Anna Knezevic

Greg Cohen

Some Facts on Permanents

in Finite Characteristics

Abstract:

The permanent's polynomial-time computability over fields of characteristic 3 for k-semi-unitary matrices (i.e. n×n-matrices A such that $rank(AA^T - I_n) = k$) in the case $k \leq 1$ and its $\#_3P$-completeness for any k > 1 (Ref. 9) is a result that essentially widens our understanding of the computational complexity boundaries for the permanent modulo 3. Now we extend this result to study more closely the case k > 1 regarding the (n-k)×(n-k)-sub-permanents (or permanent-minors) of a unitary n×n-matrix and their possible relations, because an (n-k)×(n-k)-submatrix of a unitary n×n-matrix is generically a k-semi-unitary (n-k)×(n-k)-matrix.

The following paper offers a way to receive a variety of such equations of different sorts, in the meantime also extending (in its second chapter divided into subchapters) this direction of research to reviewing all the set of polynomial-time permanent-preserving reductions and equations for a generic matrix's sub-permanents they might yield, including a number of generalizations and formulae (valid in an arbitrary prime characteristic) analogical to the classical identities relating the minors of a matrix and its inverse. Moreover, the second chapter also deals with the Hamiltonian cycle polynomial in characteristic 2 that surprisingly demonstrates quite a number of properties very similar to the corresponding ones of the permanent in characteristic 3.

Besides, the paper's third chapter is devoted to the computational complexity issues of the permanent and some related functions on a variety of Cauchy matrices and their certain generalizations, including constructing a polynomial-time algorithm (based on them) for the permanent of an arbitrary square matrix in characteristic 5 and conjecturing the existence of a similar scheme in characteristic 3.

Throughout the paper, we investigate various matrix compressions and transformations preserving the permanent and related functions in certain finite characteristics. And, as an auxiliary algebraic tool supposed for an application when needed in all the constructions we're going to discuss in the present article, we'll introduce and utilize a special principle involving a field's extension by a formal infinitesimal and allowing, provided a number of conditions are fulfilled, to reduce the computation of a polynomial over a field to solving a system of algebraic equations in polynomial time.

The neighboring computation principle

(for any characteristic).

(We'll denote by $\delta(q)$ the natural function in the natural variable q that equals unity if q is zero and equals zero otherwise, and by $\mathfrak{J}(b(t), t)|_{t=g}$ the Jacobian matrix of the vector-function b(t) on the vector-variable t computed in the point t = g. For a field F, we'll denote by $F(\varepsilon)$ F's extension by the formal infinitesimal variable ε, i.e. $F(\varepsilon) := \{a = \sum_{k=order_\varepsilon(a)}^{\infty} a_k \varepsilon^k, order_\varepsilon(a) \in \mathbb{Z}, a_k \in F \text{ for } k = order_\varepsilon(a), \dots, \infty\}$).

Let $f(u)$ be a polynomial in $u_1, \dots, u_{dim(u)}$ of degree d over a field F,

$\hat{u} = \hat{u}(h_1, \dots, h_{dim(h)}), v = v(h_1, \dots, h_{dim(h)})$ be two analytical vector-functions in the parameter vector-variable h such that $dim(\hat{u}) + dim(v) \leq dim(h)$ and $h^{[0]}$ be a value of this vector-variable such that $\hat{u}(h^{[0]})$ exists, $v(h^{[0]}) = \vec{0}_{dim(v)}$,

$\mathfrak{J}(\begin{pmatrix} \hat{u}(h) \\ v(h) \end{pmatrix}, h)\Big|_{h=h^{[0]}}$ exists and is nonsingular.

Then, given a value $u^{[0]}$ of the vector-variable u, over $F(\varepsilon)$ (where ε is a formal infinitesimal)

$$f(u^{[0]}) = \sum_{i=0}^{d} coef_{\varepsilon^i}(f(\hat{u}(\sum_{k=0}^{d} \varepsilon^k h^{[k]})))$$

where the dim(h)-vectors $h^{[1]}, \dots, h^{[d]}$ satisfy the equations

$$\mathfrak{J}(\begin{pmatrix} \hat{u}(h) \\ v(h) \end{pmatrix}, h)\Big|_{h=h^{[0]}} h^{[k]} = \begin{pmatrix} \delta(k-1)(-\hat{u}(h^{[0]}) + u^{[0]}) - coef_{\varepsilon^k}(\hat{u}(\sum_{i=0}^{k-1} \varepsilon^i h^{[i]})) \\ -coef_{\varepsilon^k}(v(\sum_{i=0}^{k-1} \varepsilon^i h^{[i]})) \end{pmatrix}$$

$$\text{for } k = 1, \ldots, d$$

(and thus $\hat{u}\left(\sum_{k=0}^{d} \varepsilon^k h^{[k]}\right) = \hat{u}\left(h^{[0]}\right) + \varepsilon\left(-\hat{u}\left(h^{[0]}\right) + u^{[0]}\right) + O\left(\varepsilon^{d+1}\right)$,

$v\left(\sum_{k=0}^{d} \varepsilon^k h^{[k]}\right) = O\left(\varepsilon^{d+1}\right)$)

This method will be called **the neighboring computation of the polynomial $f(u)$ in the point $u^{[0]}$ via the parameterization $\hat{u}(h)$ in the region $v(h) = \vec{0}_{\dim(v)}$ from the bearing point $h^{[0]}$.**

Therefore if $f(\hat{u}(h))$ is computable in polynomial time for any h such that $v(h) = \vec{0}_{\dim(v)}$ (including the ε-power series $h = \sum_{k=0}^{d} \varepsilon^k h^{[k]} + O\left(\varepsilon^{d+1}\right)$ whose members of degrees higher than d are the solutions of the above equations for $k > d$) and there exists a bearing point $h^{[0]}$ such that $\hat{u}\left(h^{[0]}\right)$ exists, $v\left(h^{[0]}\right) = \vec{0}_{\dim(v)}$,

$\Im\left(\begin{pmatrix}\hat{u}(h) \\ v(h)\end{pmatrix}, h\right)\Big|_{h=h^{[0]}}$ exists and is nonsingular then $f(u^{[0]})$ is computable in polynomial

time for any $u^{[0]}$ too.

In the further, for the purpose of simplicity, we'll call a system of functions S algebraically *absolutely* independent in a region R (given by a system of equations with a zero right part) if and only if the joint system of functions consisting of S and the left part of the system representing R is algebraically independent at some point of R.

We'll also define a computational circuit as ***arithmetically polynomial-time*** over a field if it consists of a polynomial-time number of arithmetic operations over the field.

> *The above principle hence tells us that a polynomial in n variables over a field is computable in arithmetically polynomial (in n) time over the field when its calculation can be arithmetically polynomial-time (in n, over the field) reduced to finding a solution of an algebraic equation system for a polynomial (in n) number of some other variables that consists of a polynomial (in n) number of equations represented by arithmetically polynomial-time (in n, over the field) computable and analytical (over the field) functions in these new variables.*

I. Equalities for the sub-permanents of a unitary matrix over fields of characteristic 3

Definitions:

Let A be an nxn-matrix, I, J be two subsets of {1,…,n} of an equal cardinality. Then we define its *I→J-replacement* matrix $A^{[I\to J]}$ as the matrix received from A through replacing its rows with indexes from J by those with indexes from I, i.e. through replacing its i_k-th row by its j_k-th one for k =1,…,|I|.

Analogically, given two pairs I,J and K,L of subsets of {1,…,n} such that |I|=|J| and |K|=|L|, we define its *I→J,K→ L-double-replacement* matrix $A^{[I\to J,K\to L]}$ as the matrix received from A through replacing its rows with indexes from J by those with indexes from I and its columns with indexes from L by those with indexes from K.

We also define its *I,J-repeat* matrix $A^{[I,J]}$ as the matrix received from A through repeating twice its rows with indexes from I and its columns with indexes from J (while the pairs of doubled rows or columns receive neighboring indexes. i.e. the doubled rows and columns follow each other).

By $A^{(I,J)}$ we'll denote the matrix lying on the intersection of the rows with indexes from I and the columns with indexes from J, and by $A^{(\backslash I,\backslash J)}$ we'll denote the matrix received from A through removing its rows with indexes from I and its columns with indexes from J.

For the purpose of simplicity, for a 1-set {i} we'll omit the brackets {} and write just i instead.

Theorem I.1:

Let U be a unitary nxn-matrix, I, J be two disjoint subsets of {1,…,n} of an equal cardinality.

Then $\mathrm{per}\left(U^{[I\to J]}\right) = (-1)^{|I|}\mathrm{per}\left(U^{[J\to I]}\right)$

Proof:

To prove this theorem, we should effectively apply the principal equality expressing the permanent of an nxn-matrix through its "principal minor convolution", i.e.

$$(1) \quad \mathrm{per}(A) = (-1)^n \sum_{L,L\subseteq\{1,…,n\}} \det(A^{(L,L)})\, \det(A^{(\backslash L,\backslash L)})$$

First of all, as the permanent of a square matrix doesn't change after any permutation of its rows and a unitary matrix remains unitary after any permutation of its rows, we can assume I={1,3,…,2k-1}, J={2,4,…,2k} because we always can permute the rows of U

so that the latter condition is fulfilled. Therefore proving the theorem for this pair of sets I, J is equivalent to proving it for the generic case. Hence, each of the two rows of the matrix $U^{[I\to J]}$ with the indexes 2q-1, 2q are the (2q-1)-th row of U, q=1,…,k.

Since the matrix $U^{[I\to J]}$ for |I|=k has k doubled rows, the sum over T in the above equality (1) can be replaced by the sum over those T that contain exactly one element from each pair 2q-1, 2q for q=1,…,k.

And now we apply the equality expressing a minor of a square matrix A through a minor of its inverse (for L,M being subsets of {1,…,n} of an equal cardinality):

$$(2) \qquad \det(A^{(L,M)}) = \det(A)\det((A^{-1})^{(\backslash M, \backslash L)})(-1)^{\Sigma_{l\in L} l + \Sigma_{m\in M} m}$$

For a unitary U this formula just takes the form

$$(3) \qquad \det(U^{(L,M)}) = \det(U)\det(U^{(\backslash L, \backslash M)})(-1)^{\Sigma_{l\in L} l + \Sigma_{m\in M} m}$$

while in such a case the convolution equality (1) for the matrix $U^{[I\to J]}$ yields:

$$\text{per}(U^{[I\to J]}) =$$
$$(-1)^n \sum_{R, R\subseteq \{1,…,n\}\backslash\{I\cup J\}} \sum_{h\in\{0,1\}^k} \det(U^{(R\cup I, R\cup G(h))}) \det(U^{(\backslash\{R\cup J\}, \backslash\{R\cup G(h)\})})$$

where $G(h) = \{2 - h_1 \cup … \cup 2k - h_k\}$.

After the application of the formula (3) to the latter equality, we receive

$$\text{per}(U^{[I\to J]}) =$$

$$(-1)^n \sum_{R, R\subseteq \{1,…,n\}\backslash\{I\cup J\}} \sum_{h\in\{0,1\}^k} (-1)^k \det(U^{(\backslash\{R\cup I\}, \backslash\{R\cup G(h)\})}) \det(U^{(R\cup J, R\cup G(h))})$$

as all the indexes of the involved minors are doubled except 1,…,2k each of whom appears exactly once in the corresponding sum of indexes (according to the formula (3)) and their sum is equal to k modulo 2. Hence, we get the theorem.

Theorem I.2:

Let U be a unitary n×n-matrix, I,J be two subsets of {1,…,n} of an equal cardinality.

Then $\text{per}(U^{[I,J]}) = (-1)^{|I|}\text{per}(U^{(\backslash I, \backslash J)})$

Proof:

The proof of this theorem virtually repeats the proof of Theorem I.1, including the preliminary permutations of repeated rows and repeated columns that make their indexes belong to the set {1,…,2k}, where |I|=|J|= k. I.e. we can assume, beforehand, that I = J = {1,…,k} − for the same reason as in the proof of Theorem I.1, while preserving the degree of commonness. In such a case in the corresponding convolution sum all the indexes would be repeated twice when passaging to the inverse's minors, while each product of principal minors will have the coefficient $2^{|I|} = (-1)^{|I|}$

Theorem I.3: given two pairs I,J and K,L of subsets of {1,…,n} such that

$$|I|=|J|=|K|=|L|, \ I \cap J = K \cap L = \emptyset,$$

$$\mathrm{per}\big(U^{[I \to J, K \to L]}\big) = \mathrm{per}\big(U^{[J \to I, L \to K]}\big)$$

Proof:

Once again, this theorem can be easily proven in the same way as Theorems I.1, I.2, while assuming I = K = {1,3,…,2k-1}, J = L = {2,4,…,2k}.

Definition:

For an n×n-matrix A and $k \le n$, let's define its k-th *permanent-minor matrix* $P(A, k)$ as a $C_n^k \times C_n^k$-matrix whose rows and columns are indexed by k-subsets of {1,…,n} and whose I,J-entry $p_{I,J}(A, k) = \mathrm{per}\big(A^{(I,J)}\big)$ for a pair of k-subsets I,J.

Let's also define its k-th *permanent-complement matrix* $F(A, k)$ as a $C_n^k \times C_n^k$-matrix whose rows and columns are indexed by k-subsets of {1,…,n} and whose I,J-entry $f_{I,J}(A, k) = \mathrm{per}\big(A^{(\backslash I, \backslash J)}\big)$ for a pair of k-subsets I,J.

Obviously, $P(U^T, k) = P^T(U, k)$ and $F(U^T, k) = F^T(U, k)$.

Corollary I.4: let U be unitary. Then

(*) $F(U, k)P^T(U, k) = (-1)^k P(U, k)F^T(U, k) \star \{(-1)^{|I \cap J|}\}_{C_n^k \times C_n^k}$

where $\star$ denotes the Hadamard (i.e. entry-wise) product of matrices.

 Corollary I.5:

(**) $(-1)^{k+1}F(U, k) + P(U, k)F^T(U, k)P(U, k) =$

$$= -\{\sum_{s=0}^{k-1} \sum_{\substack{\hat{I} \subset I, \hat{J} \subset J \\ |\hat{I}|=|\hat{J}|=s}} (P(U,s)F^T(U,s)P(U,s))_{\hat{I},\hat{J}} \, p_{I \setminus \hat{I}, J \setminus \hat{J}}(U, k-s)\}_{C_n^k \times C_n^k} = 0$$

Both the above corollaries follow from Theorems I.1 and I.2 correspondingly and the Laplace expansions of the permanent for a set of rows and for a set of rows and a set of columns.

The equalities (*) and (**) are actually linear equations expressing the entries of F(U,k) through the entries of F(U,s) for s<k.

We can also notice that for a unitary U its replacement matrix for |I|=|J|= 1 is 1-semi-unitary and, therefore, we can compute its permanent in a polynomial time, while for a unitary U and four pair-wise distinct indexes i, j, s, r (where s<r)

(***) $\mathrm{per}\big(U^{[\{i,s\} \to \{j,r\}]}\big) - \mathrm{per}\big(U^{[\{i,r\} \to \{j,s\}]}\big) = \mathrm{per}\big(M_{s,r}U^{[i \to j]}\big)$

where $M_{s,r}$ is the identity matrix I_n where the s-th and r-th rows were left-multiplied by the unitary matrix $\begin{pmatrix} \sqrt{-1} & \sqrt{-1} \\ -\sqrt{-1} & \sqrt{-1} \end{pmatrix}$ (hence $M_{s,r}U^{[i \to j]}$ is also 1-semi-unitary as a unitary row-transformation of the 1-semi-unitary matrix $U^{[i \to j]}$).

Lemma I.6.

Let U be unitary, i<j. Then $\mathrm{per}\big(U^{[i \to j]}\big) = \mathrm{per}\big(M_{i,j}U\big)$

Lemma I.7.

Let U be unitary, i<j, s<r, |{i,j,r,s}|=4. Then

$$\mathrm{per}\big(U^{[\{i,s\} \to \{j,r\}]}\big) - \mathrm{per}\big(U^{[\{i,r\} \to \{j,s\}]}\big) = \mathrm{per}\big(M_{s,r}M_{i,j}U\big)$$

Since, by the Laplace expansion of the permanent for a set of rows, the I,J-entry of the matrix $P(U,2)F^T(U,2)$ equals $\mathrm{per}\big(U^{[I \to J]}\big)$, i.e $(P(U,2)F^T(U,2))_{I,J} = \mathrm{per}\big(U^{[I \to J]}\big)$, the equalities (***) (together with the fact that if $|I \cap J| > 0$ then $U^{[I \to J]}$ has either precisely one replaced row or no replaced rows and, accordingly, is 1-semi-unitary or unitary correspondingly) signify that the matrix $F(U,2)P^T(U,2) = (P(U,2)F^T(U,2))^T$ can be polynomial-time expressed as the sum of a known matrix and a matrix X(U) with

the following properties: $x_{I,J}(U) = x_{K,L}(U)$ if $I \cup J = K \cup L$ and $x_{I,J}(U) = 0$ if $|I \cap J| >$ 0. We'll call a matrix *super-symmetric* if its rows and columns are indexed by 2-subsets of $\{1,\dots,n\}$ and it satisfies the two latter conditions. Hence, analogically, the matrix $P^T(U,2)F(U,2) = P(U^T,2)F^T(U^T,2)$ can also be expressed as the sum of a known matrix and a super-symmetric $Y(U)$. Accordingly, if in the generic case of a unitary U the homogeneous system of linear equations for two super-symmetric matrices X and Y $P(U,2)X - YP(U,2) = 0$ is non-singular and, therefore, its only solution is zero, we can polynomial-time compute the entries of F(U,2) because the two above-mentioned expressions yield a non-singular system of linear equations for X(U), Y(U) (via expressing F(U,2) through X(U) and Y(U) correspondingly from these expressions). As an instrument for studying the equation $P(U,2)X - YP(U,2) = 0$, we can apply, for $n \equiv 0 \pmod m$, the unitary matrix $U = \text{Diag}(\{W_q\}_{q=1,\dots,\frac{n}{m}})$ where $W_1, \dots, W_{n/m}$ are unitary m×m-matrices.

Analogically, we can polynomial-time compute the differences

(****) $\text{per}(U^{[i \to j, r \to s]}) - \text{per}(U^{[i \to j, s \to r]}) = \text{per}(U^{[i \to j]})M_{s,r}$

because $\text{per}(U^{[i \to j]})M_{s,r}$ is also 1-semi-unitary for the same reason as $M_{s,r}U^{[i \to j]}$.

It provides us with even more equations for the entries of the matrix F(U,2). Similar unitary linear combinations of k-1 pair-wise disjoint pairs of rows in the matrix $U^{[i \to j]}$ would lead to some linear equations relating the entries of the matrices F(U,k),…,F(U,1). And the following question arises accordingly: whether it may appear that all those equations form a non-singular system for finding those matrices for some k>1 in the generic case of a unitary U. If it's so, we may easily reduce their computation in any special case of our interest (particularly, significant for proving P=NP) to computing those matrices in the most generic case -- hence implying P=NP. As a necessary tool for such a research, we can offer the neighboring computation principle.

Let's call the equation (***) the (i,j;r,s)-replacement-shift equation for the matrix U and the equation (****) the (i,j;r,s)-double-replacement-shift equation for U.

Let's also define the matrix $B(U,\alpha) = \begin{pmatrix} \alpha I_n & \sqrt{1-\alpha^2}U^T \\ \sqrt{1-\alpha^2}U & -\alpha I_n \end{pmatrix}$ which we'll call the α-block-composition of U where α is an element of a field. It's easy to see that $B(U,\alpha)$ is unitary when U is unitary. We'll now consider, for a ground field H the entries of U belong to, its α-extension $H(\alpha)$ whose elements are formal power series in α, i.e. having

the form $h = \sum_{t=k}^{\infty} h_t \alpha^t$ where k we'll call the smallness-order of h (or just the order of h) order(h).

Conjecture I.8.

For the generic case of a unitary nxn-matrix U, the set of (i,j;s,r)-replacement-shift equations for U and U^T and (i+n,j;r+n,s)-replacement-shift equations (considered only for the power α^2) for $B(U, \alpha)$ and $B(U^T, \alpha)$, where i,j,s,r are from {1,...,n}, form an algebraically complete (i.e. having a nonsingular Jacobian matrix) system of equations for the entries of F(U,2).

Actually, while proving the above conjecture looks yet too difficult at the present time, we can try to experimentally check it via a computer modeling on a random U.

II.   Reviewing the permanent-minors and other permanents derived from a unitary matrix from a much wider point of view.

1. The permanent-analog of the inverse's minor formula.

Let A be an n×n-matrix over a field of a prime characteristic p, α, β be two n-vectors having all their entries from the set {0,...,p-1}, i.e. $\alpha, \beta \in \{0, \ldots, p-1\}^n$. Then let's denote by $A^{(\alpha,\beta)}$ the matrix received from A through repeating α_i times its i-th row for i = 1,...,n and β_j times its j-th column for j=1,...,n (if some α_i or β_j equals zero it would mean we remove the i-th row or j-th column correspondingly). Then, in case if $A^{(\alpha,\beta)}$ is square, i.e. $\sum_{i=1}^{n} \alpha_i = \sum_{j=1}^{n} \beta_j$, the following identity holds

Theorem II.1.1 (in characteristic p):

$$\mathrm{per}(A^{(\alpha,\beta)}) = \mathrm{det}^{p-1}(A)\,\mathrm{per}((A^{-1})^{((p-1)\vec{1}_n - \beta, (p-1)\vec{1}_n - \alpha)})\frac{\prod_{i=1}^{n} \alpha_i\,!}{\prod_{j=1}^{n}(p-1-\beta_j)\,!}$$

where $\vec{1}_n$ is the n-vector all whose coordinates are equal to 1.

The above identity can be also written as

$(*)$ $\quad$ $\mathrm{per}(A^{(\alpha,\beta)}) =$

$$= \det{}^{p-1}(A)\,\mathrm{per}\big((A^{-1})^{((p-1)\vec{1}_n-\beta,\,(p-1)\vec{1}_n-\alpha)}\big)\Big(\prod_{i=1}^{n}\alpha_i\,!\Big)\Big(\prod_{j=1}^{n}\beta_j\,!\Big)(-1)^{n+\Sigma_{i=1}^{n}\alpha_i}$$

Proof:

First of all, let's prove that

(1) $\quad$ $\big(\prod_{i=1}^{n}\frac{(p-1)!}{\alpha_i!}\big)\mathrm{per}(A^{(\alpha,\beta)}) = \mathrm{per}\big(I_n^{((p-1)\vec{1}_n,\,(p-1)\vec{1}_n-\alpha)}\quad A^{((p-1)\vec{1}_n,\beta)}\big)$

where in the right side is the permanent of a matrix composed of two blocks, the first block $I_n^{((p-1)\vec{1}_n,\,(p-1)\vec{1}_n-\alpha)}$ being block-diagonal itself with diagonal blocks of sizes $(p-1)\times(p-1-\alpha_i)$, i=1,…,n. The identity (1) follows from the Laplace expansion of the right permanent by the columns corresponding to the first block (this expansion is the direct product of the Laplace expansions for its diagonal blocks).

Secondly, if B is an m×((p-1)m)-matrix and G is an m×m-matrix then

(2) $\quad$ $\mathrm{per}\big((GB)^{((p-1)\vec{1}_m,\,\vec{1}_{(p-1)m})}\big) = \det{}^{p-1}(G)\,\mathrm{per}\big(B^{((p-1)\vec{1}_m,\,\vec{1}_{(p-1)m})}\big)$

as in characteristic p the permanent doesn't change if each row of a matrix is repeated p-1 times and we add one of its (p-1)-tuples of equal rows to another (p-1)-tuple of equal rows, and is multiplied by d^{p-1} if we multiply a (p-1)-tuple of equal rows by d.

Upon applying the formula (2) to the case $B = \big(I_n^{(\vec{1}_n,\,(p-1)\vec{1}_n-\alpha)}\quad A^{(\vec{1}_n,\beta)}\big)$, $G = A^{-1}$, we'll receive an identity involving a two-blocked matrix with the second block being block-diagonal itself (like in the identity (1)) and hence analogous to the identity (1) what will give us the initial identity.

This identity is, first of all, a generalization (for an arbitrary prime characteristic p) of all the repeat-removal identities we received in characteristic 3, and, secondly, the permanent-analog of the classical formula for the matrix inverse's minor.

