is numeric function define on $\mathbb{R}^*$ to. $\mathbb{R}$ by :

$$J(x)=2x-2^{s+1}+1\cdots\cdots\cdots(8)$$

where $s=\left[\log(x)/\log(2)\right]$.

The following figure(1) represent the graph of the Josephus function on interval $\left]0;1\right]$ with the program R.

Note that the graph of this function as figure(1) somewhat resembling an electric stun !.

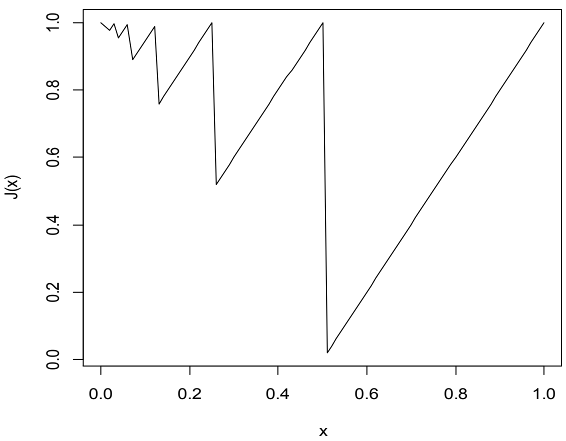

Figure 1. The graph of $J(x)$ on interval $\left]0;1\right]$.

3. Proof

About the identities (1), (2), (5), (6), we decided to omit its proof here because they can be easily proven.

The proof of (3) follows by separation then generalize.

We calculated the natural number image for $n=\{1,..,20\}$ by the Josephus function(8) without assigning value, but in each case note numbers that change between cases and then generalize it to all natural numbers as the following:

We ha have $n=20$ i.e. $n=2^s+4$ then $k=4$.

We calculated $J(n)$, by

$$\begin{cases} n=1, s=0, J(1)=2\times 1-2^1+1 \\ n=2, s=1, J(2)=2\times 2-2^2+1 \\ n=3, s=1, J(3)=2\times 3-2^2+1 \\ n=4, s=2, J(4)=2\times 4-2^3+1 \\ n=5, s=2, J(5)=2\times 5-2^3+1 \\ n=6, s=2, J(6)=2\times 6-2^3+1 \\ n=7, s=2, J(7)=2\times 7-2^3+1 \\ \vdots \\ n=20, s=4, J(20)=2\times 20-2^5+1 \end{cases}$$

With $i=\{1,..,20\}$ and $j=\{0,..,s\}$ we get

$$\sum_{i=1}^{20} J(i)=2(1+2+\ldots+20)-\sum_{j=1}^{4}2^{j-1}\times 2^j-5\times 2^5+\left(\underbrace{1+\cdots+1}_{20}\right)$$

we generalized it we find for $i=\{1,..,n\}$, and $j=\{0,..,s\}$.

and $n=2^s+k$, we have:

$$\sum_{i=1}^{n} J(i)=2(1+\cdots+n)-\sum_{j=1}^{s}2^{j-1}\times 2^j-(k+1)2^{s+1}+(1+\cdots+n)$$

divided the sum in to four
parts,

$$I_1=(1+\cdots+n), I_2=\sum_{j=1}^{s}2^{2j-1}, I_3=2^{2s+1}-2^{2s+1}-(k+1)2^{s+1}$$

$$, I_4=(1+\cdots+1).$$

we note that,

I_1, sum arithmetic sequence $u_n=n$.

I_2, sum geometric sequence $v_n=2^{2n-1}$.

$$I_3=2^{2s+1}-2^{s+1}(n+1)$$

I_4, sum arithmetic sequence $w_n=1$,

with easily calculate we get:

$$\sum_{i=1}^{n} J(i)=(n+1)\left(n-2^{s+1}\right)+\frac{2}{3}\left(2^{2s+1}+1\right)+n$$

The proof of (4) follows by separation then generalize.
we have,

$$j(n+r)=2(n+r)-2^{s+0+1}-1 \text{ if } 2^s\le n+r<2^{s+1},$$

$$j(n+r)=2(n+r)-2^{s+1+1}-1 \text{ if } 2^{s+1}\le n+r<2^{s+2},$$

$$j(n+r)=2(n+r)-2^{s+2+1}-1 \text{ if } 2^{s+2}\le n+r<2^{s+3},$$

Note that, there are number change between cases, then

$$j(n+r)=2(n+r)-2^{s+a+1}-1 \text{ if } 2^{s+a}\le n+r<2^{s+a+1}$$

The proof of (7) is the same process with proof of(4).

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Reference

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