Besides, in characteristic p there is the following pair of dual identities for an n×n-matrix A:

$$\mathrm{per}(A) = (-1)^n \sum_{J_1,\dots,J_{p-1}} \det\big(A^{(J_1,J_1)}\big)\dots\det\big(A^{(J_{p-1},J_{p-1})}\big)$$

$$\det(A) = (-1)^n \sum_{J_1,\dots,J_{p-1}} \mathrm{per}\big(A^{(J_1,J_1)}\big)\dots\mathrm{per}\big(A^{(J_{p-1},J_{p-1})}\big)$$

where in both the above formulas the summation is over all the (p-1)-tuples $J_1, \ldots, J_{p-1}$ that are partitions of the set {1,…,n} into p-1 subsets, some of them possibly empty, while p isn't obliged to be prime.

2. Permanent-preserving compressions over fields of characteristic 3

2.1 The basic compression

Let A be an n×n-matrix over a field of characteristic 3 with at least one pair of equal rows. Let i,j (i<j) be the indexes of the lexicographical minimum (index-wise) of such pairs of rows. We'll define the compression of A Comp(A) as the (n-1)×(n-1)-matrix received from A through making zero (via the Gauss algorithm) all the entries of the first column of A by its i-th row (having $a_{i,1}$ as the leading entry for the column elimination) and then removing the first column and the j-th row of the received matrix. Then

$$\text{per}(A) = -a_{i,1}\text{per}(\text{Comp}(A))$$

We'll also define the compression-closure of A $\widehat{\text{Comp}}(A)$ as the limit of the compression operator's sequential application to A or, if at some stage the received matrix is incompressible, to its transpose (i.e. the limit of actions when we compress the matrix and transpose it if no rows are equal any more but there are equal columns, until it would have no equal rows or equal columns). We can also speak about applying the compression and compression-closure operators to sets of matrices that would map them into another sets of matrices. It's obvious that, if we denote by $\mathbb{U}_k$ the set of k-semi-unitary matrices, $\widehat{\text{Comp}}(\mathbb{U}_0) = \mathbb{U}_0$, i.e. unitary matrices are incompressible because they are non-singular and can't have equal rows.

But, if we take a unitary matrix with one row replaced by another one (whose permanent we can polynomial-time compute) and multiply both copies of the repeated row by $\sqrt{-1}$, then such a matrix will be both 1-semi-unitary and compressible, and, though strange, its compression won't be 1-semi-unitary but will be 2-semi-unitary instead. Hence $\text{Comp}(\mathbb{U}_1) \subset \mathbb{U}_2$ and the latter fact raises somewhat a hope that $\widehat{\text{Comp}}(\mathbb{U}_1)$ is a set of matrices that is $\#_3P$ complete and,

in such a case, we can use the neighboring computation principle to prove $\#_3 P = P$ and, therefore, P = NP.

2.2 The generalized compression.

Let A be an n×n-matrix over a field of characteristic 3 having at least one linearly dependent triple of rows, i.e. a triple of rows with pair-wise distinct indexes i, j, k such that $a_k = ga_i + ha_j$ where g and h are some elements of the field (we also assume a_i and a_j are linearly independent). Then adding the row $ga_i - ha_j$ multiplied by any element of the field to any row of A except the i-th, j-th and k-th ones doesn't change the permanent because of its row-wise multi-linearity and the fact that the permanent of a matrix having four rows a_i, a_j, $ga_i + ha_j$, $ga_i - ha_j$ is zero in characteristic 3. Hence we can eliminate, while assuming that the first entry of the row $ga_i - ha_j$ is non-zero (or permuting A's columns for to fulfill this condition otherwise) and using it as the Gaussian column-elimination's leading entry, the first column of A except the entries a_{i1}, a_{j1}, a_{k1}. Then per(A) equals the permanent of the matrix received from A through replacing its i-th, j-th and k-th rows by the pair of rows $a_{j1}a_i - a_{i1}a_j$, $ga_i - ha_j$ and removing its first column. We'll call such a compression a triple-compression (as it involves a triple of linearly dependent rows), while the case of two linearly dependent rows $a_j = da_i$, where d is some element of the field, we'll call a pair-compression (i.e. we can divide the j-th row by d while the permanent will also be divided by d and hence we'll receive the above-described case of equal rows). In fact, the pair-compression is a partial case of the triple-compression by putting d=g, h=0, but, nevertheless, these are two cases of permanent-preserving matrix compressions we'll distinguish and in the further let's understand by Comp(A) the lexicographically least (index-wise) pair- or triple-compression of A. In the meantime, the compression-closure operator's definition won't change in this generalization of the compression-operator.

Thus the question of determining the structure of the matrix class $\widehat{\text{Comp}}(\mathbb{U}_1)$ becomes even more intriguing and challenging towards the chief mystery P versus NP. Actually we can even consider, instead of $\mathbb{U}_1$, the wider class $\mathbb{U}_{(1)} \subset \mathbb{U}_2$ of matrices received from unitary ones via replacing one row by an arbitrary vector-row (by the Laplace expansion, the permanent of such a matrix is the sum of the permanents of 1-semi-unitary matrices and hence polynomial-time computable and if the replacing vector-row is a linear combination of the matrix's two other rows then such a matrix is triple-compressible) and, accordingly, study the class $\widehat{\text{Comp}}(\mathbb{U}_{(1)})$.

It would be also useful to notice that the identity given and proven in the first section of this paper that links the generalized permanent-minors of a matrix and its inverse is, if considered only for characteristic 3, merely an application of the pair-compression operator to certain matrices. Accordingly, the following questions could be raised: what family of identities may we receive in characteristic 3 when applying the most general compression, i.e. including the triple case, and, actually, what are the possible analogs of the permanent-preserving compressions we found in characteristic 3 for other prime characteristics?

An answer to the former of these two questions might be gotten via studying an arbitrary $(3n)\times(3n)$-matrix consisting of n linearly dependent triples of rows whose compression-closure would be of size at most $(2n)\times(2n)$. It could also provide us, upon permuting the matrix's columns so that a chosen n-subset of its column set will turn into $\{1,\ldots,n\}$, with an opportunity to determine a relation between all the $(2n)\times(2n)$ matrices (thus having equal permanents) we may receive in this way. Let's call two $(2n)\times(2n)$-matrices triple-conjugate if they can be received via such a procedure from one initial $(3n)\times(3n)$-matrix consisting of n linearly dependent triples of rows. By the way, if we apply the same scheme for a $(2n)\times(2n)$-matrix consisting of n pairs of equal rows and the pair-compression operator then we'll receive $n\times n$-matrices that are partial inverses to each other. Hence, while computing the permanent in char 3, we can transfer not only to the matrix's partial inverse, but to its triple-conjugate as well (however, beforehand we should actually verify that, in fact, a triple-conjugate is (generically) not a partial inverse but its genuine generalization).

2.3 A wider generalization of permanent-preserving compressions.

Let A be a square matrix such that its first 2k rows form a matrix of rank k, i.e., generically, its rows with the indexes $k+1,\ldots,2k$ are linear combinations of its first k rows with a coefficient $k\times k$-matrix B. Then per(A) is equal to the product of per(B) and the permanent of the matrix received from A through removing its rows with the indexes $k+1,\ldots,2k$ and doubling its rows with the indexes $1,\ldots,k$. In such a case we'll receive a matrix with k pairs of equal rows to which we can apply (k times) the pair-compression operator in order to reduce its size by k. We'll call such a compression an *even* compression.

Let A be a square matrix such that its first 2k-1 (for k>1) rows form a matrix of rank k, i.e., generically, its rows with the indexes $k+1,\ldots,2k-1$ are linear combinations of its first k rows with a coefficient $(k-1)\times k$-matrix B. Then per(A) is equal to the permanent of the matrix received from A through removing its rows with the indexes $k+1,\ldots,2k-1$, doubling

its rows with the indexes 1,…,k, and, afterwards, adding one new column whose entries corresponding to both copies of the q-th row are $-\text{per}(B^{(\{1,…,k-1\},\{1,…,k\}\setminus q)})$ for q=1,…,k and all the other entries are zeros. (Please notice that the added column ensures the matrix remains square). In such a case we'll receive a matrix with k pairs of equal rows to which we can apply (k times) the pair-compression operator in order to reduce its size by k. We'll call such a compression an *odd* compression.

In both the above cases of even and odd compressions, we just should suppose, naturally, that k is fixed so that per(B) or $\text{per}(B^{(\{1,…,k-1\},\{1,…,k\}\setminus q)})$ correspondingly could be computed in a polynomial time, or that we, at least, can polynomial-time calculate these values in some way otherwise. And, apparently, we may speak about applying to a 2k- or (2k-1)-set of rows of rank k the 2k- or (2k-1)-compression operator correspondingly (as stated above) upon an appropriate permutation of the matrix's rows turning the set into {1,…,2k} or {1,…,2k-1}. We'll call k and k-1 the compression's *velocity* for an even and odd compression correspondingly as they lessen the matrix's size by k and k-1 correspondingly. We can also prove that a compression transformation doesn't change the matrix's rank (although changing, possibly, its semi-unitarity class), while transferring to a partial inverse doesn't change the matrix's semi-unitarity but, nevertheless, may change its rank. Hence, coupled together, the compression operator and all the partial inversions form a yet more perfect instrument for permanent-preservingly compressing matrix classes, first of all $\mathbb{U}_{(1)}$. If we define transferring to a partial inverse as a compression of velocity 0, we might be able to compute the permanent on a yet more reach class $\widehat{\text{Comp}}(\mathbb{U}_{(1)})$.

Actually, it's easy to notice that the earlier mentioned pair- and triple-compressions are merely partial cases of even and odd compressions correspondingly (that are, in fact, their generalizations), and, though strange, the triple-compression itself can be expressed as a double pair-compression (in a beforehand modified matrix, though). Hence since now we can start defining the compression of a matrix as its lexicographically least (index-wise) even or odd compression, with the same notion of the matrix's compression-closure.

2.4 A criterion of the permanent's equality to zero

Let A be a square matrix such that its rows with the indexes k+1,…,k+m are linear combinations of its first k rows with the coefficient m×k-matrix B such that all its m×m-subpermanents (or permanent-minors) are zero. Then per(A) = 0. In characteristic 3, an example of such a matrix B is the matrix C(x,y) where dim(y) < 2dim(x) and the joint

vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is the root vector of a polynomial that is the derivative of another polynomial. This fact is based on Lemma that is to be given further in the article.

2.5 The partial inverse equivalence and classification

of permanent-preserving compressions.

Lemma II.2.5.1 (on the permanent of a partial inverse):

over a field of characteristic 3, for A_{11}, A_{22} being square,

$$(**)\ \ \mathrm{per} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det{}^2(A_{11}) \mathrm{per} \begin{pmatrix} A_{11}^{-1} & A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

The proof of the above formula can be received via the technique applied in proving the analog of the inverse's minor formula for permanent-minors in Part 1 of this article.

Apparently, the latter formula is a generalization of the formula for the permanent of a matrix's inverse in characteristic 3, i.e., for a square non-singular A, $\mathrm{per}(A) = \det{}^2(A)\mathrm{per}(A^{-1})$. In the meantime, in a generic prime characteristic p and with the same technique's usage, we can even similarly generalize the formula (*) for permanent-minors via giving to both parts of the formula (**) their row/column multiplicity degrees:

$$(***) \qquad \mathrm{per} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{(\alpha,\beta)} =$$

$$= \det{}^{p-1}(A_{11}) \mathrm{per} \begin{pmatrix} A_{11}^{-1} & A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}^{\left((p-1)\vec{1}_n - \beta,\, (p-1)\vec{1}_n - \alpha \right)} .$$

$$\cdot \left(\prod_{i=1}^{n} \alpha_{1,i}! \right) \left(\prod_{j=1}^{n} \beta_{1,j}! \right) (-1)^{n_1 + \sum_{i=1}^{n} \alpha_{1,i}}$$

where:

for an n×m-matrix M, an n-vector x and an m-vector y, both vectors having all their entries from the set {0,...,p-1}, $M^{(x,y)}$ denotes the matrix received from M via repeating x_i times its i-th row for i = 1,...,n and y_j times its j-th column for j = 1,...,m (if some row's or column's multiplicity equals zero it would mean that the row or column was removed, and thus this notion is a generalization of the notion of submatrix);

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{(\alpha,\beta)} = \begin{pmatrix} A_{11}^{(\alpha_1,\beta_1)} & A_{12}^{(\alpha_1,\beta_2)} \\ A_{21}^{(\alpha_2,\beta_1)} & A_{22}^{(\alpha_2,\beta_2)} \end{pmatrix}, \ A_{11} \text{ is of size } n_1 \times n_1 \text{ and in the right part of}$$

the equality (***) each block-matrix is multiplicity-degreed correspondingly (while A_{11}

and $A^{(\alpha,\beta)} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{(\alpha,\beta)}$ are square).

Corollary II.2.5.2 (in characteristic 3): over a field of characteristic 3, for A_{11} of size $n_1 \times$ n_1 and invertible and A_{22} of size $n_2 \times n_2$, let the last n_2 rows of $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be linearly expressible through its first n_1 rows (what implies $A_{22} - A_{21}A_{11}^{-1}A_{12} = 0_{n_2 \times n_2}$). Then

$$\mathrm{per}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det^2(A_{11})\mathrm{per}\begin{pmatrix} A_{11}^{-1} & A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1} & 0_{n_2 \times n_2} \end{pmatrix}$$

The above corollary can be used for another interpretation of the earlier introduced even and odd compressions: if we permute A's rows so that all the linearly dependent rows we refer to in the corresponding definitions would form the second block-row in the received matrix's block-decomposition then its corresponding (to the block-decomposition) partial inverse will have the form

$$\begin{pmatrix} A_{11}^{-1} & A_{11}^{-1}A_{12} \\ (B \quad 0_{k \times (n_1 - k)}) & 0_{k \times k} \end{pmatrix} \text{ or } \begin{pmatrix} A_{11}^{-1} & A_{11}^{-1}A_{12} \\ (B \quad 0_{(k-1)\times(n_1-k)}) & 0_{(k-1)\times(k-1)} \end{pmatrix} \text{ correspondingly}$$

and, due to B being of size either k×k or (k-1)×k correspondingly, we can permanent-preservingly reduce this matrix via the Laplace-expansion for the second block-row (we'll call such compressions *primitive*). If there are several pair-wise disjoint sets of such linear dependencies we refer to by the definitions, they'll yield the direct product of corresponding primitive compressions. Hence even and odd compressions are equivalent to primitive ones via partial inverse reductions.

But there are yet compressions that are not (at least, so obviously) equivalent to primitive ones. For instance,

$$\mathrm{per}\left(\begin{pmatrix} \alpha & \beta & \alpha+\beta \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \quad A_{12} \right) = \mathrm{per}\left(\begin{pmatrix} r_{11}\alpha + r_{12}\beta & r_{21}\alpha + r_{22}\beta \\ d & e \end{pmatrix} \quad A_{12}\right)$$

where $r_{11}r_{21} = b_2$, $r_{12}r_{22} = b_1$, $r_{11}r_{22} + r_{12}r_{21} = b_1 + b_2 + b_3$,

$$r_{11}e + r_{21}d = \mathrm{per}\begin{pmatrix} 1 & 0 & 1 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

$$r_{12}e + r_{22}d = \text{per} \begin{pmatrix} 0 & 1 & 1 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

α, β are vector-columns, A_{12} is a matrix (of appropriate sizes), all the other values are elements of the ground field.

All the types of compression we discussed in Chapter 2 of the present article we'll call elementary. To summarize, we may, hence, conclude that in characteristic 3 there exists a whole variety of permanent-preserving compressions of a square matrix which, together with the set of partial inverse transformations, form the set of permanent-preserving and polynomial-time computable elementary transformations of a matrix. The problem of finding and classifying all of them is yet to be solved. And, accordingly, the compression-closure operator (understood as the closure-limit of those elementary compressions) is a pretty rich opportunity to reduce the size of a matrix whose permanent we need to know. Therefore the question of studying the compression-closure of important matrix classes we can polynomial-time compute the permanent on like $\mathbb{U}_{(1)}$ still arises as one of the chief mysteries related to the ever mysterious indefiniteness of P versus NP.

Besides, the formulae (*), (**), (***) provide us, when applied to a unitary matrix in characteristic 3, with another variety of linear equations for a unitary matrix's permanent-minors of a bounded depth (i.e. its sub-permanents received via removing k-sets of their rows and columns, with a bounded k) whose non-singularity (for a given maximum of k) is to be researched as well.

And, at last, it would be worth noting the following simple construction that, actually, is applicable in any characteristic. Let's call reducing the permanent via the Laplace expansion on a set I of k rows (or columns) containing only k or k+1 non-zero columns (or rows) **the Laplace compression on I**. If we extend an n×n-matrix whose first row has only two non-zero entries, one of them equal to 1 and another to -1, by one row and column so that

the new matrix's first column would contain only two non-zero entries, 1 and -1,

the Laplace compression on its first column would yield the original matrix,

the original matrix's first row would be involved in this extension,

and transpose the received matrix afterwards, -- then each time we can involve some arbitrary row vector as a parameter, let's call it a Laplace extension vector, while such

an extended matrix we'll call the elementary Laplace extension of the original matrix by a Laplace extension vector. Hence a generic sequence of elementary Laplace extensions will provide us with a matrix whose lower-right corner n×n-submatrix is the initial matrix or its transpose and whose permanent is the same as of the initial one. Actually such a sequence of elementary Laplace extensions could be started with an arbitrary matrix via extending it by one row and column so that the new matrix's first column would have only two non-zero entries, 1 and -1, and the new matrix's permanent would equal the permanent of the original one whose first row would be involved in the extension, and transposing the result afterwards (a Laplace extension vector is supposed here too, and we'll call it an initial Laplace extension). We'll also call the overall result of an elementary Laplace extension sequence the Laplace extension of a matrix by a sequence of Laplace extension vectors.

A special interest the Laplace extensions can present in characteristic 3 is the following question: what is the class of square matrices possessing 1-semi-unitary Laplace extensions? If, say, a generic square matrix can be Laplace-extended to a 1-semi-unitray one then this fact would yet imply, via the neighboring computational principle, the permanent's polynomial-time computability in characteristic 3. And, actually, even though earlier in this paper we somehow paralleled the elementary compressions based on row linear dependencies and those we've now called the elementary Laplace ones, still we also may investigate the above-defined (i.e. row linear dependency) compression-closure of the class of Laplace extensions generated by a given matrix or just a matrix class. We hence may conjecture that, despite the mutual expressibility of the elementary Laplace and row linear dependency compressions through each other, their self-generating chains (and, in fact, their compression-closures defined as the limits of action) pretty might appear to be nonequivalent. However, from the other hand, if we add the elementary Laplace compressions to the whole set of row linear dependency compressions (completed by the transpose and partial inverse transformations) we earlier introduced then we just may receive a much wider variety of elementary transformations whose compression-closure is to be studied – but, nevertheless, for the Laplace compression the structure of its corresponding extension (Laplace extension) perceived as its inverse modification (i.e. aka decompression) of a matrix is much more clear and can be expressed in a simple manifest form, while the row linear dependency compressions and their inverses (extensions) are apparently more difficult to express algebraically.

3. Some formulae for the hafnian
of a symmetric matrix in characteristic 3 and the Hamiltonian cycle polynomial in characteristic 2.

3.1 The even permanent

The approaches demonstrated and applied in the present article's first chapter for proving (in characteristic 3 only) a number of dependencies between the permanents of matrices received from a unitary one via certain row/column repeat/replacement modifications were in fact overlapped by the compression techniques and associated formulas that appeared in the second chapter. Nevertheless, the first chapter's methods of proof aren't yet deprived of some independent meaning as we can also use them (in characteristic 3 only as well) for proving various facts on the hafnian of a symmetric $(2n)\times(2n)$-matrix that is a generalization of the permanent of a square matrix. For this purpose, first of all let's define for a $(2n)\times(2n)$-matrix A its *even-permanent* as

$$\mathrm{per}_{\mathrm{even}}(A) = \sum_{\pi \in S_{2n}^{(\mathrm{even})}} \prod_{i=1}^{2n} a_{i,\pi(i)}$$

where $S_{2n}^{(\mathrm{even})}$ is the set of 2n-permutations having only cycles of even lengths.

Theorem II.3.1.1: let A be a $(2n)\times(2n)$-matrix. Then, in characteristic 3,

$$\mathrm{per}_{\mathrm{even}}(A) = \sum_{L,L\subseteq\{1,\dots,2n\}}(-1)^{|L|}\det(A^{(L,L)})\det(A^{(\backslash L,\backslash L)})$$

Hence, analogically to the permanent,

$$\mathrm{per}_{\mathrm{even}}(A) = \det^2(A)\mathrm{per}_{\mathrm{even}}(A^{-1})$$

Lemma II.3.1.2: let A be a symmetric $(2n)\times(2n)$-matrix. Then

$$\mathrm{haf}^2(A) = \mathrm{per}_{\mathrm{even}}(A)$$

We may also notice that the even-permanent of A doesn't depend on its diagonal entries. Secondly, in characteristic 3, if we represent a symmetric matrix in the form $\begin{pmatrix} d & b^T \\ b & M \end{pmatrix}$ where d is an element of the ground field and b is a (2n-1)-vector then $\mathrm{haf}\begin{pmatrix} d & b^T \\ b & M \end{pmatrix} = \mathrm{haf}\begin{pmatrix} d & b^T \\ b & M + \alpha bb^T \end{pmatrix}$ for any scalar coefficient α. The latter fact implies

the analogical (to the permanent) relation between the hafnians and even-permanents of a symmetric matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and its symmetric partial inverse:

$$\operatorname{haf}^2 \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det^2(A_{11}) \operatorname{haf}^2 \begin{pmatrix} A_{11}^{-1} & A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

and

$$\operatorname{per}_{\text{even}} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det^2(A_{11}) \operatorname{per}_{\text{even}} \begin{pmatrix} A_{11}^{-1} & A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

We can expect that, as a generalization of the permanent, the hafnian probably does possess its own types of compression, some of them analogical to certain types we've earlier found for the permanent, while others, perhaps, not. The one we would call primitive is to be applied to a symmetric matrix having the form $\begin{pmatrix} 0_{m,m} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where A_{12}'s number of non-zero columns equals m or m+1.

3.2 The case of characteristic 2: the Hamiltonian cycle polynomial.

Definition: let A be an n×n-matrix. Then its Hamiltonian cycle polynomial (or, shortly, the Hamiltonian of A) is defined as $\operatorname{ham}(A) := \sum_{\pi \in H_n} \prod_{i=1}^{n} a_{i,\pi(i)}$ where H_n is the set of Hamiltonian n-permutations, i.e. n-permutations having only one cycle.

Theorem II.3.2.1: let A be an n×n-matrix. Then

1) in an arbitrary characteristic:
$$\operatorname{ham}(A) := \sum_{I \in \{2,\dots,n\}} \det(-A^{(I,I)}) \operatorname{per}(A^{(\{1,\dots,n\}\backslash I, \{1,\dots,n\}\backslash I)})$$

2) in a finite characteristic p (not necessarily prime):
$$\operatorname{ham}(A) = (-1)^{n+1} \sum_{J_1,\dots,J_p} \det(A^{(J_1,J_1)}) \dots \det(A^{(J_p,J_p)}) =$$
$$= \sum_{J_1,\dots,J_p} \operatorname{per}(A^{(J_1,J_1)}) \dots \operatorname{per}(A^{(J_p,J_p)})$$

where the summation is over all the p-tuples $J_1, \dots, J_p$ that are partitions of the set {1,…,n} into p subsets (some of them possibly empty) such that $1 \in J_1$.

Theorem II.3.2.2 (in characteristic 2):

let A be an n×n-matrix. Then

$$\operatorname{ham}(A) := \sum_{I \in \{2,\dots,n\}} \det(A^{(I,I)}) \det(A^{(\{1,\dots,n\}\backslash I, \{1,\dots,n\}\backslash I)}) .$$

Theorem II.3.2.3 (in characteristic 2):

1) let U be a unitary n×n-matrix, i. e. such that $UU^T = I_n$. Then $\mathrm{ham}(U) = \det^2(U + I_n + C_{1,1})$ where $C_{1,1}$ is the n×n-matrix whose 1,1-th entry is 1 and all the others are zero.

2) let A be an involuntary n×n-matrix, i. e. such that $A^2 = I_n$.
   Then $\mathrm{ham}(A) = \det^2(A + I_n + C_{1,1}) = 0$ for n > 1.

The above theorem implies that, when n > 2, the Hamiltonian of an n×n-matrix having either three identical rows or a pair of indexes i,j such that its i-th and j-th rows are identical and its i-th and j-th columns are identical too equals zero.

While the former property generates, in this characteristic, a Hamiltonian-preserving compression of the Gaussian type (analogical to the simplest pair-compression for the permanent in characteristic 3 that possesses the same feature), the latter one (specific only for the Hamiltonian modulo 2) implies the following identity generating a type of Hamiltonian-preserving compressions applicable to certain structured unitary matrices (what makes the unitary class Hamiltonian-compressible in characteristic 2 like the 1-semi-unitary class is permanent-compressible in characteristic 3):

Theorem II.3.2.4 (in characteristic 2):

1) $\mathrm{ham}\left(\begin{pmatrix} V & V+D & A \\ V+D^{-1} & V+D^{-1}+D & A \\ B & B & U \end{pmatrix}\right) = \det(D+D^{-1})\,\mathrm{ham}\left(\begin{pmatrix} V & A \\ B & U \end{pmatrix}\right)$ where D is diagonal, V, U are square;

2) if U is unitary of size n×n, V is of size m×m, $VD + DV^T + AA^T = I_m + D^2$ then
   the matrix $\begin{pmatrix} V & V+D & A \\ V+D^{-1} & V+D^{-1}+D & A \\ UA^TD^{-1} & UA^TD^{-1} & U \end{pmatrix}$ is unitary and

   $\mathrm{ham}\left(\begin{pmatrix} V & V+D & A \\ V+D^{-1} & V+D^{-1}+D & A \\ UA^TD^{-1} & UA^TD^{-1} & U \end{pmatrix}\right) = \det^2\left(\begin{pmatrix} V+I_m+C_{1,1} & V+D & A \\ V+D^{-1} & V+D^{-1}+D+I_m & A \\ UA^TD^{-1} & UA^TD^{-1} & U+I_n \end{pmatrix}\right)$

   We'll call the passage of the theorem's part (1) the (multiple) **two-sided** pair-compression.

   As, upon multiplying the first block-column of the matrix $\begin{pmatrix} V & A \\ UA^TD^{-1} & U \end{pmatrix}$ by D, we'll receive the matrix $\begin{pmatrix} W & A \\ UA^T & U \end{pmatrix}$ where $W = VD$ whose Hamiltonian is $\det(D)\,ham\left(\begin{pmatrix} V & A \\ UA^TD^{-1} & U \end{pmatrix}\right)$ and is hence also polynomial-time computable and

therefore we get the following generalization of the theorem that the Hamiltonian of a unitary matrix is polynomial-time computable in characteristic 2:

Corollary II.3.2.5 (in characteristic 2): let U be unitary of size n×n, W, D be of size m×m, D be non-singular diagonal such that $D + D^{-1}$ is non-singular, $W + W^T + AA^T = I_m + D^2$. Then

$$\text{ham}\left(\begin{pmatrix} W & A \\ UA^T & U \end{pmatrix}\right) = \frac{\det(D)}{\det(D+D^{-1})} \det^2\left(\begin{pmatrix} WD^{-1} + I_m + C_{1,1} & WD^{-1} + D & A \\ WD^{-1} + D^{-1} & WD^{-1} + D^{-1} + D + I_m & A \\ UA^T D^{-1} & UA^T D^{-1} & U + I_n \end{pmatrix}\right)$$

Theorem II.3.2.6: let $A_{11}, \begin{pmatrix} A_{11} & A_{12} \\ A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be square matrices, $\det(A_{11}) \neq 0$. Then

$$\text{ham}\left(\begin{pmatrix} A_{11} & A_{12} \\ A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\right) = \det^2(A_{11})\text{ham}\left(\begin{pmatrix} A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1}A_{12} + A_{22} \end{pmatrix}\right)$$

We'll call the theorem's passage the (multiple) pair-compression.

We can also add that, like the even permanent in characteristic 3, the Hamiltonian of a square matrix naturally doesn't depend on its diagonal elements and $\text{ham}(A) = \det^2(A)\text{ham}(A^{-1})$

This Hamiltonian-preserving compressibility and variability analogically yields the conjecture that the compression-closure (defined by the analogy with the permanent in characteristic 3) of the unitary class (which, as we've just showed above, is compressible unlike the case of the permanent in characteristic 3) is the whole set of square matrices that likewise implies the polynomial-time computability of the Hamiltonian in characteristic 2.

Besides, in characteristic 2 the Hamiltonian possesses replacement identities for a unitary matrix U similar to the earlier introduced relations for the permanent in characteristic 3.

Definition: for a square matrix X, a pair I,J of equally sized sets of its row-indexes with a bijection $f_1 \colon I \xrightarrow{f_1} J$ and a pair K,L of equally sized sets of its column-indexes with a bijection $f_2 \colon K \xrightarrow{f_2} L$, let's define ***the $I \xrightarrow{f_1} J, K \xrightarrow{f_2} L$-double-replacement matrix*** $X^{\left[I \xrightarrow{f_1} J, K \xrightarrow{f_2} L\right]}$ as the matrix received from X through replacing, for each $i \in I$, its $f_1(i)$-th row by its i-th row and, for each $k \in K$, its $f_2(k)$-th column by its k-th column.

Theorem II.3.2.7 (in characteristic 2):

Let A be a square matrix, I,J be sets of its row-indexes and K,L be sets of its column-indexes, $|I|=|J|$, $|K|=|L|$, f_1, f_2 be bijections $I \xrightarrow{f_1} J$, $K \xrightarrow{f_2} L$ correspondingly. Then

$$\text{ham}(A^{\left[I\xrightarrow{f_1}J, K\xrightarrow{f_2}L\right]}) = \det^2(A)\text{ham}(((A^{-1})^T)^{\left[J\xrightarrow{f_1^{-1}}I, L\xrightarrow{f_2^{-1}}K\right]})$$

where f_1^{-1}, f_2^{-1} are the inverse bijections $J \xrightarrow{f_1^{-1}} I$, $L \xrightarrow{f_2^{-1}} K$ correspondingly.

Like for the permanent in characteristic 3, the proof of this identity can be received by means of using the fact that in characteristic 2 (where we have no signs +/-) any minor of a unitary matrix equals its algebraic complement, with the only essential difference that a square matrix's rows and columns can be Hamiltonian-preservingly permuted only by an arbitrary pair of identical permutations (unlike the permanent that allows independent arbitrary permutations of rows and columns).

Corollary II.3.2.8 (in characteristic 2):

Let U be a unitary matrix, I,J be sets of its row-indexes and K,L be sets of its column-indexes, $|I|=|J|$, $|K|=|L|$, f_1, f_2 be bijections $I \xrightarrow{f_1} J$, $K \xrightarrow{f_2} L$ correspondingly. Then

$$\text{ham}(U^{\left[I\xrightarrow{f_1}J, K\xrightarrow{f_2}L\right]}) = \text{ham}(U^{\left[J\xrightarrow{f_1^{-1}}I, L\xrightarrow{f_2^{-1}}K\right]})$$

where f_1^{-1}, f_2^{-1} are the inverse bijections $J \xrightarrow{f_1^{-1}} I$, $L \xrightarrow{f_2^{-1}} K$ correspondingly.

Definition:

Let A be an $n\times n$-matrix, ε be a formal infinitesimal. Then we'll call the matrix formal power series $U = \sum_{k=0}^{\infty} \varepsilon^k U_k$, where each U_k is an $n\times n$-matrix over a ground field F, $U_0 = I_n$ and $U_1 = A$, an $\boldsymbol{\varepsilon\text{-unitarization}}$ of A over $F(\varepsilon)$ if $U(A)$ is unitary as a matrix formal power series in ε.

It's easy to see that an ε-unitarization U of A exists in characteristic 2 if and only if $A = A^T$, while for a pair $i, j \in \{1, ..., n\}$ $\text{coef}_{\varepsilon^{n-1}}\text{ham}(U^{(\backslash i, \backslash j)}) = \text{ham}(A^{(\backslash i, \backslash j)})$ and thus it's $\#_2P$-complete as a function in the edge weights of the weighted digraph corresponding to A which is identically equal to zero if and only if this graph has no Hamiltonian path between the vertices i and j. Hence, taking into account the fact that for a unitary U the matrix $U^{(\backslash i, \backslash j)}$ is 1-semi-initary, we conclude that computing the Hamiltonian of a 1-semi-unitary matrix in characteristic 2 is a $\#_2P$-complete problem. It also implies,

likewise, the #$_2$P-completeness of computing the Hamiltonian of a unitary matrix over a ring of characteristic 4.

If for a an n×n-matrix A we define the matrix $H(A) = \{ham(A^{(\backslash i, \backslash j)})\}_{n \times n}$ then we'll receive, based on the above relation $ham(U^{\left[I \xrightarrow{f_1} J, K \xrightarrow{f_2} L\right]}) = ham(U^{\left[J \xrightarrow{f_1^{-1}} I, L \xrightarrow{f_2^{-1}} K\right]})$, the identity $UH^T(U) = H(U)U^T$.

And we may add that the partial inverse relation also concerns the Hamiltonian in characteristic 2:

Lemma II.3.2.9 (in characteristic 2):

For square A_{11}, A_{22},

$$ham(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}) = \det{}^2(A_{11})ham(\begin{pmatrix} A_{11}^{-1} & A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1} & A_{22} + A_{21}A_{11}^{-1}A_{12} \end{pmatrix})$$

It has the following

Corollary II.3.2.10 (in characteristic 2):

Let X, Y, Z be n×n-matrices. Then

$$ham\begin{pmatrix} X & XZ \\ YX & YXZ \end{pmatrix} = \det{}^2(X)ham(\begin{pmatrix} 0_{n \times n} & Z \\ Y & 0_{n \times n} \end{pmatrix})$$

Besides, when speaking about the Hamiltonian in characteristic 2 that is a direct algebraic representation of a fundamental NP-complete problem, it would be worth noting the existence of a non-trivial class of digraphs whose arcs could be given non-zero weights over a field of characteristic 2 making their weighted adjacency matrices unitary. Let's call them **weight-unitarazable** over a ground field F, while such a system of arc weights we'll call a digraph's **weight-unitarization** over F. Let's consider several examples of digraphs weight-unitarizable over fields of characteristic 2.

One partial case is a system of pair-wise vertex-disjoint simple directed cycles whose arc set is partitioned into pairs of vertex-disjoint arcs (a,b) and (c,d) connected by two additional arcs (c,b) and (d,a) so that the four arcs form the anti-cycle $a \to b \leftarrow c \to d \leftarrow a$ and their weights satisfy the following system of equations:

weight(a,b)weight(c,b) = weight(a,d)weight(c,d),

weight(a,b) + weight(a,d) = weight(c,b) + weight(c,d) = 1.

Those systems are variable-disjoint for different anti-cycles and are solvable in linear time, while leaving, for each anti-cycle, one independent weight-variable as a parameter. In the case of its planarity, particularly, such a digraph depicts a city with a system of two way streets between one way cyclic roads around squares where the digraph's vertices correspond to the crossroads.

Another interesting example is the arc-digraph of a digraph (received via taking the initial digraph's arc set as the new vertex set, while two new vertices are connected if and only if they form, as initial arcs, a path of length 2) where some connections between new vertices (i.e. initial arcs) are removed so that for each initial vertex the remained connections form a weight-unitarizable digraph. This example generates a direct algebraic representation of a constrained Eulerian cycle problem where some passages between adjacent arcs are forbidden.

Tournaments can be conjectured to be weight-unitarizable in characteristic 2 as well.

Moreover, in characteristic 2 the Hamiltonian has a generalization that is also computable in polynomial time for unitary and involuntary matrices.

Definition: let A be an n×n-matrix, w be an n-vector. Then its ***cycle polynomial*** is $\mathrm{cycle}(A, w) := \sum_{\pi \in S_n} \prod_{C \in \mathbb{C}(\pi)} (1 + \prod_{i \in C} w_i) \prod_{i=1}^{n} a_{i,\pi(i)}$ where $\mathbb{C}(\pi)$ is the set of π's cycles.

Theorem II.3.2.11: let A be a unitary or involuntary matrix, w be an n×n-vector. Then

$$\mathrm{cycle}(A, w) = \det(A^{\star 2} + \mathrm{Diag}(w))$$

As this function is polynomial-time computable for involuntary matrices as well, we may analogically (with weight-unitarizable ones) define ***weight-involuntarizable*** digraphs.

Hence the cycle polynomial can be considered for digraphs weight-unitarizable or weight-involuntarizable in characteristic 2 and presumably serve as a direct algebraic representation of a number of problems on digraphs.

III.   The Schur complement compression on informationally sparse classes

Given an n×n-matrix class defined by a system of matrix-functions in a set of parameters, let's define its algebraic rank as the system's algebraic rank. And, in case if the class is defined by an algebraic equation system, we'll define its algebraic rank as the difference of n^2 and the algebraic equation system's algebraic rank. In both cases we'll call it the algebraic n-rank of such a matrix class and we'll also call such a matrix class **algebraically definable** – in fact, it's an exact analogy of the notion of a smooth manifold in characteristic 0. We may also assume that both the above-mentioned forms of a matrix class's definition are reducible to each other and yield the same algebraic rank.

In the present article's two previous chapters we discussed some matrix compressions that polynomial-time reduce the permanent of a matrix to the permanent of a derived matrix of a smaller size, and we dealt with either arbitrary matrices or k-semi-unitary ones etc., i.e. classes of n×n-matrices whose algebraic rank is either $n^2 + o(n)$ or $n^2/2 + o(n)$. Let's call an algebraically definable matrix class **informationally dense** if for any n the ratio of its algebraic n-rank and n^2 (that we'll call the **informational n-density** of the class) is bigger than a nonzero constant, and **informationally sparse** otherwise.

In this chapter we're going to study some informationally sparse matrix classes (particularly, those of informational n-density $1/o(n)$) built via Cauchy and Cauchy-like matrices, as well as certain matrix compression operators (particularly, the Schur complement compression) that polynomial-time reduce one function to another on those classes, while still acting as genuine self-reducing compressions for certain introduced functions. And, once again throughout the chapter, the neighboring computation principle is supposed to serve as a chief algebraic instrument the principal below-introduced polynomial-time reductions would be impossible without. In this regard, we'll also need the following related definition:

> **Definition:**
> For a field F, a formal infinitesimal ε, and F's ε-extension $F(\varepsilon) := \{u = \sum_{k=\text{order}_\varepsilon(u)}^{\infty} u_k \varepsilon^k, \text{order}_\varepsilon(u) \in \mathbb{Z}, \ u_k \in F \text{ for } k = \text{order}_\varepsilon(u), \dots, \infty\}$, let's define
> for $u \in F(\varepsilon)$ $\lim_{\varepsilon \to 0} u := \begin{bmatrix} u_o & \text{if order}_\varepsilon(u) \geq 0 \\ \text{a nonexistent (infinitely big) element} & \text{otherwise} \end{bmatrix}$
> And we'll call $\text{order}_\varepsilon(u)$ the order of u on ε (or, shortly, the ε-order of u).

Denotation: for a matrix A, a subset I of its row set and a subset J of its column set, by $\text{Schur}_{I,J}(A)$ we'll denote (for the purpose of simplicity) the Schur complement $A/A^{(I,J)}$.

Definition:

Let a be an n-vector and b be an m-vector. Then its Kronecker sum is defined as

$$a \ddot{+} b := a \otimes \vec{1}_m + \vec{1}_n \otimes b = \begin{pmatrix} a_1 \vec{1}_{\dim(m)} + b \\ \dots \\ a_n \vec{1}_{\dim(m)} + b \end{pmatrix}$$

Definition:

$$\varphi_{p,h}(u, w, v, \gamma) := \sum_{J,|J|=\dim(w)} \left(\sum_{j \in J} \prod_{i=1}^{\dim(u)} \frac{1}{(u_i - v_j)^p} \right) \det{}^h C(w, v_J) \prod_{j \in J} \gamma_j$$

and we'll call it the **Cauchy determinant base-sum**, while calling the vector u the **Cauchy base-vector** and p the **Cauchy base-degree**.

Definition: let A, B be two skew-symmetric 2n×2n-matrices. Then

$$\xi_m(A, B) := \sum_{I \subseteq \{1,\dots,2n\}, |I|=m} \text{Pf}(A^{(I,I)}) \, \text{Pf}(B^{(I,I)})$$

Theorem III.1: let A, B be two skew-symmetric 2n×2n-matrices. Then, for an even m, we define

$$\xi_m(A, B) = \text{coef}_{\omega^{m/2}} \text{Pf}\left(\begin{pmatrix} \omega A & I_{2n} \\ -I_{2n} & B \end{pmatrix} \right)$$

Definition: for a rational number r, a natural number k and two vectors z, d of equal dimension, we define

$$\eta_{r,m}(z, d) := \sum_{I, |I|=m} \det{}^r(\tilde{C}(z_I)) \prod_{i \in I} d_i$$

Theorem III.2: (in characteristic p)

$$\eta_{\frac{p^q+1}{2}, m}(z, d) = \xi_m \left(\text{Diag}(d^{\star(1/2)}) \tilde{C}(z) \text{Diag}(d^{\star(1/2)}), \tilde{C}(z^{\star p^q}) \right)$$

………………………………..

Definition: for three n-vectors x, a, d, let's define the skew-symmetric n×n-block-matrix with 2×2-blocks (i.e. 2n×2n-matrix)

$$K(x, d, a) := \{K_{ij}\}_{n \times n}$$

where

$$K_{ii} := \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix} \text{ for } i = 1,\dots,n$$

$$K_{ij} := \begin{pmatrix} \dfrac{1}{x_i - x_j} & (\dfrac{\partial}{\partial x_j} + d_j \dfrac{\partial^2}{\partial x_j{}^2}) \dfrac{1}{x_i - x_j} \\ (\dfrac{\partial}{\partial x_i} + d_i \dfrac{\partial^2}{\partial x_i{}^2}) \dfrac{1}{x_i - x_j} & (\dfrac{\partial}{\partial x_i} + d_i \dfrac{\partial^2}{\partial x_i{}^2})(\dfrac{\partial}{\partial x_j} + d_j \dfrac{\partial^2}{\partial x_j{}^2}) \dfrac{1}{x_i - x_j} \end{pmatrix}$$

for $i \neq j$, $i, j \in \{1, \dots, n\}$

In the above-defined matrix, we'll call d_i **_the differentiation-weight_** and a_i **_the absence-weight_** corresponding to the denominator-value x_i.

Definition:
Given two matrices $A = \{a_{i,j}\}_{n \times m}$ and $B = \{b_{i,j}\}_{n \times m}$, by $A \star B = \{a_{i,j} b_{i,j}\}_{n \times m}$ we'll denote their Hadamard product.
Given a rational number k, by $A^{\star k} = \{a_{i,j}^k\}_{n \times m}$ we'll denote the k-th Hadamard power of A and, given a sequence of rational numbers $(k_1, \dots, k_s)$, by

$A^{\star(k_1, \dots k_s)}$ we'll denote its $(k_1, \dots, k_s)$-th Hadamard-power $\begin{pmatrix} A^{\star k_1} \\ \dots \\ A^{\star k_s} \end{pmatrix}$.

Definition:
1) For an n-vector x, let's define $\tilde{C}(x)$ as an n×n-matrix whose i,j-th entry is $\dfrac{1}{x_i - x_j}$ if $i \neq j$, $i, j \in \{1, \dots, n\}$, and 0 otherwise. We'll call it a **_Cauchy-wave matrix_**.

2) For three vectors x, y, z, let's also define $\tilde{C}(x, y, z) := \begin{pmatrix} \tilde{C}(x) & C(x, z) \\ C(y, x) & C(y, z) \end{pmatrix}$
We'll call it a **_Cauchy-waved matrix_**.

Definition:

Let $\dim(x) = n \equiv 0 \pmod k$. Then $W^{[k]}(x) = (x^T)^{\star(0 1_k^T, \dots, (\frac{n}{k} - 1) 1_k^T)}$
where 1_k^T denotes the k-sequence all whose entries are 1.

Definition: for a vector y, we denote

$$\text{Van}^{[k]}(y) := \begin{pmatrix} (y^T)^{\star 0} \\ (y^T)^{\star 1} \\ \dots \\ (y^T)^{\star(k-1)} \end{pmatrix}$$

and we also denote $\text{Van}(y) := \text{Van}^{[\dim(y)-1]}(y) = W^{[1]}(y)$.

Definition: let x, y be two vectors. Then we denote $(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{\dim(\mathbf{x})} \prod_{j=1}^{\dim(\mathbf{y})} (\mathbf{x_i} - \mathbf{y_j})$.

Theorem III.3 (The Borchardt formula, in any characteristic):

Let $\dim(x) = \dim(y)$. Then $\text{per}\big(C(x, y)\big) = \dfrac{\det(C^{\star 2}(x,y))}{\det(C(x,y))}$.

Lemma III.4 (about square Cauchy-waved matrices, in characteristic zero if not specified otherwise):

1) for $\dim(x) > 2$: $\text{ham}\left(\tilde{C}(x)\right) = 0$

2) for $\dim(y) = \dim(z) > 0$:

$$\text{ham}(\tilde{C}(x, y, z)) = \text{ham}(C(y, z)) \prod_{i=1}^{\dim(x)} \left(\sum_{j=1}^{\dim(y)} \frac{1}{y_j - x_i} - \sum_{k=1}^{\dim(z)} \frac{1}{z_k - x_i} \right)$$

3) for $\dim(y)=\dim(z)$:

$$\text{per}(\tilde{C}(x, y, z)) = \text{per}(C(y, z))\text{per}(\tilde{C}(x) + \text{Diag}(\left\{ \textstyle\sum_{j=1}^{\dim(y)} \frac{1}{y_j-x_i} - \sum_{k=1}^{\dim(z)} \frac{1}{z_k-x_i} \right\}_{\dim(x)}))$$

4) for $\dim(y) = \dim(z)$:

$$\det(\tilde{C}(x, y, z)) = \det\big(C(y, z)\big)\det(\tilde{C}(x) - \text{Diag}(\left\{ \textstyle\sum_{j=1}^{\dim(y)} \frac{1}{y_j-x_i} - \sum_{k=1}^{\dim(z)} \frac{1}{z_k-x_i} \right\}_{\dim(x)})) ,$$

$$\det\big(C(y, z)\big) = \frac{\det(\text{Van}(y))\det(\text{Van}(z))}{\text{pol}(y,z)}$$

5) for $\dim(x) = 2n$:

$$\text{Pf}(\tilde{C}(x)) = \frac{\sum_{I \subset \{1,\dots,2n\}, |I|=n} \det^2(\text{Van}(x_I))\det^2(\text{Van}(x_{\setminus I}))}{2^n \det(\text{Van}(x))}$$

$$\det(\tilde{C}(x)) = (-1)^n \text{per}(\tilde{C}(x)) = \frac{\sum_{I \subset \{1,\dots,2n\}, |I|=n} \det^4(\text{Van}(x_I))\det^4(\text{Van}(x_{\setminus I}))}{2^n \det^2(\text{Van}(x))} = \text{haf}(\tilde{C}^{\star 2}(x))$$

6) $\quad \text{Pf}(\tilde{C}(x)) = \dfrac{\text{per}^2(W^{[2]}(x))}{\det(\text{Van}(x))} \qquad$ in characteristic 3

$\text{Pf}(\tilde{C}(x)) = \dfrac{\text{per}(W^{[4]}(\binom{x}{x}))}{\det(\text{Van}(x))} \qquad$ in characteristic 5

7)   in a prime characteristic p, for dim(y) = (p-1)dim(x):

$$\text{per}\left(C\left(x \otimes \vec{1}_{p-1}, y\right)\right) = \frac{\det^{p-1}(\text{Van}(x))\text{per}(W^{[p-1]}(y))}{\text{pol}(x,y)}$$

  Proof:

The proofs of all the lemma's statements can be based on the first of them (which is well-known) and the second one for dim(x) = dim(y) = 1, as well as on the Borchardt identity.

The statement (2) is a key result that implies the statements (3) and (4): we use the fact that the determinant of a square matrix is the sum, over all its traversals, of the product of the traversal's entries multiplied by $(-1)^{|C_1|+1+\cdots+|C_k|+1}$ where $|C_1|, \ldots, |C_k|$ are the lengths of its cycles.

The first identity of the statement (5) follows from the relation

$$\text{Pf}(\tilde{C}(x)) = \frac{\sum_{I,|I|=n} \sigma(\pi(I))\det(C(x_I, x_{\backslash I}))}{2^n} \quad \text{where for } I = \{i_1, \ldots, i_n\}, 1 \le i_1 < \cdots < i_n \le 2n,$$

$\sigma(\pi(I))$ is the sign of the 2n-permutation $\pi(I) = \begin{pmatrix} 1 \ldots n & n+1 \ldots 2n \\ i_1 \ldots i_n & \hat{i}_1 \quad \ldots \quad \hat{i}_n \end{pmatrix}$ where $\hat{I} = \{1, \ldots, 2n\}\backslash I = \{\hat{i}_1, \ldots, \hat{i}_n\}, 1 \le \hat{i}_1 < \cdots < \hat{i}_n \le 2n$ (this formula is a partial case of the identity for a skew-symmetric 2n×2n-matrix A $\text{Pf}(A) = \frac{\sum_{I,|I|=n} \sigma(\pi(I))\det(A^{(I,\backslash I)})}{2^n}$) because $\sigma(\pi(I))$ is also the ratio $\frac{\det(\text{Van}(x_I))\text{pol}(x_I, x_{\backslash I})\det(\text{Van}(x_{\backslash I}))}{\det(\text{Van}(x))}$

and its second identity follows from the relation $\det(\tilde{C}(x)) = \frac{\sum_{I,|I|=n}\det^2(C(x_I, x_{\backslash I}))}{2^n}$ (that is a partial case of the identity for a skew-symmetric 2n×2n-matrix A $\det(A) = \frac{\sum_{I,|I|=n}\det^2(A^{(I,\backslash I)})}{2^n}$).

The statement (6) for characteristic 3 is due to the identity (for this characteristic)

$$\text{per}^2\left(W^{[2]}(x)\right) = \text{per}\left(\begin{pmatrix} 0_{n\times n} & (W^{[2]}(x))^T \\ W^{[2]}(x) & 0_{n\times n} \end{pmatrix}\right) =$$

$$= 2^n \sum_{I \subset \{1,\ldots,2n\}, |I|=n} \det^2(\text{Van}(x_I))\det^2\left(\text{Van}(x_{\backslash I})\right)$$

And for characteristic 5 it follows from the fact that in this characteristic there holds

the identity $\lim\limits_{\varepsilon \to 0}(\varepsilon \dfrac{\mathrm{per}(W^{[4]}(\begin{pmatrix} y \\ u \\ u+\varepsilon \\ y \\ u \\ u+\varepsilon \\ y \\ u \\ u+\varepsilon \end{pmatrix}))}{\det(\mathrm{Van}(\begin{pmatrix} y \\ u \\ u+\varepsilon \end{pmatrix}))}) = \dfrac{\mathrm{per}(W^{[4]}(\begin{pmatrix} y \\ y \end{pmatrix}))}{\det(\mathrm{Van}(y))}$ and $\mathrm{Pf}(\tilde{C}(x))$ satisfies the same

functional equation, while $\mathrm{Pf}(\tilde{C}(x))$ is a fraction whose denominator is $\det(\mathrm{Van}(x))$ and whose numerator is a homogenous polynomial in x of degree $\dfrac{\dim^2(x) - 2\dim(x)}{2}$ that is the degree of the homogenous polynomial $\mathrm{per}(W^{[4]}(\begin{pmatrix} X \\ X \end{pmatrix}))$.

The statement (7) can be received via multiplying the j-th column of $C(x \otimes \vec{1}_{p-1}, y)$ by $\prod_{i=1}^{\dim(x)}(x_i - y_j)$ and turning this matrix, via linear operations with (p-1)-tuples of rows $C(x_i \vec{1}_{p-1}, y)$, into $W^{[p-1]}(y)$.

Lemma III.5 (about rectangular Cauchy matrices)

1) In characteristic zero:

$$\mathrm{per}(C(y,z)) = \mathrm{per}(\tilde{C}(y) + \mathrm{Diag}(\{\textstyle\sum_{k,k \neq j}\frac{1}{y_j - y_k} - \sum_{k=1}^{\dim(z)}\frac{1}{y_j - z_k}\}_{\dim(y)}))$$

2) In characteristic 3: $\mathrm{per}(C(y,z)) = \mathrm{per}(C(y,x))$ for any vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $\mathrm{pol}''(v, \begin{pmatrix} x \\ y \\ z \end{pmatrix}) \equiv 0$ (identically as a polynomial in v).

Proof:

The statement (1) follows directly from the Borchardt formula

The statement (2) is implied by the statement (1).

Definition:

Let A be a square matrix, then $\mathrm{per}_\lambda(A) := \sum_{\pi \in S_n} \lambda^{c(\pi)} \prod_{i=1}^{n} a_{ij}$

where $c(\pi)$ is the number of cycles in the permutation π .

Theorem III.6 (in any prime characteristic p > 2):

$$(\textstyle\prod_{i=1}^{n}(1 + d_i \frac{\partial}{\partial x_i}))\mathrm{per}_{\frac{1}{4}}(\tilde{C}^{\star 2}(x) + \mathrm{Diag}(a)) = \frac{1}{2^n}\mathrm{Pf}(K(x, d, a))$$

Proof:

first let's prove the validness of this identity for the case of $a = d = \vec{0}_n$. In an arbitrary prime characteristic p, let's consider the expression

$$
(*) \quad \lim_{\varepsilon \to 0} \frac{\sum_{I \subseteq \{1,\dots,2n\}} (-1)^{|I|/(p+1)} \det^{\frac{p+1}{2}}(\varepsilon \tilde{C}(\begin{pmatrix} x_1 \\ x_1+\varepsilon \\ \dots \\ x_n \\ x_n+\varepsilon \end{pmatrix})^{(I,I)})}{\varepsilon^{2n}}
$$

that, due to the anti-symmetry of the matrix $\tilde{C}(\begin{pmatrix} x_1 \\ x_1 + \varepsilon \\ \dots \\ x_n \\ x_n + \varepsilon \end{pmatrix})$ and the well-known fact that

the determinant of a skew-symmetric matrix is the square of its Pfaffian, is identical to the expression

$$
(**) \quad \lim_{\varepsilon \to 0} \frac{\sum_{I \subseteq \{1,\dots,2n\}} (-1)^{|I|/(p+1)} Pf^{p+1}(\varepsilon \tilde{C}(\begin{pmatrix} x_1 \\ x_1+\varepsilon \\ \dots \\ x_n \\ x_n+\varepsilon \end{pmatrix})^{(I,I)})}{\varepsilon^{2n}} = Pf\left(K(x, \vec{0}_n, \vec{0}_n)\right).
$$

Let's show that the former expression (*) is $2^n per_{\frac{1}{4}}(\tilde{C}^{\star 2}(x))$.

First of all, we know that for any vector z $\det(\tilde{C}(z)) = haf(\tilde{C}^{\star 2}(z))$ and, hence, we can re-write the expression (*) as

$$
(***) \quad \lim_{\varepsilon \to 0} \frac{\sum_{I \subseteq \{1,\dots,2n\}} (-1)^{|I|/(p+1)} haf^{\frac{p+1}{2}}(\varepsilon^2 \tilde{C}^{\star 2}(\begin{pmatrix} x_1 \\ x_1+\varepsilon \\ \dots \\ x_n \\ x_n+\varepsilon \end{pmatrix})^{(I,I)})}{\varepsilon^{2n}}
$$

Secondly, due to the summation and the limit $\lim_{\varepsilon \to 0}$ (producing a "weight" $O(\varepsilon^2)$ for each infinitely-close pair $x_i, x_i + \varepsilon$),

for i = 1,…,n, among the $\frac{p+1}{2}$ multipliers each of whom is $haf(\varepsilon^2 \tilde{C}^{\star 2}(\begin{pmatrix} x_1 \\ x_1 + \varepsilon \\ \dots \\ x_n \\ x_n + \varepsilon \end{pmatrix})^{(I,I)}$

there should be exactly one where the term $\frac{\varepsilon^2}{(x_i - x_i - \varepsilon)^2} = 1$ isn't to be taken what hence implies the appearance (exactly in one of the multipliers) of a cycle $\mathfrak{C}$ connecting (by the taken terms $\frac{\varepsilon^2}{(x_i - x_j + O(\varepsilon))^2}$) those "untaken singularities" associated with pairs $x_i, x_i + \varepsilon$. The limit $\lim_{\varepsilon \to 0}$ and the denominator of the fraction

$$\frac{\sum_{I\subseteq\{1,\ldots,2n\}}(-1)^{|I|/(p+1)}\mathrm{haf}^{\frac{p+1}{2}}(\varepsilon^2\widetilde{C}^{\star 2}(\begin{pmatrix}x_1\\x_1+\varepsilon\\\cdots\\x_n\\x_n+\varepsilon\end{pmatrix}))^{(I,I)}}{\varepsilon^{2n}}$$ turn the product of $\mathfrak{C}$'s terms into the product

of the corresponding terms $\frac{1}{(x_i-x_j)^2}$. The whole expression (***) hence turns into

$\sum_{I_1,\ldots,I_{(p+1)/2}}\prod_{q=1}^{(p+1)/2}\mathrm{haf}(A^{(I_q,I_q)})$ where A is the matrix $\widetilde{C}^{\star 2}(\begin{pmatrix}x_1\\x_1+\varepsilon\\\cdots\\x_n\\x_n+\varepsilon\end{pmatrix})$ with all the

"infinitely big" entries $\frac{1}{(x_i-x_i-\varepsilon)^2}$ replaced by zeros and each I_q is a subset of the set of

pairs $\{(2i-1,2i), i=1,\ldots,n\}$, while the $\frac{p+1}{2}$-tuple $I_1,\ldots,I_{(p+1)/2}$ runs over all its

partitions (possibly including empty sets). Besides, the cycle $\mathfrak{C}$ can be considered as

directed and is, in fact, the corresponding directed cycle in $\mathrm{per}_{1/4}(\widetilde{C}^{\star 2}(x))$ with its

coefficient ¼ that is multiplied, in (*), by 2^l (where l is its length considered as the

number of singularities it connects) because:

(with each denominator-value x_i or $x_i+\varepsilon$ we'll further associate a vertex in the

corresponding weighted graph with the weighted adjacency matrix A; such a vertex is to

"appear" exactly in one of $\mathrm{haf}(A)'s$ (p+1)/2 copies)

for $l > 2$ $\mathfrak{C}$'s direction is determined by the connection of the vertex $x_{\min(\mathfrak{C})}$ of $\mathfrak{C}$'s

lexicographically minimal ε-close pair $x_{\min(\mathfrak{C})}, x_{\min(\mathfrak{C})}+\varepsilon$ (i.e. by the ε-close pair whose

vertex is connected with $x_{\min(\mathfrak{C})}$ in $\mathfrak{C}$), while for each direction (including the case of $l=$

2 when there is only one direction) there are 2^{l-1} variants of $\mathfrak{C}$-forming systems of

connections (as in the pair $x_{\min(\mathfrak{C})}, x_{\min(\mathfrak{C})}+\varepsilon$ we already cannot choose a vertex, while

in all the other ε-close pairs of $\mathfrak{C}$ we choose one vertex from a pair when coming to it

from the preceding pair while traversing $\mathfrak{C}$). Besides, there are $\frac{p+1}{2}\equiv\frac{1}{2}(\mathrm{mod}\ p)$ variants

of locating $\mathfrak{C}$ (independently of other connecting cycles) in one of the multipliers what,

altogether, gives the overall combinatorial coefficient $2^{l-1}\frac{1}{2}=\frac{1}{4}2^l$ for each connecting

cycle. We just should add that we hence built a natural bijection between the

singularity-connecting directed cycles and the directed cycles of $\mathrm{per}_{1/4}(\widetilde{C}^{\star 2}(x))$.

And, as well, it's easy to realize that the expression (**) is $\mathrm{Pf}(K(x,\vec{0}_n,\vec{0}_n))$ because in

the multiplier $\mathrm{Pf}^p(\varepsilon\widetilde{C}(\begin{pmatrix}x_1\\x_1+\varepsilon\\\cdots\\x_n\\x_n+\varepsilon\end{pmatrix}))$ we should take all the singularity terms $\frac{\varepsilon^p}{(x_i-x_i-\varepsilon)^p}$ due

to obtaining, in the numerator of (**), "at least" the "weight" $O(\varepsilon^{2p-1})$ otherwise for each singularity where it's not taken. Let's explain it even in the "best" case when in

$$\text{Pf}(\varepsilon\tilde{C}(\begin{pmatrix} x_1 \\ x_1 + \varepsilon \\ \cdots \\ x_n \\ x_n + \varepsilon \end{pmatrix}))$$ we take the term $\frac{\varepsilon}{x_i - x_i - \varepsilon}$ for this singularity: those untaken

singularities of $\text{Pf}^p(\varepsilon\tilde{C}(\begin{pmatrix} x_1 \\ x_1 + \varepsilon \\ \cdots \\ x_n \\ x_n + \varepsilon \end{pmatrix}))$ would also form cycles with connecting terms of the

type $O(\varepsilon^p)$ and, besides, in each untaken singularity in $\text{Pf}^p(\varepsilon\tilde{C}(\begin{pmatrix} x_1 \\ x_1 + \varepsilon \\ \cdots \\ x_n \\ x_n + \varepsilon \end{pmatrix}))$ the edge

corresponding to $x_i + \varepsilon$ should be replaced by its differential on ε to prevent the "resulting" Pfaffian from having a pair of identical rows and a pair of identical columns, while in the numerator of (**) for each singularity we're supposed to get, due to the limit and the denominator ε^{2n}, the "weight" not "smaller" than $O(\varepsilon^2)$ (because each singularity necessarily produces it due to the summation).

And now let's prove the theorem's identity for arbitrary a, d.

For i = 1,…,n, differentiating $\text{Pf}(K(x, \vec{0}_n, \vec{0}_n))$ on the variable x_i is equivalent, due to the Pfaffian's general nature, to differentiating (with the differentiation weight coefficient d_i) the corresponding (i.e. containing the term x_i) blocks of $K(x, \vec{0}_n, \vec{0}_n)$ on this variable as it's shown in the theorem's formula (due to receiving the Pfaffian of a matrix having a pair of identical rows and a pair of identical columns otherwise). And putting a_i in the corresponding diagonal block of $K(x, \vec{0}_n, \vec{0}_n)$ generates the case of "removing" (with the absence-weight coefficient a_i) all the terms containing x_i from the Pfaffian's sum expansion.

Sparse compressions in characteristic 5

Corollary III.7:

in characteristic 5,

$$(@) \qquad \left(\prod_{i=1}^{n}(1 + d_i \frac{\partial}{\partial x_i})\right)\det(\tilde{C}^{\star 2}(x) + \mathrm{Diag}(a)) = \frac{1}{2^n}\mathrm{Pf}\big(K(x,d,a)\big)$$

Theorem III.8:

Let G be an $n{\times}m$-matrix of a rank unexceeding k, $\dim(x) = n$, $\dim(y) = m$. Then in the matrix $G \star C(x,y)$ the Schur complement of the block lying on a set of rows I and a set of columns J such that $|I|=|J|$ is a matrix of the form $\breve{G} \star C(x_{\backslash I}, y_{\backslash J})$ where $\breve{G}$ is an (n-$|I|$)$\times$(m-$|J|$)-matrix of a rank unexceeding k.

Proof:

This theorem can be easily proven by the induction on $|I|$ because the Schur complement of any block can be represented as the result of a chain of consequent elementary Schur complement compressions for blocks of size $1{\times}1$.

Indeed, let's consider, for $i = 2,\dots,\dim(x)$ and $j = 2,\dots,\dim(y)$, the determinant

$$\det\begin{pmatrix} \dfrac{\alpha_1^T\beta_1}{x_1-y_1} & \dfrac{\alpha_1^T\beta_j}{x_1-y_j} \\ \dfrac{\alpha_i^T\beta_1}{x_i-y_1} & \dfrac{\alpha_i^T\beta_j}{x_i-y_j} \end{pmatrix}$$ where $\alpha_1, \beta_1, \alpha_i, \beta_j$ are k-vectors. We can represent it as

$$\alpha_i^T\beta_1\alpha_1^T\beta_j\det\begin{pmatrix} \dfrac{1}{x_1-y_1} & \dfrac{1}{x_1-y_j} \\ \dfrac{1}{x_i-y_1} & \dfrac{1}{x_i-y_j} \end{pmatrix} + \frac{\alpha_1^T\beta_1\alpha_i^T\beta_j - \alpha_i^T\beta_1\alpha_1^T\beta_j}{x_1-y_1}\frac{1}{x_i-y_j} =$$

$$= \alpha_i^T\beta_1\alpha_1^T\beta_j\frac{(x_i-x_1)(y_j-y_1)}{(x_i-y_1)(x_1-y_1)(x_1-y_j)}\frac{1}{x_i-y_j} + \frac{\alpha_1^T\beta_1\alpha_i^T\beta_j - \alpha_i^T\beta_1\alpha_1^T\beta_j}{x_1-y_1}\frac{1}{x_i-y_j} =$$

$$= \frac{1}{(x_1-y_1)}\frac{\alpha_i^T\beta_1\frac{x_i-x_1}{x_i-y_1}\cdot\frac{y_j-y_1}{x_1-y_j}\alpha_1^T\beta_j + \alpha_i^T(\alpha_1^T\beta_1 I_k - \beta_1\alpha_1^T)\beta_j}{x_i-y_j} \quad .$$

Thus we receive $\mathrm{Schur}_{\{1\},\{1\}}\left(\begin{pmatrix} \dfrac{\alpha_1^T\beta_1}{x_1-y_1} & \dfrac{\alpha_1^T\beta_j}{x_1-y_j} \\ \dfrac{\alpha_i^T\beta_1}{x_i-y_1} & \dfrac{\alpha_i^T\beta_j}{x_i-y_j} \end{pmatrix}\right) = \dfrac{\det\begin{pmatrix} \dfrac{\alpha_1^T\beta_1}{x_1-y_1} & \dfrac{\alpha_1^T\beta_j}{x_1-y_j} \\ \dfrac{\alpha_i^T\beta_1}{x_i-y_1} & \dfrac{\alpha_i^T\beta_j}{x_i-y_j} \end{pmatrix}}{\dfrac{\alpha_1^T\beta_1}{x_1-y_1}} =$$

$$= \frac{1}{\alpha_1^T\beta_1}\frac{\alpha_i^T\beta_1\dfrac{x_i-x_1}{x_i-y_1}\cdot\dfrac{y_j-y_1}{x_1-y_j}\alpha_1^T\beta_j + \alpha_i^T(\alpha_1^T\beta_1 I_k - \beta_1\alpha_1^T)\beta_j}{x_i-y_j} =$$

$$= \frac{u_i v_j + \alpha_i^T(\frac{1}{\alpha_1^T \beta_1}(\alpha_1^T \beta_1 I_k - \beta_1 \alpha_1^T))\beta_j}{x_i - y_j}$$

$$\text{where } u_i = \frac{\alpha_i^T \beta_1}{\sqrt{\alpha_1^T \beta_1}} \frac{x_i - x_1}{x_i - y_1}, v_j = \frac{y_j - y_1}{x_1 - y_j} \frac{\alpha_1^T \beta_j}{\sqrt{\alpha_1^T \beta_1}}$$

Since the rank of the matrix $\frac{1}{\alpha_1^T \beta_1}(\alpha_1^T \beta_1 I_k - \beta_1 \alpha_1^T)$ doesn't exceed k-1 and we hence can represent it as $A_1 B_1$ where A_1 is a k×(k-1)-matrix and B_1 is a (k-1)×k-matrix, we get $\alpha_i^T(\frac{1}{\alpha_1^T \beta_1}(\alpha_1^T \beta_1 I_k - \beta_1 \alpha_1^T))\beta_j = \alpha_i^T A_1 B_1 \beta_j$ and, therefore, the compressed matrix's entries are $\frac{u_i v_j + \alpha_i^T A_1 B_1 \beta_j}{x_i - y_j}$ (where $\alpha_i^T A_1$ is a (k-1)-row and $B_1 \beta_j$ is a (k-1) column) what completes the proof.

Definition:

We'll call two elements of a field's extension by the infinitesimal ε **infinitely close** on ε (or, shortly, ε-close) if their difference's order on ε is bigger than zero.

Definition: let's call an n×m-matrix A that can be represented in the form $G \star C(x, y)$ where G is an n×m-matrix of rank k, dim(x) = n, dim(y) = m, a matrix of **Cauchy-rank k**; and we also define, for this representation, G as A's **numerator-matrix** and x,y as A's **row** and **column** (or left and right) **denominator-value** vectors **(**or sets**)** correspondingly. If G = LR, where L is an n×k-matrix and R is a k×m-matrix, we'll call the i-th row of L its i-th **numerator-row** and the j-th column of R its j-th **numerator-column**, while x_i, y_j will be called its i-th **row** and j-th **column denominator-values** correspondingly (for i = 1,…n, j = 1,…,m). In case if n = m and for each i = 1,…,n its i-th row-numerator equals its transposed column-numerator right-multiplied, optionally, by an k×k-matrix M that we'll call the **multiplication matrix**, the i-th numerator-column will be further called its i-th **numerator-vector**.

The above theorem hence tells us that the class of matrices of a Cauchy-rank unexceeding k is closed under the Schur complement compression operator.

We'll also consider the following generalization of a matrix of Cauchy-rank k:

Definition: let ε be an infinitesimal the ground field is extended by, L be a dim(x)×k-matrix and R be a k×dim(y)-matrix and in $LR \star C(x, y)$ some row and column denominator-values form ε-close families, hence forming row- and column-disjoint blocks whose entries have denominators of ε-order 1, i.e. of the type $O(\varepsilon)$ (let's call

them singular entries), while all the denominator-values, numerator-rows and numerator-columns are of ε-order 0 or bigger. In case if the matrix $LR \star C(x,y)$ has no entries of ε-order smaller than zero (infinitely big entries) then we'll call its entry-wise limit on ε $\lim\limits_{\varepsilon \to 0}(LR \star C(x,y))$ a **singularized** matrix of Cauchy-rank k.

Theorem III.8.1 Each Schur complement compression turns a singularized matrix of Cauchy-rank k into a singularized matrix of a Cauchy-rank unexceeding k, and in such a matrix any numerator-row and any numerator-column corresponding to equal row and column denominator-values correspondingly are orthogonal.

==A polynomial-time algorithm for computing the permanent in characteristic 5==

Definition:

Let A be a 2n×2n-matrix. Then its **alternate determinant** is

$$\text{altdet}(A) := \sum_{I \subseteq \{1,...n\}} \det(A^{(I \cup \{n+1,...,2n\} \setminus \hat{I}, \{1,...n\} \setminus I \cup \hat{I})})$$

where $\hat{I}$ is the subset of {n+1,...,2n} received via adding n to each element of I.

Definition: let $P, \widehat{P}$ be 4×n-matrices, $z, g^{(11)}, g^{(12)}, g^{(21)}, g^{(22)}$ be n-vectors. Then

in the matrix $\begin{pmatrix} P^T M P \star \tilde{C}(z) + \text{Diag}(g^{(11)}) & P^T M \widehat{P} \star \tilde{C}(z) + \text{Diag}(g^{(12)}) \\ \widehat{P}^T M P \star \tilde{C}(z) + \text{Diag}(g^{(21)}) & \widehat{P}^T M \widehat{P} \star \tilde{C}(z) + \text{Diag}(g^{(22)}) \end{pmatrix}$

we'll call, for j = 1,...,n, the j-th columns of P and $\widehat{P}$ (p_j and $\hat{p}_j$ correspondingly) the **numerator-vector** and **alternate numerator-vector** correspondingly and $g_j^{(12)} + g_j^{(21)}$ the **absence-weight** of the denominator-value z_j.

Theorem III.9 (in characteristic 5):

Let $P, \widehat{P}$ be 4×n-matrices, t, g be n-vectors, $M = \begin{pmatrix} 0_{2\times2} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0_{2\times2} \end{pmatrix}$. Then

a) $\text{altdet}(\begin{pmatrix} P^T M P \star \tilde{C}(t^{\star 5}) & P^T M \widehat{P} \star \tilde{C}(t^{\star 5}) \\ \widehat{P}^T M P \star \tilde{C}(t^{\star 5}) + \text{Diag}(g) & \widehat{P}^T M \widehat{P} \star \tilde{C}(t^{\star 5}) \end{pmatrix}) =$

$$= \frac{1}{2^n} \operatorname{coef}_{\lambda^n} \operatorname{Pf}(K(\binom{x}{t}, \binom{\vec{0}_m}{\lambda \vec{1}_n}, \binom{\alpha}{\beta})))$$

where

for $i = 1,2,3,4, \; j = 1, \ldots, n$

$$
\begin{cases}
\frac{1}{\Delta} \det\left(\tilde{C}^{\star 2}(x, v_i, t_j) + \operatorname{Diag}(\binom{\alpha}{0})\right) = r_i^T M p_j / (v_i - t_j)^5 \\
\frac{1}{\Delta} \frac{\partial}{\partial t_j} \det\left(\tilde{C}^{\star 2}(x, v_i, t_j) + \operatorname{Diag}(\binom{\alpha}{0})\right) = r_i^T M \hat{p}_j / (v_i - t_j)^5 \\
\qquad \frac{1}{\Delta} \frac{\partial}{\partial t_j} \det\left(\tilde{C}^{\star 2}(\binom{x}{t_j}) + \operatorname{Diag}(\binom{\alpha}{t_j \beta_j})\right) = g_j
\end{cases}
$$

for $i_1, i_2 = 1,2,3,4$

$$\frac{1}{\Delta} \det\left(\tilde{C}^{\star 2}(x, v_{i_1}, v_{i_2}) + \operatorname{Diag}(\binom{\alpha}{0})\right) = r_{i_1}^T M r_{i_2} / (v_{i_1} - v_{i_2})^5 \quad \text{if} \quad i_1 \neq i_2$$

where $p_j, \hat{p}_j$ are the j-th rows of P, $\hat{P}$ correspondingly, v_1, v_2, v_3, v_4 are generic scalars, r_1, r_2, r_3, r_4 are some 4-vectors, $\Delta = \det\left(\tilde{C}^{\star 2}(x) + \operatorname{Diag}(\alpha)\right)$

    b) the set of functions in x, α, β that are the left parts of the above system of equations is a system of functions whose algebraic rank is $11n + 6$ and it implies that the entries of $p_1, \hat{p}_1 \ldots, p_n, \hat{p}_n$ is a system of functions maximally algebraically independent under the condition $p_j^T M \hat{p}_j = 0$ for j = 1,…,n (fulfilled for any x, α, β).

Proof:

Part (1). According to Corollary III.7, the right side of the theorem's first part's equality is

$$(\prod_{i=1}^n \frac{\partial}{\partial t_j}) \det(\tilde{C}^{\star 2}(\binom{x}{t})) + \operatorname{Diag}(\binom{\alpha}{\beta}))) =$$

$$= \sum_{I \in \mathcal{J}} \det \begin{pmatrix} \tilde{C}^{\star 2}(x) + \operatorname{Diag}(\alpha) & C^{\star 2}(x, t) & C^{\star 2}(x, t)D \\ C^{\star 2}(t, x) & \tilde{C}^{\star 2}(t) & \tilde{C}^{\star 2}(t)D \\ DC^{\star 2}(t, x) & D\tilde{C}^{\star 2}(t) + \operatorname{Diag}(\beta) & D\tilde{C}^{\star 2}(t)D \end{pmatrix}^{(\backslash I, \backslash \bar{I})}$$

where $\mathcal{J} = \{m + 1, m + n + 1\} \times \ldots \times \{m + n, m + 2n\}$, $\bar{I}$ is the set received from I via taking the other element from each set $\{m + k, m + n + k\}$ for k = 1,…,n, $D = \operatorname{Diag}(\{\frac{\partial}{\partial t_j}\}_n)$.

and it's equal to

$$\Delta \text{altdet}(\text{Schur}_{\{1,...,m\},\{1,...,m\}}\begin{pmatrix} \tilde{C}^{\star 2}(x) + \text{Diag}(\alpha) & C^{\star 2}(x,t) & C^{\star 2}(x,t)D \\ C^{\star 2}(t,x) & \tilde{C}^{\star 2}(t) & \tilde{C}^{\star 2}(t)D \\ DC^{\star 2}(t,x) & D\tilde{C}^{\star 2}(t) + \text{Diag}(\beta) & D\tilde{C}^{\star 2}(t)D \end{pmatrix}) =$$

$$= \Delta \text{altdet}(\text{Schur}_{\{1,...,m\},\{1,...,m\}}\begin{pmatrix} \tilde{C}^{\star 2}(x) + \text{Diag}(\alpha) & C^{\star 2}(x,t) & C^{\star 2}(x,t)D \\ C^{\star 2}(t,x) & \tilde{C}^{\star 2}(t) & \tilde{C}^{\star 2}(t)D \\ DC^{\star 2}(t,x) & D\tilde{C}^{\star 2}(t) & D\tilde{C}^{\star 2}(t)D \end{pmatrix} +$$

$$+ \begin{pmatrix} 0_{n\times n} & 0_{n\times n} \\ \text{Diag}(\beta) & 0_{n\times n} \end{pmatrix}))$$

where $\Delta = \det(\tilde{C}^{\star 2}(x) + \text{Diag}(\alpha))$.

The matrix $\begin{pmatrix} \tilde{C}^{\star 2}(x) + \text{Diag}(\alpha) & C^{\star 2}(x,t) & C^{\star 2}(x,t)D \\ C^{\star 2}(t,x) & \tilde{C}^{\star 2}(t) & \tilde{C}^{\star 2}(t)D \\ DC^{\star 2}(t,x) & D\tilde{C}^{\star 2}(t) & D\tilde{C}^{\star 2}(t)D \end{pmatrix}$ is symmetric and of Cauchy-

rank 4 for the denominator-value set $\begin{pmatrix} \{x_k^5\}_m \\ \{t_j^5\}_n \\ \{t_j^5\}_n \end{pmatrix}$ because for two independent

indeterminates u, v there holds (for M defined in the theorem)

$$\frac{1}{(u-v)^2} = \frac{(u-v)^3}{(u-v)^5} = \frac{(1 \quad u \quad 3u^2 \quad u^3)M\begin{pmatrix} 1 \\ v \\ 3v^2 \\ v^3 \end{pmatrix}}{(u-v)^5}$$

$$\frac{\partial}{\partial u}\frac{1}{(u-v)^2} = \frac{\frac{\partial}{\partial u}(1 \quad u \quad 3u^2 \quad u^3)M\begin{pmatrix} 1 \\ v \\ 3v^2 \\ v^3 \end{pmatrix}}{(u-v)^5}$$

$$\frac{\partial}{\partial v}\frac{1}{(u-v)^2} = \frac{(1 \quad u \quad 3u^2 \quad u^3)M\frac{\partial}{\partial v}\begin{pmatrix} 1 \\ v \\ 3v^2 \\ v^3 \end{pmatrix}}{(u-v)^5}$$

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v}\frac{1}{(u-v)^2} = \frac{\frac{\partial}{\partial u}(1 \quad u \quad 3u^2 \quad u^3)M\frac{\partial}{\partial v}\begin{pmatrix} 1 \\ v \\ 3v^2 \\ v^3 \end{pmatrix}}{(u-v)^5} \ .$$

Hence its Schur complement on $\{1, ..., m\}, \{1, ..., m\}$ is also symmetric and of a Cauchy-rank unexceeding 4 for the same denominator-value set and, accordingly, has the form

$$\begin{pmatrix} P^T MP \star \tilde{C}(t^{\star 5}) + D_{11} & P^T M\hat{P} \star \tilde{C}(t^{\star 5}) + D_{12} \\ \hat{P}^T MP \star \tilde{C}(t^{\star 5}) + D_{21} & \hat{P}^T M\hat{P} \star \tilde{C}(t^{\star 5}) + D_{22} \end{pmatrix}$$ where $D_{11}, D_{12}, D_{21}, D_{22}$ are diagonal, $D_{12} = D_{21}$, and the diagonal entries of $P^T MP$, $P^T M\hat{P}$, $\hat{P}^T MP$, $\hat{P}^T M\hat{P}$ are zeros (as it's supposed in matrices of any Cauchy-rank for entries with equal row and column denominator-values).

Part (2). As, according to Corollary III.7, the right side of the theorem's first part's equality is $(\prod_{i=1}^n \frac{\partial}{\partial t_j}) \det(\tilde{C}^{\star 2}(\begin{pmatrix} X \\ t \end{pmatrix})) + \text{Diag}(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}))$, let's consider the following two identities:

1) $\lim_{\varepsilon \to 0} \det(\tilde{C}^{\star 2}(\begin{pmatrix} x^{(0)} \\ y \\ y + \varepsilon \vec{1}_{\dim(y)} \\ t \end{pmatrix}) + \text{Diag}(\begin{pmatrix} \alpha^{(0)} \\ \varepsilon^{-2}\vec{1}_{\dim(y)} + \varepsilon^2 \alpha^{(1)} \\ \varepsilon^{-2}\vec{1}_{\dim(y)} \\ \beta \end{pmatrix})) =$

$$= \det(D\tilde{C}^{\star 2}(\begin{pmatrix} x^{(0)} \\ y \\ t \end{pmatrix})D + \text{Diag}(\begin{pmatrix} \alpha^{(0)} \\ \gamma \\ \beta \end{pmatrix}))$$

$$\text{where } D = \text{Diag}(\begin{pmatrix} I_{m_0} \\ \text{Diag}(\{\frac{\partial}{\partial y_k}\}_{\dim(y)}) \\ I_n \end{pmatrix})$$

and

2) $\det(\breve{D}\tilde{C}^{\star 2}(\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \\ t \end{pmatrix})\breve{D} + \text{Diag}(\begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \\ \beta \end{pmatrix})) =$

$$= \lim_{\varepsilon \to 0}(\varepsilon^{2m_2} \det(D\tilde{C}^{\star 2}(\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \\ x^{(2)} + \varepsilon\vec{1}_{m_2} \\ t \end{pmatrix})D + \text{Diag}(\begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \varepsilon^{-4}\vec{1}_{m_2} + \varepsilon^2 \alpha^{(2)} \\ \varepsilon^{-4}\vec{1}_{m_2} \\ \beta \end{pmatrix})))$$

$$\text{where } \breve{D} = \text{Diag}\left(\begin{pmatrix} I_{m_0} \\ \text{Diag}(\{\frac{\partial}{\partial x_k^{(1)}}\}m_1) \\ \text{Diag}(\{\frac{\partial^2}{\partial (x_k^{(2)})^2}\}m_2) \\ I_n \end{pmatrix}\right), \ D = \text{Diag}\left(\begin{pmatrix} I_{m_0} \\ \text{Diag}(\{\frac{\partial}{\partial x_k^{(1)}}\}m_1) \\ \text{Diag}(\{\frac{\partial}{\partial x_k^{(2)}}\}m_2) \\ \text{Diag}(\{\frac{\partial}{\partial (x_k^{(2)}+\varepsilon)}\}m_2) \\ I_n \end{pmatrix}\right).$$

$$\dim(x^{(q)}) = \dim(\alpha^{(q)}) = m_q \text{ for q = 0,1,2}$$

Hence, for proving the theorem's second part, it's sufficient to replace, as by a partial case of x, α (yielding a generalization, though), its first part's system of equations by the following system:

for $i = 1,2,3,4, \ j = 1, \dots, n$

$$\begin{cases} \frac{1}{\Delta}\det(\breve{D}\tilde{C}^{\star 2}(\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \end{pmatrix}, v_i, t_j)\breve{D} + \text{Diag}(\begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \\ 0 \end{pmatrix})) = r_i^T M p_j/(v_i - t_j)^5 \\[3em] \frac{1}{\Delta}\frac{\partial}{\partial t_j}\det(\breve{D}\tilde{C}^{\star 2}(\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \end{pmatrix}, v_i, t_j)\breve{D} + \text{Diag}(\begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \\ 0 \end{pmatrix})) = r_i^T M \hat{p}_j/(v_i - t_j)^5 \\[3em] \frac{1}{\Delta}\frac{\partial}{\partial t_j}\det(\breve{D}\tilde{C}^{\star 2}(\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \\ t_j \end{pmatrix})\breve{D} + \text{Diag}(\begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \\ t_j\beta_j \end{pmatrix})) = g_j \end{cases}$$

for $i_1, i_2 = 1,2,3,4$

$$\frac{1}{\Delta}\det(\breve{D}\tilde{C}^{\star 2}(\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \end{pmatrix}, v_{i_1}, v_{i_2})\breve{D} + \text{Diag}(\begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \\ 0 \end{pmatrix})) = r_{i_1}^T M r_{i_2}/(v_{i_1} - v_{i_2})^5 \quad \text{if} \quad i_1 \neq i_2$$

where $\breve{D} = \text{Diag}(\begin{pmatrix} I_{m_0} \\ \text{Diag}(\{\frac{\partial}{\partial x_k^{(1)}}\}m_1) \\ \text{Diag}(\{\frac{\partial}{\partial x_k^{(2)}}\}m_2) \\ I_n \end{pmatrix})$, $p_j, \hat{p}_j$ are the j-th rows of P, $\widehat{P}$

correspondingly, v_1, v_2, v_3, v_4 are generic scalars, r_1, r_2, r_3, r_4 are generic 4-vectors, $\Delta = $

$$\det(\breve{D}\widetilde{C}^{\star 2}(\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \end{pmatrix})\breve{D} + \text{Diag}(\begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \end{pmatrix}))$$

Now let's choose a partial case of the vectors (sets) $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}$ where all of them can be partitioned into subvectors (subsets) of sizes divisible by 5 each of whom consists of entries (elements) $\lambda_k/(\varepsilon d_{k,r})$ (with λ_k distinct for different subsets), where ε is a formal infinitesimal, k is the index of the subset and r is the index of the element in the subset. We'll call each λ_k a **uniting value**, while by $U(\lambda_k) = \{x_{k,1}, \dots, x_{k,|U(\lambda_k)|}\}$ we'll denote the

subvector (subset) of $\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \end{pmatrix}$ corresponding to λ_k and we'll call it the family of

denominator-values united by λ_k (thus each $x_{k,r}$ is an entry of either $x^{(0)}$ or $x^{(1)}$ or $x^{(2)}$).

Let's show that this partial case turns, for a generic fixed $\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \end{pmatrix}$, the left parts of our

system of equations into a system of functions in the uniting values λ_k and the entries of β whose algebraic rank is 11n + 6. Because the absence-weight equation for the variable β_j

$$\frac{1}{\Delta}\frac{\partial}{\partial t_j}\det(\breve{D}\widetilde{C}^{\star 2}(\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \\ t_j \end{pmatrix})\breve{D} + \text{Diag}(\begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \\ t_j\beta_j \end{pmatrix})) = g_j$$

is solvable for any fixed $\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \end{pmatrix}$, $\begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \end{pmatrix}$ such that $\Delta \neq 0$, it's sufficient to prove that the

algebraic rank of the system of functions received from the above-mentioned system

via excluding all the absence-weights $\frac{1}{\Delta}\frac{\partial}{\partial t_j}\det(\breve{D}\widetilde{C}^{\star 2}(\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \\ t_j \end{pmatrix})\breve{D} + \text{Diag}(\begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \\ t_j\beta_j \end{pmatrix}))$ is 7n +

6.

Due to the divisibility of each united family's size by 5, the derivatives of the functions

$$f_{i,j}(\lambda) = \frac{1}{\Delta} \det(\breve{D}\tilde{C}^{\star 2}(\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \end{pmatrix}, v_i, t_j)\breve{D} + \mathrm{Diag}(\begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \\ 0 \end{pmatrix}))) \ ,$$

$$\hat{f}_{i,j}(\lambda) = \frac{1}{\Delta} \frac{\partial}{\partial t_j} \det(\breve{D}\tilde{C}^{\star 2}(\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \end{pmatrix}, v_i, t_j)\breve{D} + \mathrm{Diag}(\begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \\ 0 \end{pmatrix}))$$

$$\text{and } h_{i_1,i_2}(\lambda) = \frac{1}{\Delta} \det(\breve{D}\tilde{C}^{\star 2}(\begin{pmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \end{pmatrix}, v_{i_1}, v_{i_2})\breve{D} + \mathrm{Diag}(\begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \\ 0 \end{pmatrix}))$$

on any uniting value λ_k are of ε-order 1 or bigger. Hence it's sufficient to prove the

equality $\mathrm{rank}(\lim_{\varepsilon \to 0}(\varepsilon^{-1} \mathfrak{J}(\begin{pmatrix} f \\ \hat{f} \\ h \end{pmatrix}, \lambda))) = 7n + 6$ where $\mathfrak{J}(\begin{pmatrix} f \\ \hat{f} \\ h \end{pmatrix}, \lambda)$ is their Jacobian

matrix on the uniting values.

We'll say that a uniting value λ_k is of **auxiliary differentiation order** q if $U(\lambda_k)$ is a subset of $x^{(q)}$, for q = 0,1,2. Then we receive for λ_k of auxiliary differentiation order q, for i = 1,2,3,4, j = 1,…,n:

$$\lim_{\varepsilon \to 0} \frac{\partial}{\partial \lambda_k}(\varepsilon^{-1} f_{i,j}(\lambda)) = \sum_{r=1}^{|U(\lambda_k)|} d_{k,r}(\frac{\partial^q}{\partial x_{k,r}^q} \frac{1}{(v_i - x_{k,r})^2})(\frac{\partial^q}{\partial x_{k,r}^q} \frac{1}{(x_{k,r} - t_j)^2})$$

$$\lim_{\varepsilon \to 0} \frac{\partial}{\partial \lambda_k}(\varepsilon^{-1} \hat{f}_{i,j}(\lambda)) = \frac{\partial}{\partial t_j} \sum_{r=1}^{|U(\lambda_k)|} d_{k,r}(\frac{\partial^q}{\partial x_{k,r}^q} \frac{1}{(v_i - x_{k,r})^2})(\frac{\partial^q}{\partial x_{k,r}^q} \frac{1}{(x_{k,r} - t_j)^2})$$

$$\lim_{\varepsilon \to 0} \frac{\partial}{\partial \lambda_k}(\varepsilon^{-1} h_{i_1,i_2}(\lambda)) = \sum_{r=1}^{|U(\lambda_k)|} d_{k,r}(\frac{\partial^q}{\partial x_{k,r}^q} \frac{1}{(v_{i_1} - x_{k,r})^2})(\frac{\partial^q}{\partial x_{k,r}^q} \frac{1}{(x_{k,r} - v_{i_2})^2})$$

For i = 1,2,3,4, j = 1,…,n, the first two above sums are linear combinations of the sums $\sum_{r=1}^{|U(\lambda_k)|} \frac{d_{k,r}}{(v_i - x_{k,r})^w}$ and $\sum_{r=1}^{|U(\lambda_k)|} \frac{d_{k,r}}{(t_j - x_{k,r})^w}$ for w = 2,3,4,5, while the third one is a linear combination of $\sum_{r=1}^{|U(\lambda_k)|} \frac{d_{k,r}}{(v_{i_1} - x_{k,r})^w}$ and $\sum_{r=1}^{|U(\lambda_k)|} \frac{d_{k,r}}{(v_{i_2} - x_{k,r})^w}$ for w = 2,3,4,5. Let's consider the case when for each uniting value λ_k $\sum_{r=1}^{|U(\lambda_k)|} \frac{d_{k,r}}{(v_i - x_{k,r})^w} = \sum_{r=1}^{|U(\lambda_k)|} \frac{d_{k,r}}{(t_j - x_{k,r})^w} = 0$ for all

i, j except either exactly one index $i(\lambda_k)$ or exactly one index $j(\lambda_k)$ and for all w except exactly one degree $w(\lambda_k)$. Let's call such a uniting value λ_k a v_i,w-**supporting** and t_j,w-**supporting** uniting value correspondingly. We additionally put $\sum_{r=1}^{|U(\lambda_k)|} \frac{d_{k,r}}{(c-x_{k,r})^w} = 1$ and $\sum_{r=1}^{|U(\lambda_k)|} \frac{d_{k,r}}{(v_i-x_{k,r})^w} = 1$ if λ_k supports t_j,w and v_i,w correspondingly. Then, for j = 1,...n, we take

one t_j,w-supporting uniting value of auxiliary differentiation order q for each of the following pairs (w,q): (5,2), (4,1), (3,0), (1,1), (2,1), (1,0), (2,0);

one v_1,w-supporting uniting value of auxiliary differentiation order 1 for each of w = 1,2,3;

one v_2,1-supporting uniting value of auxiliary differentiation order 0 for each of w = 1,2;

one v_3,1-supporting uniting value of auxiliary differentiation order 0.

Then, upon multiplying its columns by non-zero constants from the set {1,2,3,4},

$\lim_{\varepsilon \to 0}(\varepsilon^{-1} \, \Im(\begin{pmatrix} f \\ \hat{f} \\ f \\ h \end{pmatrix}, \lambda))$ will be the block-triangular matrix

$$\begin{pmatrix} \text{Diag}(\{T(t_j)\}_n) & P \\ 0_{6\times 7n} & \begin{pmatrix} \{\frac{1}{(v_i - v_1)^{5-s}}\}_{\substack{i=2,3,4 \\ s=1,2,3}} & B_{12} & B_{13} \\ 0_{2\times 3} & \{\frac{1}{(v_i - v_2)^{3-s}}\}_{\substack{i=3,4 \\ s=1,2}} & B_{23} \\ 0_{1\times 3} & 0_{1\times 2} & \frac{1}{(v_3 - v_4)^3} \end{pmatrix} \end{pmatrix}$$

where for j = 1,...,n $T(t_j) = \begin{pmatrix} \{\frac{1}{(v_i - t_j)^{5-s}}\}_{\substack{i=1,2,3,4 \\ s=1,2,3}} & A_j \\ 0_{4\times 3} & \{\frac{1}{(v_i - t_j)^{5-s}}\}_{\substack{i=1,2,3,4 \\ s=1,2,3,4}} \end{pmatrix}$, A_j is a 4×4-matrix,

P is an 7n×6-matrix, B_{12} is a 3×2-matrix, B_{13} is a 3×1-matrix, B_{23} is a 2×1-matrix.

Taking into account the fact that each of the above matrix's first n diagonal 8×7-blocks is of rank 7 and the last 3 ones are of ranks 3,2,1 correspondingly, we complete the theorem's proof.

The above theorem implies that in characteristic 5 we can polynomial-time reduce computing the alternate determinant of a symmetric singularized matrix of Cauchy-rank 4 such that all its denominator-values are alternate-wise doubled and $p_j^T M \hat{p}_j = 0$ for j = 1,…,n to computing a Pfaffian via the use of the neighboring computation principle (because the alternate determinant of such a matrix is a polynomial in the entries of its numerator-vectors, alternate numerator-vectors and its alternate-wise block-diagonal entries) and, hence, the alternate determinant of such a matrix is computable in polynomial time.

Definition:

We'll call **proper** a directed cycle that isn't a loop.

Definition: let dim(y) = dim(g) = n, A be an n×n-matrix, $B^{(1)}, …, B^{(n)}$ be m×m-matrices. Then we define the **trace-determinant** of A on $B^{(1)}, …, B^{(n)}$ as

$$\det\nolimits_{tr}\left(A, \{B^{(i)}\}_n\right) := \sum_{\pi \in S_n} \prod_{\mathcal{C} \in PC(\pi)} ((-1)^{|\mathcal{C}|+1} tr(\prod_{q=1}^{|\mathcal{C}|} B^{(i_q(\mathcal{C}))})) \prod_{i=1}^{n} a_{i,\pi_i}$$

where $PC(\pi)$ is the set of π's proper cycles, while for each proper cycle $\mathcal{C} = (i_1(\mathcal{C}), …, i_{|\mathcal{C}|}(\mathcal{C})) \in PC(\pi)$ (represented with the lexicographically minimal starting vertex $i_1(\mathcal{C})$) the multiplication order in the matrix product $\prod_{q=1}^{|\mathcal{C}|} B^{(i_q(C))} = B^{(i_1(\mathcal{C}))} … B^{(i_{|\mathcal{C}|}(\mathcal{C}))}$ is $\mathcal{C}$'s order.

In case if $A = \tilde{C}(x) + \text{Diag}(g)$, we'll call $B^{(i)}$ the **_matrix-weight_** and g_i the absence-weight of the denominator-value x_i. Further we'll often consider, when dealing with the trace-determinant, any denominator-value together with these two parameters as the triple $(x_i, B^{(i)}, g_i)$, while assuming the notions of infinitesimal-closeness and limit on an infinitesimal only for the values of x_i themselves when speaking about infinitesimal-close denominator-values or/and their limit.

Theorem III.10: Let $P, \widehat{P}$ be 4×n-matrices, t, g be n-vectors, $M = \begin{pmatrix} 0_{2\times2} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0_{2\times2} \end{pmatrix}$.

Then

$$\det\nolimits_{tr}\left(\tilde{C}(t) + \text{Diag}(g), \{(p_i\hat{p}_i^T + \hat{p}_i p_i^T)M\}_n\right) =$$

$$= \text{altdet}\left(\begin{pmatrix} P^T M P \star \tilde{C}(t^{\star 5}) & P^T M \hat{P} \star \tilde{C}(t^{\star 5}) \\ \hat{P}^T M P \star \tilde{C}(t^{\star 5}) + \text{Diag}(g) & \hat{P}^T M \hat{P} \star \tilde{C}(t^{\star 5}) \end{pmatrix}\right)$$

As a corollary, due the fact that for any 4×2-matrix such that $F^T M F = 0_{2 \times 2}$ the matrix FF^T can be represented as $p\hat{p}^T + \hat{p}p^T$ for some 4-vectors p, $\hat{p}$ such that $pM\hat{p}^T = 0$, we get

Theorem III.10.1:

$\det_{tr}\left(\tilde{C}(t) + \text{Diag}(g), \{F^{(i)}(F^{(i)})^T M\}_n\right)$ is polynomial-time computable for any 4×2-matrices $F^{(1)}, \dots, F^{(n)}$ such that $(F^{(i)})^T M F^{(i)} = 0_{2 \times 2}$ for i = 1,…,n.

Definition: let dim(y) = dim(g) = n, A be an n×n-matrix, $B^{(1)}, \dots, B^{(n)}$ be m×m-matrices. Then we define the **open trace-determinant** of A on $B^{(1)}, \dots, B^{(n)}$ as

$$\widetilde{\det}_{tr}\left(A, \{B^{(i)}\}_n\right) :=$$

$$\sum_{\pi \in \check{S}_n} \left(\frac{(-1)^{|\mathcal{P}_\pi|+1}}{a_{i_{|\mathcal{P}_\pi|}(\mathcal{P}_\pi), i_1(\mathcal{P}_\pi)}} \prod_{q=1}^{|\mathcal{P}_\pi|} B^{(i_q(\mathcal{P}_\pi))}\right) \prod_{\mathcal{C} \in \widetilde{PC}(\pi)} \left((-1)^{|\mathcal{C}|+1} \text{tr}\left(\prod_{q=1}^{|\mathcal{C}|} B^{(i_q(\mathcal{C}))}\right)\right) \prod_{i=1}^n a_{i,\pi_i}$$

where $\check{S}_n$ is the set on n-permutations where exactly one cycle is considered as broken and turned into a path $\mathcal{P}_\pi = (i_1(\mathcal{P}_\pi), \dots, i_{|\mathcal{P}_\pi|}(\mathcal{P}_\pi))$ (including, as an option, the case of a loop $\mathcal{P}_\pi = (i_1(\mathcal{P}_\pi))$), $\widetilde{PC}(\pi)$ is the set of π's unbroken proper cycles, while the multiplication order in the matrix product $\prod_{q=1}^{|\mathcal{C}|} B^{(i_q(\mathcal{C}))} = B^{(i_1(\mathcal{C}))} \dots B^{(i_{|\mathcal{C}|}(\mathcal{C}))}$ is $\mathcal{C}$'s order for each unbroken proper cycle $\mathcal{C} = (i_1(\mathcal{C}), \dots, i_{|\mathcal{C}|}(\mathcal{C})) \in \widetilde{PC}(\pi)$ (represented with the lexicographically minimal starting vertex $i_1(\mathcal{C})$) and the multiplication order in the matrix product $\prod_{q=1}^{|\mathcal{P}_\pi|} B^{(i_q(\mathcal{P}_\pi))}$ is $\mathcal{P}_\pi$'s order.

Comment: the above definition remains actual also in the case $a_{i_1(\mathcal{P}_\pi), i_{|\mathcal{P}_\pi|}(\mathcal{P}_\pi)} = 0$ due to the presence of the term $a_{i_1(\mathcal{P}_\pi), i_{|\mathcal{P}_\pi|}(\mathcal{P}_\pi)}$ in $\prod_{i=1}^n a_{i,\pi_i}$.

Theorem III.10.2 (in an arbitrary characteristic)

For i = 1,…,n, let x_i be a scalar, χ_i, h_i be k_i-vectors and $L^{(i,1)}, \dots, L^{(i,k_i)}$ be m×m-matrices of ε-order 0 or bigger such that $\lim_{\varepsilon \to 0} \dfrac{\det_{tr}\left(\tilde{C}(\chi_i) + \text{Diag}(\varepsilon^{-1} h_i), \begin{pmatrix} L^{(i,1)} \\ \dots \\ L^{(i,n_i)} \end{pmatrix}\right)}{\varepsilon^{-k_i}} = 0$ for i =1,…n. Then

$$\det_{tr}(\tilde{C}(x) + \text{Diag}(\{g_j\}_n), \{B^{(i)}\}_n) =$$

$$= \lim_{\varepsilon \to 0} \frac{\det_{tr}(\tilde{C}(\{x_i \vec{1}_{n_i} + \varepsilon\chi_i\}_n) + \text{Diag}(\{\varepsilon^{-1}h_i\}_n), \{\begin{pmatrix} L^{(i,1)} \\ \cdots \\ L^{(i,k_i)} \end{pmatrix}\}n)}{\varepsilon^{n-(k_1+\cdots+k_n)}}$$

where for i = 1,…,n
$$g_i = \lim_{\varepsilon \to 0} \frac{\det_{tr}(\tilde{C}(\chi_i) + \text{Diag}(\varepsilon^{-1}h_i), \begin{pmatrix} L^{(i,1)} \\ \cdots \\ L^{(i,k_i)} \end{pmatrix})}{\varepsilon^{1-k_i}},$$

$$B^{(i)} = \widetilde{\det}_{tr}(\tilde{C}(\chi_i) + \text{Diag}\left(\lim_{\varepsilon \to 0} h_i\right), \lim_{\varepsilon \to 0} \begin{pmatrix} L^{(i,1)} \\ \cdots \\ L^{(i,n_i)} \end{pmatrix})$$

Proof:

This statement follows from the definitions of the trace-determinant and the open trace-determinant because the "common" limit $\lim_{\varepsilon \to 0}$ provides, for i = 1,…,n, the "opening" of the weighted sub-digraph corresponding to the ε-close denominator-value family $x_i\vec{1}_{n_i} + \varepsilon\chi_i$ (whose arcs are of ε-order -1, while all the matrix-weights are of ε-order 0) , while the i-th absence weight g_i is obtained under the "common" limit as the limit of this sub-digraph's trace-determinant divided by ε^{1-k_i} (i.e. via the case of this sub-digraph remaining "closed" in the trace-determinant's traversals).

By the above theorem, we hence introduced one more type of compression (actual for the trace-determinant) when a family of pair-wise infinitesimal-close denominator-values contracts, via the limit on the infinitesimal, into a new denominator-value that is their common limit on the infinitesimal and, accordingly, this family generates its limit's matrix-weight and absence-weight. Therefore it's also a compression of a family of m×m-matrices into another m×m-matrix. Hence, given a family of m×m-matrices, it contracts (for all the possible "accompanying" families of denominator-values and absence-weights providing the corresponding matrix's trace-determinant's equality to zero, i.e. the "opening" of the trace-determinant) into a set of new m×m-matrices (depending on the chosen families of denominator-values and absence-weights). Let's call it the open trace-determinant compression of a family of m×m-matrices. Hence, given a class of m×m-matrices, it generates, via the open trace-determinant compression of its subsets (families), a wider class and accordingly we can also speak, once again, about the compression-closure of the class for this operator.

Comment: in the above theorem, the ε-orders can also be considered fractional or/and yield a non-existing (infinitely big) expression.

Theorem III.11 (in characteristic 5):

Let $B^{(1)}, \ldots, B^{(n)}$ be symmetric 4×4-matrices such that for i = 1,…,n $B^{(i)}M$ has not more than two eigenvalues, $M = \begin{pmatrix} 0_{2\times2} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0_{2\times2} \end{pmatrix}$, $F^{(i)}, F^{(n+i)}$ be 4×2-matrices for i = 1,…,n, t, g be n-vectors. Then

1) $\det_{tr}\left(\tilde{C}(x) + \text{Diag}(g), \{B^{(i)}M\}_n\right) = \lim\limits_{\varepsilon \to 0} \dfrac{\det_{tr}(\tilde{C}(\binom{x}{x+\varepsilon})) + \text{Diag}(\binom{\varepsilon^{-1/2}g}{\varepsilon^{-1/2}\vec{1}_n}), \binom{\{F^{(i)}(F^{(i)})^TM\}_n}{\{F^{(n+i)}(F^{(n+i)})^TM\}_n})}{\varepsilon^{-n}}$

where for i = 1,…,n
$$\begin{cases} Y^{(i)} \begin{pmatrix} 0_{2\times2} & Z^{(i)} \\ (Z^{(i)})^T & 0_{2\times2} \end{pmatrix} (Y^{(i)})^T = B^{(i)} \\ (Y^{(i)})^T M Y^{(i)} = \begin{pmatrix} 0_{2\times2} & -Z^{(i)} \\ (Z^{(i)})^T & 0_{2\times2} \end{pmatrix} \\ Y^{(i)} = \begin{pmatrix} F^{(i)} & F^{(n+i)} \end{pmatrix} \end{cases}$$

for a 2×2-matrix $Z^{(i)}$ such that $\text{tr}(Z^{(i)}(Z^{(i)})^T) = 0$.

The above system of equations for the variables $Y^{(i)}, Z^{(i)}, F^{(i)}$ is solvable for an arbitrary symmetric 4×4-matrix $B^{(i)}$ such that $B^{(i)}M$ has not more than two eigenvalues.

2) this theorem's Part (1)'s conditions imply $(F^{(i)})^T M F^{(i)} = 0_{2\times2}$ for i = 1,…,2n and, accordingly, $\det_{tr}\left(\tilde{C}(x) + \text{Diag}(g), \{B^{(i)}M\}_n\right)$ is polynomial-time computable for arbitrary symmetric 4×4-matrices $B^{(1)}, \ldots, B^{(n)}$ such that for i = 1,…,n $B^{(i)}M$ has not more than two eigenvalues (including the partial case of $B^{(i)}M = F^{(i)}(F^{(i)})^T M$ having exactly one eigenvalue equal to zero when $(F^{(i)})^T M F^{(i)} = 0_{2\times2}$).

Proof:
The main formula of this theorem (in Part (1)) is a direct implication of Theorem III.10.2 as its conditions imply the "opening" of the corresponding weighted sub-digraph for each ε-close pair of denominator-values $x_i, x_i + \varepsilon$. And now let's prove the theorem's statements on solvability and computability.

Let's use the fact that the system of equations $\begin{cases} YAY^T = B \\ Y^T\breve{A}Y = \breve{B} \end{cases}$ for the m×m-matrix variable Y, where A, B are symmetric m×m-matrices and $\breve{A}, \breve{B}$ are skew-symmetric m×m-matrices, is solvable if and only if $A\breve{B}$ and $B\breve{A}$ have equal eigenvalue spectrums, while the product of a symmetric matrix and a skew-symmetric one (of the same size) has the eigenvalue spectrum of a skew-symmetric matrix, i.e. partitionable into pairs of opposite eigenvalues.

We hence conclude that the eigenvalue spectrums' equality for the matrices
$$\begin{pmatrix} 0_{2\times2} & Z^{(i)} \\ (Z^{(i)})^T & 0_{2\times2} \end{pmatrix}\begin{pmatrix} 0_{2\times2} & -Z^{(i)} \\ (Z^{(i)})^T & 0_{2\times2} \end{pmatrix} = \begin{pmatrix} Z^{(i)}(Z^{(i)})^T & 0_{2\times2} \\ 0_{2\times2} & -(Z^{(i)})^TZ^{(i)} \end{pmatrix} \text{ and } B^{(i)}M$$
is equivalent to the solvability of the system of equations
$$\begin{cases} Y^{(i)}\begin{pmatrix} 0_{2\times2} & Z^{(i)} \\ (Z^{(i)})^T & 0_{2\times2} \end{pmatrix}(Y^{(i)})^T = B^{(i)} \\ (Y^{(i)})^TMY^{(i)} = \begin{pmatrix} 0_{2\times2} & -Z^{(i)} \\ (Z^{(i)})^T & 0_{2\times2} \end{pmatrix} \\ Y^{(i)} = \begin{pmatrix} F^{(i)} & F^{(n+i)} \end{pmatrix} \end{cases}$$
for the variables $Y^{(i)}, Z^{(i)}, F^{(i)}$, while the condition $\text{tr}(Z^{(i)}(Z^{(i)})^T) = 0$ implies that
$\begin{pmatrix} Z^{(i)}(Z^{(i)})^T & 0_{2\times2} \\ 0_{2\times2} & -(Z^{(i)})^TZ^{(i)} \end{pmatrix}$ has not more than two eigenvalues.

Theorem III.11.1 (in characteristic 5):

Let $B^{(1)}, \ldots, B^{(n)}$ be symmetric 4×4-matrices such that for i = 1,…,n $B^{(i)}M$ has not more than two eigenvalues, $M = \begin{pmatrix} 0_{2\times2} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0_{2\times2} \end{pmatrix}$, $F^{(1)}, \ldots, F^{(n)}$ be 4×2-matrices, t, g be n-vectors. Then

$$\det_{tr}\left(\tilde{C}(x) + \text{Diag}(g), \left\{(B^{(i)} + F^{(i)}(K^{(i)} - \text{tr}(K^{(i)})I_2)(F^{(i)})^T)M\right\}_n\right) =$$

$$= \lim_{\varepsilon\to0}\lim_{\varepsilon_1\to0} \frac{\det_{tr}(\tilde{C}(\begin{pmatrix} x\ddot{+}\varepsilon \\ x \\ x\ddot{+}\varepsilon_1 \end{pmatrix})) + \text{Diag}(\begin{pmatrix} -g \\ \varepsilon^{-1}\vec{1}_n \\ -\varepsilon^{-1}\vec{1}_n \end{pmatrix}), \begin{pmatrix} \{B^{(i)}M\}_n \\ \{F^{(i)}(F^{(i)})^TM\}_n \\ \{F^{(i)}(F^{(i)})^TM\}_n \end{pmatrix})}{\varepsilon^{-2n}}$$

where for i = 1,…,n $K^{(i)} = (F^{(i)})^TMB^{(i)}MF^{(i)}$

Proof:

This statement follows from Theorem III.10.2 for the same reasons as Theorem III.11, with the only difference that first we take $\lim\limits_{\varepsilon_1 \to 0}$ for to receive, for i =1,…,n, the pair of equal denominator-values x_i, x_i with equal matrix-weights and opposite absence-weights that "make" them to be present and absent only "together". And, when "appearing" together, they "join" the denominator-value $x_i + \varepsilon$ for to form an ε-close denominator-value family (with two identical "twin"-members) whose trace-determinant "opens".

Theorem III.11.2 (in characteristic 5):

1) $\det_{tr}\left(\tilde{C}(x) + \text{Diag}(g), \left\{(-B^{(i)} + F^{(i)}(K^{(i)} - tr(K^{(i)})I_2)(F^{(i)})^T)M\right\}_n\right)$, where for i = 1,…,n $K^{(i)} = (F^{(i)})^T MB^{(i)}MF^{(i)}$, is polynomial-time computable if for i = 1,…,n $B^{(i)}$ is an arbitrary symmetric 4×4-matrix such that $B^{(i)}M$ has not more than two eigenvalues

and $F^{(i)}$ is an 4×2-matrix such that $(F^{(i)})^T MF^{(i)} = 0_{2\times2}$, $M = \begin{pmatrix} 0_{2\times2} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0_{2\times2} \end{pmatrix}$.

2) $\det_{tr}\left(\tilde{C}(x) + \text{Diag}(g), \left\{G^{(i)}M\right\}_n\right)$ is polynomial-time computable for arbitrary symmetric 4×4-matrices $G^{(1)}, …, G^{(n)}$.

Proof:

Part (2). This statement follows from Theorem III.11.1 and the fact that any symmetric 4×4-matrix G can be represented as $-B + F(K - tr(K)I_2)F^T$ where B is a symmetric 4×4-matrix such that $B^{(i)}M$ has not more than two eigenvalues, F is an 4×2-matrix such that $F^T MF = 0_{2\times2}$, and $K = F^T MBMF$.

Let's also notice that the class of matrices of the form BM, where B is a symmetric 4×4-matrix and $M = \begin{pmatrix} 0_{2\times2} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0_{2\times2} \end{pmatrix}$, is closed under the open trace-determinant compression operator. Hence above we've proven that its subclass of matrices of the

form FF^TM, where F is an 4×2-matrix such that $F^TMF = 0_{2\times2}$, generates the whole class via this operator's closure.

Theorem III.12 (in characteristic 5): let $\dim(x) = n$, $\dim(y) = \dim(\lambda) = m$. Then

$$\text{per}\big(C^{\star 2}(x,y)\text{Diag}(\lambda)\big) =$$

$$= (\prod_{j=1}^{m}\lambda_j)\det_{tr}(\tilde{C}(\begin{smallmatrix}x\\y\end{smallmatrix})) + \text{Diag}(\begin{pmatrix}\vec{0}_n\\\lambda^{\star(-1)}\end{pmatrix})), (\begin{smallmatrix}\{GM\}_n\\\{\widehat{GM}\}_m\end{smallmatrix}))$$

where

$$M = \begin{pmatrix} 0_{2\times2} & \begin{matrix}0 & 1\\-1 & 0\end{matrix} \\ \begin{matrix}0 & 1\\-1 & 0\end{matrix} & 0_{2\times2} \end{pmatrix}, G = \begin{pmatrix} 0_{2\times2} & \begin{matrix}0 & 1\\\sqrt{-1} & 0\end{matrix} \\ \begin{matrix}0 & \sqrt{-1}\\1 & 0\end{matrix} & 0_{2\times2} \end{pmatrix}, \widehat{G} = \begin{pmatrix} 0_{2\times2} & \begin{matrix}0 & -1\\\sqrt{-1} & 0\end{matrix} \\ \begin{matrix}0 & \sqrt{-1}\\-1 & 0\end{matrix} & 0_{2\times2} \end{pmatrix}$$

This theorem hence provides, due to the previous theorem regarding the polynomial-time computability of $\det_{tr}\big(\tilde{C}(x) + \text{Diag}(g), \{G^{(i)}M\}_n\big)$ for any symmetric matrices $G^{(i)}$ in characteristic 5, the polynomial-time computability of $\text{per}\big(C^{\star 2}(x,y)\text{Diag}(\lambda)\big)$ in characteristic 5. Further, in Theorem III.32, we'll prove its $\#_5P$-completeness.

Lemma III.12.1 (in a prime characteristic p):

Let ω be a p-vector whose entries are all the elements of $GF(p)$. Then

1) Let $f(u_1, \dots, u_{p-k})$ be a symmetric homogeneous polynomial in p-k variables of degree q. Then $\sum_{I\subseteq\{1,\dots,p\},|I|=k} \det\big(\tilde{C}(\omega_I)\big) f(\omega_{\backslash I}) = 0$ if $q \le k < p-1$;

2) $\sum_{i=1}^{p} \det\big(\tilde{C}(\omega_{\backslash i})\big) \omega_i^q = \begin{bmatrix} -1, q = p-1 \\ 0, q < p-1 \end{bmatrix}$, for q = 0,...,p-1

   Proof:

Part (1). This lemma can be proven via the use of Lemma III.4 as in characteristic p

$$\det\big(\tilde{C}(\omega_I)\big) = \det(\tilde{C}(\omega_I) + \text{Diag}(\{\sum_{j\in I, j\neq i}\frac{1}{\omega_i-\omega_j} - (p-1)\sum_{j\notin I}\frac{1}{\omega_i-\omega_j}\}_{i\in I})) =$$

$$= \text{per}(C(\omega_I, \omega_{\backslash I} \otimes \vec{1}_{p-1})$$

where $\{\sum_{j\in I,j\neq i}\frac{1}{\omega_i-\omega_j} - (p-1)\sum_{j\notin I}\frac{1}{\omega_i-\omega_j}\}_{i\in I}$ is the $|I|$-vector indexed by the elements

of I and having, for $i \in I$, its i-th entry equal to $\sum_{j\in I,j\neq i}\frac{1}{\omega_i-\omega_j} - (p-1)\sum_{j\notin I}\frac{1}{\omega_i-\omega_j}$. This

vector is zero if ω is a p-vector whose entries are all the elements of GF(p).

Let's also take into account the fact that for independent indeterminates

$u_1, ..., u_m$ there hold the identities

(*) $\sum_{r=1}^{m}\frac{u_r^d}{\prod_{w,w\neq r}(u_r-u_w)} = 0$ if $d < m-1$

and

(**) $\sum_{r=1}^{m}\frac{u_r^{m-1}}{\prod_{w,w\neq r}(u_r-u_w)} = 1$.

For proving the lemma's first part it's sufficient to consider just a symmetric polynomial

of the form $\frac{\sum_{\pi\in S_{p-k}}u_{\pi_1}^{q_1}...u_{\pi_{p-k}}^{q_{p-k}}}{(-1)^q q_1!...q_{p-k}!}$, where $q_1, ..., q_{p-k}$ are non-zero integers such that $q_1 +$

$\cdots + q_{p-k} = q$, because any symmetric polynomial is a linear combination of such

polynomials. Hence for proving Part (1) it's sufficient to show that the identity

$$\sum_{I\subseteq\{1,...,p\},|I|=k} \text{per}(C(\omega_I, \omega_{\backslash I} \otimes \vec{1}_{p-1}) \sum_{\pi\in S_{GF(p)\backslash I}} \omega_{\pi_1}^{q_1} ... \omega_{\pi_{p-k}}^{q_{p-k}} = 0$$

(where for a set J S_J denotes the set of permutations on it)

holds for $q \leq k < p-1$.

Due to the above-mentioned facts, for $q < k$ it equals zero because of (*) and for $q = k$,

because of (**), it's the number of GF(p)'s partitions into subsets of cardinalities $q_1 +$

$1, ..., q_{p-k} + 1$ that is zero when $k < p-1$.

(Part 2) . It follows from (**).

Theorem III.14 (in characteristic 5):
let $P, \widehat{P}$ be 4×n-matrices, 5dim(u) + dim(v) + dim(w) = n. Then

$$\varphi_{5,2}(u, w, v, \gamma) = \frac{(-1)^{\dim(w)}}{2^{\dim(u)}} (\prod_{j=1}^{\dim(v)} \gamma_j)\lim_{\varepsilon\to 0} \text{coef}_{\lambda^{\dim(v)-\dim(w)}}$$

$$\text{det}_{tr}(\tilde{C}(\begin{pmatrix} u\ddot{+}\varepsilon\omega \\ v \\ w \end{pmatrix})) + \text{Diag}(\begin{pmatrix} -C(u\otimes\vec{1}_5, w)\vec{1}_{\dim(w)} \\ \lambda\gamma^{\star(-1)} \\ 0_{\dim(v)} \end{pmatrix}), \{(p_i\hat{p}_i^T + \hat{p}_i p_i^T)M\}_n)$$

$$\text{where: } \omega = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}; \; M = \begin{pmatrix} 0_{2\times2} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0_{2\times2} \end{pmatrix};$$

all the numerator-vectors corresponding to $\ddot{u}\dotplus\varepsilon\omega$ are f and all the alternate numerator-vectors corresponding to $\ddot{u}\dotplus\varepsilon\omega$ are $\hat{f}$; all the numerator-vectors corresponding to w and v are equal to their alternate numerator-vectors and for w they are f, for v they are $\hat{f}$, where f, $\hat{f}$ are arbitrary 4-vectors satisfying the relation $f^T M\hat{f} = 1$

Proof:

In the present proof, with the considered trace-determinant we'll associate a weighted digraph whose vertices will be associated with the denominator-values $x = \begin{pmatrix} \ddot{u}\dotplus\varepsilon\omega \\ v \\ w \end{pmatrix}$, while by the weight of a proper cycle $\mathcal{C} = (x_{i_1}, \ldots, x_{i_{|\mathcal{C}|}})$ we'll understand the cycle's **trace-weight** $(-1)^{|\mathcal{C}|+1}\text{tr}\left(\prod_{q=1}^{|\mathcal{C}|} B^{\left(i_q(\mathcal{C})\right)}\right)\right) \prod_{q=1}^{|\mathcal{C}|} \frac{1}{x_{i_q}-x_{\pi_{i_q}}}$, by a loop (x_{i_1})'s weight -- the corresponding absence-weight g_i (i.e. the corresponding diagonal entry of the matrix

$$\tilde{C}\left(\begin{pmatrix} \ddot{u}\dotplus\varepsilon\omega \\ v \\ w \end{pmatrix}\right) + \text{Diag}\left(\begin{pmatrix} -C\left(u\otimes\vec{1}_5, w\right)\vec{1}_{\dim(w)} \\ \lambda\gamma^{*(-1)} \\ 0_{\dim(v)} \end{pmatrix}\right)), \text{ and by a cycle system's weight -- the}$$

product of its cycles' weights. We'll also call a vertex absent (in a spanning cycle system) if it's covered by its loop, and **present** otherwise.

As we have $B^{(i)} = p_i\hat{p}_i^T + \hat{p}_i p_i^T$, the considered trace-determinant is the corresponding alternate determinant

$$\text{altdet}\left(\begin{pmatrix} P^T M P \star \tilde{C}(t^{\star 5}) & P^T M\hat{P} \star \tilde{C}(t^{\star 5}) \\ \hat{P}^T M P \star \tilde{C}(t^{\star 5}) + \text{Diag}(g) & \hat{P}^T M\hat{P} \star \tilde{C}(t^{\star 5}) \end{pmatrix}\right) =$$

with the numerator-vectors and alternate numerator-vectors p_i and $\hat{p}_i$ correspondingly and the absence-weights g_i.

Let's call the denominator-values (as well as the corresponding vertices of the weighted digraphs we're going to build in this proof) of $\hat{u} = u \ddot{+} \varepsilon \omega$ **regular**, of w **active**, of v **passive**.

In each transversal summand of the considered matrix, let's consider its proper cycle system spanning its set of present vertices. We'll call a regular vertex **busy** if it's located in a cycle having not only regular vertices (and we'll call such a cycle **non-regular**), and **free** otherwise (and we'll accordingly call a cycle regular if it consists of regular vertices only). Besides, a pair of vertices whose denominator values' difference's ε-order is bigger than zero will be called **ε-close** (of a specified ε-order, if necessary to detail).

First of all, let's notice that we can consider only regular cycles of length 2 because the sum of the weights of all the regular cycles covering a set of regular vertices of a cardinality bigger than 2 equals zero due to the fact that, for each regular vertex, its numerator-vector is f and its alternate numerator-vector is $\hat{f}$ and hence, due to the theorem's condition $f^T M \hat{f} = 1$, the weight of a cycle $\mathcal{C}$ covering a regular vertex set $\hat{u}_I$ equals $((\hat{f}^T M f)^l + (\hat{f} M f^T)^l) \prod_{(i_1,i_2) \in \mathcal{C}} \frac{1}{\hat{u}_{i_1} - \hat{u}_{i_2}}$ (where l is the cycle's length) and hence it is zero when $l > 2$ because of the earlier proven fact that $\text{ham}\left(\tilde{C}(x)\right) = 0$ if dim(x) > 2. Besides, due to the identity $\text{ham}\left(\tilde{C}(x,y,z)\right) = \frac{1}{y-z} \prod_{i=1}^{\dim(x)} (\frac{1}{y-x_i} - \frac{1}{z-x_i})$ for dim(y) = dim(z) = 1 (that is a partial case of Lemma III.4) and the values of the numerator-vectors and alternate numerator-vectors given in the theorem (providing that active and passive vertices should alternate in any cycle of a non-zero weight if we don't take into account the regular vertices between them), any non-regular cycle of a non-zero weight should contain equal quantities of active and passive vertices and its weight equals the weight of the cycle received from it by removing all its regular vertices multiplied by $\prod_{i \in I} (\sum_{k \in K} \frac{2}{w_k - \hat{u}_i} - \sum_{j \in J} \frac{2}{v_j - \hat{u}_i})$ where I, J, K are the sets of its regular, passive and active vertices correspondingly. The latter relation will remain true if we replace the word "cycle" by "cycle system". Altogether, due to the given absence-weights of our denominator-values and upon taking the given coefficient at $\lambda^{\dim(v) - \dim(w)}$ (providing exactly dim(w) present passive vertices, while all the active ones are present due to having zero absence-weights), we receive the expression

$$\lim_{\varepsilon \to 0} \sum_K 2^{\frac{\dim(\hat{u}_{\backslash K})}{2}} \det\left(\tilde{C}(\hat{u}_{\backslash K})\right) \sum_{J, |J| = \dim(w)} (\prod_{i \in I} (\sum_{k=1}^{\dim(w)} \frac{2}{w_k - \hat{u}_i} - \sum_{j \in J} \frac{2}{v_j - \hat{u}_i})) \cdot$$

$$\cdot \det{}^2(C(w, v_J)) \prod_{l\in\{1,\dots,\dim(v)\}\backslash J} \frac{1}{\gamma_l}$$

Let's now show that, in the above expression, for each i = 1,…,dim(u) the family of denominator-values $u_i\vec{1}_5 + \varepsilon\omega$ yields, under this limit, the Cauchy-base multiplier $\sum_{j\in J}\frac{1}{(u_i-v_j)^5}$ multiplied by -1. It follows from Theorem III.12.1 because the minimal ε-order we receive for this family is zero and we get it either when four denominator-values of the family are free and one is busy or when all its denominator-values are busy -- and those two cases together give us the multiplier $\sum_{k=1}^{\dim(w)}\frac{1}{(u_i-w_k)^5} - \sum_{j\in J}\frac{1}{(u_i-v_j)^5}$, -- or when all of them are absent, i.e. covered by their loops each of whom has the weight $-\sum_{k=1}^{\dim(w)}\frac{1}{(u_i-w_k)^5}$.

We hence obtain, altogether, the expression

$$\sum_{J,|J|=\dim(w)} (\prod_{i\in I}(-\sum_{j\in J}\frac{1}{(u_i-v_j)^5})) \det{}^2(C(w, v_J)) \prod_{r\in\{1,\dots,\dim(v)\}\backslash J} \frac{1}{\gamma_r}$$

what completes the proof.

Theorem III.15 (in a prime characteristic p):
Let A be an n×n-matrix, h be an even number. Then $\mathrm{per}(A) =$
$\varphi_{p,h}(z\otimes\vec{1}_{p^{q-1}}, w, v, \gamma)$

where: $\dim(w) = \dim(z) = n$, $\dim(v) = \dim(\gamma) > h(n^2 + n)$
$C^{\star p^q}(z, v)\mathrm{Diag}(\gamma)C^{\star s}(v, w) = 0_{n\times n}$ for s = 1,…,h-1
$C^{\star p^q}(z, v)\mathrm{Diag}(\gamma)C^{\star h}(v, w) = A$
$C^{\star s}(w, v)\mathrm{Diag}(\gamma) = \vec{0}_n$ for s = 1,…,h

while the above system of linear equations for γ is nonsingular in the generic case if $p^q > h(n^2 + n)$.

Comment: $\varphi_{p,h}(z\otimes\vec{1}_{p^{q-1}}, w, v, \gamma)$ can be polynomial-time computed as $\lim_{\varepsilon\to 0} \varphi_{p,h}(z\otimes\vec{1}_{p^{q-1}} + \varepsilon\zeta, w, v, \gamma)$ where ζ is an arbitrary $p^{q-1}\dim(z)$-vector with pair-wise distinct entries.

Proof:

This theorem is based on the following generalization of the Binet-Cauchy identity (about the determinant of the product of two matrices), valid in an arbitrary characteristic:

Let $A^{(1)}, \ldots, A^{(h)}$ be n×m-matrices, h be non-zero even, B be a k×m-matrix. Then

$$\left(\prod_{v=1}^{h} \det\left((A^{(v)})^{(\{1,\ldots,n\},J)}\right)\right) \prod_{r=1}^{k} \sum_{j\in J} b_{r,j} =$$

$$= \sum_{\substack{\pi^{(2)},\ldots,\pi^{(h)}\in S_n, \\ (R_1,\ldots,R_n)\in\mathcal{P}_n(\{1,\ldots,k\})}} \sigma(\pi^{(2)}\ldots\pi^{(h)}) \prod_{i=1}^{n}\sum_{j=1}^{m} a_{i,j}^{(1)} a_{\pi_i^{(2)},j}^{(2)} \ldots a_{\pi_i^{(h)},j}^{(h)} \prod_{r\in R_i} b_{r,j}$$

where $\mathcal{P}_n(\{1,\ldots,k\})$ is the set of partitions of the set $\{1,\ldots,k\}$ into n subsets (some of them possibly empty) and $\sigma(\pi^{(2)}\ldots\pi^{(h)})$ is the sign of the permutation $\pi^{(2)}\ldots\pi^{(h)}$.

In our case we have, by the definition, $\varphi_{p,h}\left(z\otimes\vec{1}_{5^{q-1}}, w, v, \gamma\right) =$

$$= \sum_{\substack{J\subseteq\{1,\ldots,m\} \\ |J|=n}} \det\left((C(w,v))^{(\{1,\ldots,n\},J)}\right) \ldots \det\left((C(w,v))^{(\{1,\ldots,n\},J)}\right) \prod_{r=1}^{k}\sum_{j\in J}\frac{\gamma_j}{(z_r-v_j)^{p^q}} \qquad .$$

Hence, while considering the entries of the vectors w, z as "constants" and the entries of the vectors v, γ as "variables", we can say that in our case each expression $\sum_{j=1}^{m} a_{1,j}^{(1)} a_{\pi_i^{(2)},j}^{(2)} \ldots a_{\pi_i^{(n)},j}^{(h)} \prod_{r\in R_i} b_{r,j}$ is a linear combination of the sums $\sum_{j=1}^{m}\frac{\gamma_j}{(w_i-v_j)^s}$ with i = 1,...,n, s = 1,...,h and $\sum_{j=1}^{m}\frac{\gamma_j}{(z_r-v_j)^s}$ with r = 1,...,n, s = 1,...,p^q . Due to the fact that, according to the theorem's conditions, all the former ones are equal to zero, in our case each $\sum_{j=1}^{m} a_{1,j}^{(1)} a_{\pi_i^{(2)},j}^{(2)} \ldots a_{\pi_i^{(n)},j}^{(h)} \prod_{r\in R_i} b_{r,j}$ is a linear combination of the sums $\sum_{j=1}^{m}\frac{\gamma_j}{(z_r-v_j)^s}$ only and isn't zero only if R_i isn't empty; hence we can consider only partitions $(R_1,\ldots,R_n)$ where all the subsets R_i are of cardinality 1. Therefore we can consider only the expressions $\sum_{j=1}^{m}\frac{\gamma_j}{(w_{i_1}-v_j)\ldots(w_{i_h}-v_j)(z_r-v_j)^{p^q}}$ that are linear combinations of the sums

$$\sum_{j=1}^{m}\frac{\gamma_j}{(w_i-v_j)^s(z_r-v_j)^{p^q}} = \sum_{t=0}^{p^q-1}\frac{(-s)\ldots(-s-t+1)}{t!(w_i-z_r)^{s+t}}\sum_{j=1}^{m}\frac{\gamma_j}{(z_r-v_j)^{p^q-t}}$$ with s =1,...,h, i = 1,...,n,

r = 1,...,n and form a non-singular system of hn^2 linear functions in the sums $\sum_{j=1}^{m}\frac{\gamma_j}{(z_r-v_j)^{p^q-t}}$ with r = 1,...,n, t = 1,..., p^q-1. According to the theorem's conditions, the sum $\sum_{j=1}^{m}\frac{\gamma_j}{(w_i-v_j)^s(z_r-v_j)^{p^q}}$ is zero when s < h and equals $a_{i,r}$ when s = h. The latter case implies $w_{i_1} = \cdots = w_{i_h}$ and hence we can consider only the case $\pi^{(2)} = \cdots = \pi^{(h)} = \begin{pmatrix} 1,\ldots,n \\ 1,\ldots,n \end{pmatrix}$ and, because $\sigma\left(\pi^{(2)}\ldots\pi^{(h)}\right) = 1$ in such a case, we eventually get per(A), while the sums $\sum_{j=1}^{m}\frac{\gamma_j}{(z_r-v_j)^{p^q-t}}$ and $\sum_{j=1}^{m}\frac{\gamma_j}{(w_i-v_j)^s}$ with r, i = 1,...,n and t =

$0,\dots,p^q-1$ generically form a nonsingular system of linear functions in $\gamma_1,\dots,\gamma_m$ provided $p^q > h(n^2 + n)$.

Let's also formulate the Binet-Minc identity's generalization we used in this proof, even in a wider form:

Theorem III.15.1 (in any characteristic):
Let $A^{(1)},\dots,A^{(h)},A^{(h+1)},\dots,A^{(h+d)}$ be $n\times m$-matrices, h be non-zero even, B be a $k\times m$-matrix. Then

$$\sum_{\substack{J\subseteq\{1,\dots,m\}\\|J|=n}}\left(\prod_{v=1}^{h}\det((A^{(v)})^{(\{1,\dots,n\},J)})\right)\left(\prod_{v=h+1}^{h+d}\operatorname{per}((A^{(v)})^{(\{1,\dots,n\},J)})\right)\prod_{r=1}^{k}\sum_{j\in J}b_{r,j}=$$

$$=\sum_{\substack{\pi^{(2)},\dots,\pi^{(h+d)}\in S_n,\\(R_1,\dots,R_n)\in\mathcal{P}_n(\{1,\dots,k\})}}\sigma(\pi^{(2)}\dots\pi^{(h)})\prod_{i=1}^{n}\sum_{j=1}^{m}a_{i,j}^{(1)}a_{\pi_i^{(2)},j}^{(2)}\dots a_{\pi_i^{(h+d)},j}^{(h+d)}\prod_{r\in R_i}b_{r,j}$$

where $\mathcal{P}_n(\{1,\dots,k\})$ is the set of partitions of the set $\{1,\dots,k\}$ into n subsets (some of them possibly empty) and $\sigma(\pi^{(2)}\dots\pi^{(h)})$ is the sign of the permutation $\pi^{(2)}\dots\pi^{(h)}$.

Additionally, we can also formulate

Theorem III.16 (in characteristic 5): let $\dim(z) = n$, $\dim(y) = m$. Then

$$\left(\prod_{i=1}^{n}(1 + d_i\frac{\partial}{\partial z_i})\right)\det\left(\left\{\frac{\alpha_i-\alpha_j + (\beta_i + \beta_j)(z_i - z_i)}{(z_i - z_j)^5}\right\}_{n\times n} + \operatorname{Diag}(h)\right)$$

$$= \operatorname{coef}_{\lambda^n}\lim_{\varepsilon_1\to 0}\lim_{\varepsilon\to 0}\frac{\operatorname{Pf}(K(\underbrace{\left(y\ddot{+}\binom{0}{\varepsilon}\ddot{+}\binom{0}{\varepsilon_1}\right)}_{z},\underbrace{(\vec{0}_{4m})}_{d},\underbrace{\left((\varepsilon\varepsilon_1)^2(\lambda g)^{*(-1)}\ddot{+}(\varepsilon^{-2} + \frac{\varepsilon^2}{2}\varepsilon_1^{-4})\vec{1}_4\right)}_{h}))}{(-2)^{n+4m}\varepsilon_1^{-2m}}$$

where $\alpha_i = \alpha(z_i) = \sum_{k=1}^{m}\frac{g_k}{(y_k-z_i)^3}$, $\beta_i = \beta(z_i) = \sum_{k=1}^{m}\frac{g_k}{(y_k-z_i)^4}$ for $i = 1,\dots,n$

****

Theorem III.17 (in characteristic 5): Let $\dim(x) = n$, A be a nonsingular skew-symmetric $4n\times 4n$-matrix, $\dim(\zeta) = m$ .Then

$$\text{per}(C(x \otimes \vec{1}_4, \zeta \otimes \vec{1}_2)\text{Diag}(d \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix})) =$$

$$= \frac{\text{coef}_{\gamma^{4n}}\det^4(\text{Van}(x))\text{Pf}(\begin{pmatrix} \gamma D\tilde{C}(\zeta)D & I_n \\ -I_n & (\text{Van}^{[4n]}(\zeta))^T A\text{Van}^{[4n]}(\zeta) \end{pmatrix}))}{\text{Pf}(A)}$$

where $D = \text{Diag}(\{\sqrt{\dfrac{-d_j}{\text{pol}(\zeta_j,x)}}\}_m)$

Proof:

This statement is due to Lemma III.4. The numerator of the theorem's equality's right side is

$$\det^4(\text{Van}(x))\sum_{J,|J|=2n}\text{Pf}(\tilde{C}(\zeta_J))\text{Pf}(\text{Van}^{[4n]}(\zeta_J))^T A\text{Van}^{[4n]}(\zeta_J)\prod_{j\in J}\frac{-d_j}{\text{pol}(\zeta_j,x)} =$$

$$= \det^4(\text{Van}(x))\sum_{J,|J|=2n}\text{Pf}(\tilde{C}(\zeta_J))\text{Pf}(\text{Van}(\zeta_J))^T A\text{Van}(\zeta_J)\prod_{j\in J}\frac{-d_j}{\text{pol}(\zeta_j, x)}$$

$$= \det^4(\text{Van}(x))\sum_{J,|J|=2n}\text{Pf}(\tilde{C}(\zeta_J))\det(\text{Van}(\zeta_J))\,\text{Pf}(A)\prod_{j\in J}\frac{-d_j}{\text{pol}(\zeta_j,x)} =$$

$$= \det^4(\text{Van}(x))\sum_{J,|J|=2n}\frac{\det(W^{[4]}(\begin{pmatrix}\zeta_J \\ \zeta_J\end{pmatrix}))}{\det(\text{Van}(\zeta_J))}\det(\text{Van}(\zeta_J))\,\text{Pf}(A)\prod_{j\in J}\frac{-d_j}{\text{pol}(\zeta_j, x)} =$$

$$= \text{Pf}(A)\sum_{J,|J|=2n}\det^4(\text{Van}(x))\det(W^{[4]}(\begin{pmatrix}\zeta_J \\ \zeta_J\end{pmatrix}))\prod_{j\in J}\frac{-d_j}{\text{pol}(\zeta_j, x)} =$$

$$= \text{Pf}(A)\sum_{J,|J|=2n}\text{per}(C(x \otimes \vec{1}_4, \zeta_j \otimes \vec{1}_2)\prod_{j\in J}(-d_j) \ ,$$

while the left side is $\sum_{J,|J|=2n}\text{per}(C(x \otimes \vec{1}_4, \zeta_j \otimes \vec{1}_2)\prod_{j\in J}(-d_j)$

****

Theorem III.18 (the Binet-Minc identity, for any characteristic)

Let A be an n×m-matrix, then

$$\text{per}(A) = (-1)^n\sum_{P\in\text{Part}(\{1,\dots,n\})}\prod_{I\in P}(-(|I|-1)!\sum_{j=1}^{m}\prod_{i\in I}a_{ij})$$

where Part({1,,...,n}) is the set of partitions of the set {1,...,n} into non-empty subsets.

Sparse compressions in characteristic 3

Theorem III.19 (in any characteristic). Let dim(z) = 2dim(x). Then

$$\det(C^{\star(1,2)}(x,z)) = (-1)^{\dim(x)} \frac{\det^4(\mathrm{Van}(x))\det(\mathrm{Van}(z))}{\mathrm{pol}^2(x,z)}$$

Theorem III.20 (in any characteristic). Let dim(z) = 2dim(x). Then

$$\det(C^{\star(2,3)}(x,z)) = (1/2)^{\dim(x)}\mathrm{per}(C(\begin{pmatrix}X\\X\end{pmatrix},z)\det(C^{\star(1,2)}(x,z))$$

Proof:

This statement follows directly from the Borchardt identity as

$$\det(C^{\star(2,3)}(x,z)) = \lim_{\varepsilon\to 0}\frac{\det(C^{\star 2}(\begin{pmatrix}X\\x\ddot{+}\varepsilon\end{pmatrix},z))}{(-2\varepsilon)^{\dim(x)}} = \lim_{\varepsilon\to 0}\frac{\det(C(\begin{pmatrix}X\\x\ddot{+}\varepsilon\end{pmatrix},z))\mathrm{per}(C(\begin{pmatrix}X\\x\ddot{+}\varepsilon\end{pmatrix},z))}{(-2\varepsilon)^{\dim(x)}} =$$

$$= \frac{\mathrm{per}(C(\begin{pmatrix}X\\X\end{pmatrix},z)(-1)^{\dim(x)}\det(C^{\star(1,2)}(x,z))}{(-2)^{\dim(x)}}$$

==*A conjectured polynomial-time algorithm for computing the permanent in characteristic 3*==

Theorem III.21 (in characteristic 3):

Let A be a nonsingular skew-symmetric 2n×2n-matix, dim(x) = n, dim(y) = m. Then

$$\mathrm{per}^3(C(\begin{pmatrix}X\\X\end{pmatrix},y)\mathrm{Diag}(d)) = 2^n \frac{\mathrm{coef}_{\gamma^n}\mathrm{Pf}(\begin{pmatrix}\gamma D\tilde{C}(y)D & I_m\\ -I_m & (C^{\star(2,3)}(x,y))^T A C^{\star(2,3)}(x,y)\end{pmatrix})}{\mathrm{Pf}(A)}$$

where $D = \mathrm{Diag}^3(d)$.

Proof:

The proof of this theorem is based on Lemma III.4 as $\mathrm{Pf}(\tilde{C}(y_J)) = \frac{\mathrm{per}^2(W^{[2]}(y_J))}{\det(\mathrm{Van}(y_J))}$ and

$$\mathrm{Pf}((C^{\star(2,3)}(x,y_J))^T A C^{\star(2,3)}(x,y_J)) = \det(C^{\star(2,3)}(x,y_J))\mathrm{Pf}(A) =$$

$$= 2^n \mathrm{per}(C(\begin{smallmatrix}X\\x\end{smallmatrix}),y_J)\det(C^{\star(1,2)}(x,y_J))\mathrm{Pf}(A) =$$

$$= \mathrm{per}(C(\begin{smallmatrix}X\\x\end{smallmatrix}),y_J)\frac{\det^4(\mathrm{Van}(x))\det(\mathrm{Van}(y_J))}{\mathrm{pol}^2(x,y_J)}\mathrm{Pf}(A) \ ,$$

while the numerator of the theorem's equality's right side is

$$\sum_{J,|J|=2n}\mathrm{Pf}(\tilde{C}(y_J))\mathrm{Pf}((C^{\star(2,3)}(x,y_J))^T A C^{\star(2,3)}(x,y_J))\prod_{j\in J}d_j^3 =$$

$$= \sum_{J,|J|=2n}\frac{\mathrm{per}^2(W^{[2]}(y_J))}{\det(\mathrm{Van}(y_J))}\mathrm{per}(C(\begin{smallmatrix}X\\x\end{smallmatrix}),y_J)\frac{\det^4(\mathrm{Van}(x))\det(\mathrm{Van}(y_J))}{\mathrm{pol}^2(x,y_J)}\prod_{j\in J}d_j^3$$

Taking into account the fact that, according to Lemma III.4, $\mathrm{per}^2(C(\begin{smallmatrix}X\\x\end{smallmatrix}),y_J) = \frac{\det^4(\mathrm{Van}(x))}{\mathrm{pol}^2(x,y_J)}\mathrm{per}^2(W^{[2]}(y_J))$, we complete the proof.

Accordingly, via the reduction

$$\lim_{\varepsilon\to 0}(\varepsilon^{\dim(x)}\mathrm{per}(C(\begin{smallmatrix}X\\x\end{smallmatrix}),(\begin{smallmatrix}x+\varepsilon\vec{1}_{\dim(x)}\\z\end{smallmatrix}))\mathrm{Diag}((\begin{smallmatrix}\vec{1}_{\dim(x)}\\\lambda\end{smallmatrix})))) =$$

$$= 2^{\dim(x)}\mathrm{per}(C(x,z)\mathrm{Diag}(\lambda)), \text{ we receive also}$$

Theorem III.22 (in characteristic 3):

$\mathrm{per}(C(x,z)\mathrm{Diag}(\lambda))$ is computable in polynomial time for arbitrary .

Definition: for $\dim(x) = \dim(d) = \dim(a)$,

$$\rho(x,d,a) := (\prod_{i=1}^{n}(1+d_i\frac{\partial}{\partial r_i}))\det(\tilde{C}(x)+\mathrm{Diag}(a))$$

Theorem III.23 (in characteristic 3): $\rho(x,d,a) = \mathrm{per}(C(x^{\star 3},z)\mathrm{Diag}(\lambda))$

where $C(x^{\star 3},z)\lambda^{\star q} = \delta(q-1)a - \delta(q-2)(x+d)$ for q = 1,2,3 and this system of equations for z, λ is generically algebraically nonsingular.

Theorem III.24 (in characteristic 3):

$$(\prod_{i=1}^{\dim(t)} \frac{\partial}{\partial t_i})\det(\tilde{C}(\begin{pmatrix}x\\t\end{pmatrix})) + \mathrm{Diag}\begin{pmatrix}\alpha\\t\star\beta\end{pmatrix})) = \det(\tilde{C}(x) + \mathrm{Diag}(\alpha)) \cdot$$

$$\cdot\, \mathrm{altdet}(\mathrm{Schur}_{T,T}\begin{pmatrix} \tilde{C}(x) + \mathrm{Diag}(\alpha) & C(x,t) & C^{\star 2}(x,t) \\ C(t,x) & \tilde{C}(t) + D_{11} & C^{\star 2}(t) + D_{12} \\ -C^{\star 2}(t,x) & -C^{\star 2}(t) + D_{21} & C^{\star 3}(t) + D_{22} \end{pmatrix})) =$$

$$= \mathrm{altdet}\begin{pmatrix} P_1^T P_2 \star \tilde{C}(t^{\star 9}) + \breve{D}_{11} & P_1^T \hat{P}_2 \star \tilde{C}(t^{\star 9}) + \breve{D}_{12} \\ \hat{P}_1^T P_2 \star \tilde{C}(t^{\star 9}) + \breve{D}_{21} & \hat{P}_1^T \hat{P}_2 \star \tilde{C}(t^{\star 9}) + \breve{D}_{22} \end{pmatrix} =$$

$$= \det_{\mathrm{tr}}(\tilde{C}(t) + \mathrm{Diag}(g), \{p_{1,i}\hat{p}_{2,i}^T + \hat{p}_{1,i}p_{2,i}^T\}_n)$$

where T={1,…,dim(x)} , $P_1, \hat{P}_1, P_2, \hat{P}_2$ are some n×9-matrices such that for i = 1,…n $p_{1,i}^T p_{2,i} = \hat{p}_{1,i}^T p_{2,i} = p_{1,i}^T \hat{p}_{2,i} = \hat{p}_{1,i}^T \hat{p}_{2,i} = 0$; for k, l = 1,2 $D_{k,l}, \breve{D}_{k,l}$ are diagonal,

$D_{12} + D_{21} = \mathrm{Diag}(\beta)$, $\breve{D}_{12} + \breve{D}_{21} = \mathrm{Diag}(g)$.

Conjecture III.25: let T={1,…,dim(v)}, $D_{k,l}$ be diagonal matrices for k, l = 1,2. Then the class of matrices of the form

$$\mathrm{Schur}_{\{1,…,\dim(x)\},\{1,…,\dim(x)\}}\begin{pmatrix} \tilde{C}(x) + \mathrm{Diag}(\alpha) & C(x,t) & C^{\star 2}(x,t) \\ C(t,x) & \tilde{C}(t) + D_{11} & C^{\star 2}(t) + D_{12} \\ -C^{\star 2}(t,x) & -C^{\star 2}(t) + D_{21} & C^{\star 3}(t) + D_{22} \end{pmatrix})$$

(where $D_{k,l}$ are diagonal for k, l = 1,2)

is generically the class

$$\begin{pmatrix} P_1^T P_2 \star \tilde{C}(t^{\star 9}) + \breve{D}_{11} & P_1^T \hat{P}_2 \star \tilde{C}(t^{\star 9}) + \breve{D}_{12} \\ \hat{P}_1^T P_2 \star \tilde{C}(t^{\star 9}) + \breve{D}_{21} & \hat{P}_1^T \hat{P}_2 \star \tilde{C}(t^{\star 9}) + \breve{D}_{22} \end{pmatrix}$$

(where $P_1, \hat{P}_1, P_2, \hat{P}_2$ are n×9-matrices, $\breve{D}_{k,l}$ are diagonal for k, l=1,2)

of singularized matrices of Cauchy-rank 9 all whose denominator-values are alternate-wise doubled and $p_{1,i}^T p_{2,i} = \hat{p}_{1,i}^T p_{2,i} = p_{1,i}^T \hat{p}_{2,i} = \hat{p}_{1,i}^T \hat{p}_{2,i} = 0$ for i = 1,…,n.

The above conjecture is an analogue of Theorem III.9 in characteristic 5 and it's based on Theorem III.23. If it's true we can, analogically, generate certain families of infinitesimal-close denominator-values with their non-symmetric (due to the considered matrix's non-symmetry in this case) matrix-weights and absence-weights, while using Theorem III.10.2 for compressing those families into denominator-values whose matrix-

weights and absence-weights can be *conjectured* arbitrary. If the later conjecturing doesn't fail too then we can use the fact that in any characteristic we have an exact analog of Theorem III.14 for left and right numerator-vectors and alternate numerator-vectors (row- and column-numerators and alternate ones) of dimension 9, with any nonsingular multiplication matrix M. There holds also an analog of Theorem III.12 providing polynomial-time generating, by the trace-determinant of $\tilde{C}(t) + \text{Diag}(g)$ on

arbitrary 9×9-matrix-weights, per $\begin{pmatrix} C^{\star 2}(y^{(1)}, y)\text{Diag}(\lambda^{(1)}) \\ ... \\ C^{\star 2}(y^{(4)}, y)\text{Diag}(\lambda^{(4)}) \end{pmatrix}$ that is reducible, as it will

further be shown in Theorem III.31, to per $\begin{pmatrix} C^{\star 3}(y^{(1)}, y)\text{Diag}(\lambda^{(1)}) \\ ... \\ C^{\star 3}(y^{(4)}, y)\text{Diag}(\lambda^{(4)}) \end{pmatrix}$ and $\#_3 P$-complete.

Thus we come to the following conclusion:

Theorem III.26:

Over fields of characteristic 3, computing the trace-determinant of $(\tilde{C}(t) + \text{Diag}(g))$ on arbitrary 9×9-matrix-weights $\#_3$-P-complete.

We can comment, however, that in fact Theorem III.14 can be re-formulated even for 2×2-matrix-weights (i.e. numerator-vectors and alternate numerator-vectors of dimension 2, with any non-singular skew-symmetric multiplication 2×2-matrix).

Theorem III.28 (in characteristic p):

Let h be an even natural number bigger than 2 that isn't a square modulo p. Then

$$\varphi_{p,h}(u, w, v, \gamma) = \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon \to 0} \frac{\varphi_{0,h}\left(\emptyset, \begin{pmatrix} w \\ \hat{u} \end{pmatrix}, \left(\hat{u} \overset{\cdots}{\mp} \varepsilon \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)^v, \left(\vec{1}_{\dim(\hat{u})} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \varepsilon h C(\hat{u}, w)\vec{1}_{\dim(w)} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)^\gamma\right)}{(2h)^{\dim(u)} \varepsilon^{\dim(\hat{u})}}$$

where $\hat{u} = u \overset{\cdots}{\mp} \varepsilon_1 \begin{pmatrix} 0 \\ 1 \\ ... \\ p-1 \end{pmatrix}$

Proof:

The proof of this theorem is based on Theorem III.12.1 and Lemma III.4.

The first limit $\lim_{\varepsilon \to 0}$ turns the fraction in the theorem's equality's right side into

$$\frac{1}{(2h)^{\dim(u)}} \sum_{\substack{J \subseteq \{1,\ldots,\dim(v)\} \\ |J|=\dim(w)}} \sum_{I,I^{(1)},\ldots,I^{(h)}} \left(\prod_{i \in I}(hC(\hat{u}_i, w)\vec{1}_{\dim(w)})\right) \left(\prod_{q=1}^{h} \det{}^h(\tilde{C}(\hat{u}_{I^{(q)}}, w, v_J))\right) \prod_{j \in J} \gamma_j$$

where the summation is over all the (h+1)-tuples $I, I^{(1)}, \ldots, I^{(h)}$ that are partitions of the set $\{1,\ldots,\dim(\hat{u})\}$ into h+1 subsets, some of them possibly empty.

According to the statement (3) of Lemma III.4, this expression is equal to

$$\frac{1}{(2h)^{\dim(u)}} \sum_{\substack{J \subseteq \{1,\ldots,\dim(v)\} \\ |J|=\dim(w)}} \sum_{K \subseteq \{1,\ldots,\dim(\hat{u})\}} h^{\frac{\dim(\hat{u})-|I|}{2}} \det(\tilde{C}(\hat{u}_{\backslash I}))\left(\prod_{i \in I}\sum_{j \in J}\frac{h}{\hat{u}_i - v_j}\right) \det{}^h(\tilde{C}(w, v_J)) \prod_{j \in J} \gamma_j \ .$$

Therefore, due to the structure of the vector $\hat{u}$, the second limit $\lim_{\varepsilon_1 \to 0}$ provides the correctness of the theorem's identity because of the same argument (referring to Theorem III.12.1) that was applied in the proof of Theorem III.14.

Corollary III.30:
For an arbitrary prime characteristic p, $\varphi_{0,h}(\emptyset, w, v, \beta)$ is $\#_p P$-complete for any even h > 2 that isn't a square modulo p.
Proof: this corollary from Theorem III.28 is based on Theorem III.15 proving the $\#_p P$-completeness of $\varphi_{p,h}(u, w, v, \gamma)$.

Theorem III.31: let p be a prime number bigger than 5. Then computing $per(C(x, z)\mathrm{Diag}(\lambda))$ over fields of characteristic p is $\#_p P$-complete.

Proof:
For the case when -1 isn't a square modulo p, it follows immediately from Corollary III.30 and the fact that

$$\varphi_{0,p-1}\left(\emptyset, w, v, \lambda^{\star(p-1)}\right) = per\left(C\left(w \otimes \vec{1}_{p-1}, v \otimes \vec{1}_{p-1}\right)\mathrm{Diag}\left(\lambda \otimes \begin{pmatrix} 1 \\ \cdots \\ p-1 \end{pmatrix}\right)\right)$$

However, this theorem can be proven in a different way (common for all the prime characteristics bigger than 5) based on the Binet-Minc identity.

We'll say that the left denominator-value x_i is of multiplicity $\mathrm{mult}(x_i)$ if it's repeated $\mathrm{mult}(x_i)$ times in the vector x. Then, according to the Binet-Minc identity for characteristic p, $per(C(x, z)\mathrm{Diag}(\lambda))$ is a polynomial in the values $\sum_{j=1}^{\dim(z)}\frac{\lambda_j^r}{(x_i-z_j)^s}, s = 1, \ldots, \mathrm{mult}(x_i), r = 1, \ldots, p$, that are a system of algebraically independent functions in z, λ upon excluding those of them where r and s are both divided by p. Let's call the sum

$\sum_{j=1}^{\dim(z)} \frac{\lambda_j^r}{(x_i - z_j)^s}$ the ***r,s-row-weight*** of the left denominator-value x_i and the maximum set $\{x_1, \ldots, x_m\}$ of pair-wise distinct left (row) denominator-values the ***left denominator-value spectrum*** of $C(x, z)$. When the vectors z, λ are presented as $\begin{pmatrix} z^{(1)} \\ z^{(2)} \end{pmatrix}$ and $\begin{pmatrix} \lambda^{(1)} \\ \lambda^{(2)} \end{pmatrix}$, $\dim(z^{(1)}) = \dim(\lambda^{(1)})$, we'll call $z^{(1)}, \lambda^{(1)}$ the main parts of z, λ correspondingly and $z^{(1)}, \lambda^{(1)}$ the ***prolonged*** parts, while a row-weight will accordingly be the sum of its main part $\sum_{j=1}^{\dim(z)} \frac{(\lambda_j^{(1)})^r}{(x_i - z_j^{(1)})^s}$ and its prolonged part $\sum_{j=1}^{\dim(z)} \frac{(\lambda_j^{(2)})^r}{(x_i - z_j^{(2)})^s}$, and the latter we'll also call the ***prolonged row-weight***.

Hence, upon putting $\sum_{j=1}^{\dim(z)} \frac{(\lambda_j^{(2)})^{\mathrm{mult}(x_i)}}{(x_i - z_j^{(2)})^{\mathrm{mult}(x_i)}} = \frac{1}{d_i}$ and all the other prolonged row-weights equal to zero, we receive the relation

$$\mathrm{per}\left(C(\{x_i \vec{1}_{\mathrm{mult}(x_i)}\}_{i \in I}, z)\mathrm{Diag}(\lambda)\right) = \frac{\sum_{I \subseteq \{1, \ldots, m\}}(\prod_{i \in I} d_i)\mathrm{per}(C(\{x_i \vec{1}_{\mathrm{mult}(x_i)}\}_{i \in I}, z^{(1)})\mathrm{Diag}(\lambda^{(1)}))}{\prod_{i=1}^{\dim(x)} d_i}$$

where d_i we'll call the ***summation weight*** of x_i. If $p > 5$ this expression polynomial-time yields, upon taking a number of infinitely close (on some infinitesimal) pairs of left-spectral denominator-values of multiplicity $p-2$ and their summation weights of infinitesimal-order -1, the expression

$$\left(\prod_{t \in T} \frac{\partial}{\partial x_t}\right)\mathrm{per}(C(x \otimes \vec{1}_{p-2}, z)\mathrm{Diag}(\lambda))$$

where T is a subset of $\{1, \ldots, m\}$. Let's now consider, as a generalization of the above expression (up to multiplying its rows by constants), the expression:

$$\mathrm{per}\left(\begin{pmatrix} C^{\star\gamma_1}(x_1, z) \\ \ldots \\ C^{\star\gamma_m}(x_m, z) \end{pmatrix}\mathrm{Diag}(\lambda)\right)$$

where $\gamma_1, \ldots, \gamma_m$ are natural sequences (Hadamard vector-degrees) which we'll call the ***valences*** of the left-spectral denominator-values $x_1, \ldots, x_m$ correspondingly.

Particularly, the already above-considered expression $(\prod_{t \in T} \frac{\partial}{\partial x_t})\mathrm{per}(C(\{x \otimes \vec{1}_{p-2}, z)\mathrm{Diag}(\lambda))$ can be written as the expression $\mathrm{per}\left(\begin{pmatrix} C^{\star\gamma_1}(x_1, z) \\ \ldots \\ C^{\star\gamma_m}(x_m, z) \end{pmatrix}\mathrm{Diag}(\lambda)\right)$ (multiplied by a constant) where some valences are $(\vec{1}_{p-2}^T)$ and others (i.e. those from the set T) are $(\vec{1}_{p-3}^T, 2)$.

Given $a \in F(\varepsilon)$, where F is the ground field and ε is an infinitesimal, let's further say that we apply the limit technique to a when we compute $\lim\limits_{\varepsilon \to 0} \dfrac{a}{e^{\mathrm{order}_\varepsilon a}}$.

If $p > 5$ then, due to the algebraic independence of all the r,s-row-weights where r and s are not both divided by p, we can transform the valence $(\vec{1}^{\mathrm{T}}_{p-3}, 2)$ into (2) via taking the prolonged 1,1-row-weights of those "differentiated" left denominator-values equal to a formal variable (while taking all the other prolonged row-weights equal to zero) and calculating the coefficient at its power of the maximal degree. Then, via the limit technique, we can polynomial-time receive (for an arbitrary subset of the left denominator-value spectrum) the valence $(2\vec{1}^{\mathrm{T}}_{p-1}, \ldots, q\vec{1}^{\mathrm{T}}_{p-1})$, q = 2,…,p, (via taking infinitely close denominator-value families of sizes (q-1)(p-1)) which, in turn, can correspondingly polynomial-time generate the valences (2), (3), …, (5) (via taking, for the valence (q), the prolonged 2,(q+1)-weight equal to a formal variable and all the other prolonged row-weights equal to zero). Each of the above-mentioned steps is provided by one polynomial-time reduction via calculating either the limit on a new infinitesimal of the expression multiplied by an appropriate power of the infinitesimal (i.e. via using the limit technique) or the maximal degree power's coefficient of a new formal variable. Eventually, taking into account the non-differentiated denominator-values whose valences are $(\vec{1}^{\mathrm{T}}_4)$ which we can turn into (1) via taking the corresponding prolonged (p-2),(p-2)-row-weights equal to a formal variable, we can polynomial-time compute, for any left denominator-value spectrum $x = \{x_1, \ldots, x_m\}$, the expression $\mathrm{per}(\begin{pmatrix} C^{\star \gamma_1}(x_1, z) \\ \ldots \\ C^{\star \gamma_m}(x_m, z) \end{pmatrix} \mathrm{Diag}(\lambda))$ for any valences whose elements are taken from the set $\{1, \ldots, p\}$.

Let's consider its partial case when the denominator-value spectrum-vector x can be partitioned into subvectors $y^{(1)}, \ldots, y^{(p)}$ of valences (1),…,(p) correspondingly and y of valence (p) such that for j = 1,…,dim(y) the 1,p-row-weight of y_j is 1 and its 2,s-row-weight is $\lambda_j^{(s)} \omega$, s = 1,…,p (where ω is another formal variable), while all the other row-weights of all the left denominator-values are zeros (hence $y^{(1)}, \ldots, y^{(p)}$ have all their row-weights equal to zero). Then, upon calculating the coefficient at ω's power of the minimal degree, we'll polynomial-time get the permanent $\mathrm{per}\begin{pmatrix} C(y^{(1)}, y)\mathrm{Diag}(\lambda^{(1)}) \\ \ldots \\ C(y^{(p)}, y)\mathrm{Diag}(\lambda^{(p)}) \end{pmatrix}$ which, in turn, generates (upon taking infinitely close families of left denominator-

values of size p) per $\begin{pmatrix} C(y^{(1)}, y)\mathrm{Diag}(\lambda^{(1)}) \\ \ldots \\ C(y^{(p)}, y)\mathrm{Diag}(\lambda^{(p)}) \end{pmatrix}^{\star(\vec{1}^T_{p-1}, 2)}$. If we take $y^{(1)} = \cdots = y^{(p)} = v$

then we'll obtain per $\begin{pmatrix} C(v, y)\mathrm{Diag}(\lambda^{(1)}) \\ \ldots \\ C(v, y)\mathrm{Diag}(\lambda^{(p)}) \end{pmatrix}^{\star(\vec{1}^T_{p-1}, 2)}$ that is a polynomial in the sums

$\sum_{j=1}^{\dim(y)} \frac{(\lambda_j^{(1)})^{r_1} \ldots (\lambda_j^{(p)})^{r_p}}{(v_i - y_j)^s}$ (let's call such a sum, by analogy, the $r_1, \ldots, r_p, \textbf{\textit{s-row-weight}}$ of

v_i, while correspondingly defining the prolonged $r_1, \ldots, r_p, s$-row-weight). We can also analogically notice that the set of row-weights such that not all of the numbers $r_1, \ldots, r_p, s$ are multiples of p is an algebraically independent system of functions. And, at last, upon taking, for a formal variable ω, the v_i-th prolonged $1, \ldots, 1, 2p$-row-weight equal to $\alpha_i \omega^p$ and all the prolonged $r_1, \ldots, r_p, (p+1)-$row-weights where one of the numbers $r_1, \ldots, r_p$ is p and all the others are zeros equal to ω (while all the other prolonged row-weights are to be taken equal to zero), we'll obtain, after calculating the coefficient at ω's power of the maximal degree, the sum

$\sum_I (\prod_{i \in I} \alpha_i) \, \mathrm{per} \begin{pmatrix} C(v_I, y)\mathrm{Diag}(\lambda^{(1)}) \\ \ldots \\ C(v_I, y)\mathrm{Diag}(\lambda^{(p)}) \end{pmatrix}^{\star(\vec{1}^T_{p-1})}$.

For any prime characteristic, by the limit technique this sum yields, via taking pairs of infinitely close denominator-values and summation-weights of infinitesimal-order -2 (opposite for each pair), the partial derivative

$(\prod_{t=1}^{\dim(v)} \frac{\partial}{\partial v_t}) \mathrm{per} \begin{pmatrix} C(v, y)\mathrm{Diag}(\lambda^{(1)}) \\ \ldots \\ C(v, y)\mathrm{Diag}(\lambda^{(p)}) \end{pmatrix}^{\star(\vec{1}^T_{p-1})}$ which is equal (due to Lemma III.4, and the

next passage is due to it too) to the expression

$\sum_{J, |J| = p(p-1)\dim(v)} \mathrm{per} \begin{pmatrix} C(v, y_J)\mathrm{Diag}(\lambda^{(1)}) \\ \ldots \\ C(v, y_J)\mathrm{Diag}(\lambda^{(p)}) \end{pmatrix}^{\star(\vec{1}^T_{p-1})} \prod_{i=1}^{\dim(v)} \sum_{j \in J} \frac{1}{v_i - y_j} =$

$$= \frac{\sum_{J, |J| = p(p-1)\dim(v)} \mathrm{per} \begin{pmatrix} C(v^{(1)}, y_J)\mathrm{Diag}(\lambda^{(1)} \star \{\frac{\mathrm{pol}(v^{(1)}, y_j)}{\mathrm{pol}(v, y_j)}\}_{\dim(y)}) \\ \ldots \\ C(v^{(p)}, y_J)\mathrm{Diag}(\lambda^{(p)} \star \{\frac{\mathrm{pol}(v^{(p)}, y_j)}{\mathrm{pol}(v, y_j)}\}_{\dim(y)}) \end{pmatrix}^{\star(\vec{1}^T_{p-1})} \prod_{i=1}^{\dim(v)} \sum_{j \in J} \frac{1}{v_i - y_j}}{(\prod_{q=1}^{p} \det^{p-1}(\mathrm{Van}(v^{(q)})))/\det^{p(p-1)}(\mathrm{Van}(v))}$$

$$= \sum_{J,|J|=p(p-1)\dim(v)} \mathrm{per}\begin{pmatrix} C(v^{(1)},y_J)\mathrm{Diag}(\hat{\lambda}^{(1)}) \\ \dots \\ C(v^{(p)},y_J)\mathrm{Diag}(\hat{\lambda}^{(p)}) \end{pmatrix}^{\star(\vec{1}_{p-1}^T)} \prod_{i=1}^{\dim(v)} \sum_{j\in J} \frac{1}{v_i - y_j}$$

(for arbitrary $\dim(v)$-vectors $v^{(1)},\dots,v^{(p)}$ and, because of the arbitrariness of $\lambda^{(1)},\dots,\lambda^{(p)}$, arbitrary $\hat{\lambda}^{(1)},\dots,\hat{\lambda}^{(p)}$). The latter expression is easy to turn, via taking, for

an infinitesimal ε, y of the generic form $\begin{pmatrix} v^{(1)}_{\backslash\{1,\dots,\dim(u)\}} \otimes \overset{w}{\vec{1}_{p-1}} \\ v^{(2)} \otimes \vec{1}_{p-1} \\ \dots \\ v^{(p)} \otimes \vec{1}_{p-1} \end{pmatrix} + O(\varepsilon)$ (with no other

indeterminates involving ε) and applying the limit technique, into

$$\sum_{J,|J|=(p-1)\dim(u)} \mathrm{per}\big(C(u,w_J)\mathrm{Diag}(\alpha)\big)^{\star(\vec{1}_{p-1}^T)} \prod_{i=1}^{\dim(v)} \sum_{j\in J} \frac{1}{v_i - w_j}$$

where $u = v^{(1)}_{\{1,\dots,\dim(u)\}}$, α is the first $\dim(w)$ entries of $\hat{\lambda}^{(1)}$,

$\sum_{q=1}^{p}\sum_{j=1}^{\dim(v)} \frac{1}{v_i - v_j^{(q)}} - \sum_{k=1}^{\dim(u)} \frac{1}{v_i - u_k} = 0$ for i = 1,…,dim(v).

Upon putting $w = \hat{w} \otimes \vec{1}_{p-1}$ and $\alpha = \hat{\alpha} \otimes \begin{pmatrix} 1 \\ \dots \\ p-1 \end{pmatrix}$, the latter expression eventually

turns into $\varphi_{1,p-1}\big(v,u,\hat{w},\hat{\alpha}^{\star(p-1)}\big)$ (for arbitrary $v, u, \hat{w}, \hat{\alpha}$) what completes the proof.

We can also add that all the above proof's polynomial-time reductions from $\mathrm{per}\begin{pmatrix} C(y^{(1)},y)\mathrm{Diag}(\lambda^{(1)}) \\ \dots \\ C(y^{(p)},y)\mathrm{Diag}(\lambda^{(p)}) \end{pmatrix}$ to the very end remain valid in characteristics 3 and 5 as well what makes this permanent $\#_p$P-complete for any odd prime. This fact is equivalent to the $\#_p$P-completeness of the permanent of a rectangular matrix of Cauchy-rank p in any odd prime characteristic p.

Hence we've shown that the permanent of a "column-weighted" rectangular Cauchy matrix is polynomial-time computable in characteristic 3 and $\#_p$P-complete for any prime p > 5. In the case of all the column-weights equal to unity (i.e. of a "non-column-weighted" rectangular Cauchy matrix) this permanent is polynomial-time computable in

any characteristic according to Lemma III.1. The question also arises whether in characteristic 3 there is a likewise polynomial-time manipulation with denominator-values' grouping and row-weights that generates the valence (2).

Besides, as it was said earlier, Theorem III.12 provides the polynomial-time computability of $\text{per}\big(C^{\star 2}(x,y)\text{Diag}(\lambda)\big)$ in characteristic 5. In this regard, let's prove the following theorem:

Theorem III.32.

$\text{per}\big(C^{\star 2}(x,y)\text{Diag}(\lambda)\big)$ is $\#_5$P-complete

Proof:

The expression $\text{per}\big(C^{\star 2}(x,y)\text{Diag}(\lambda)\big)$ polynomial-time generates

$$\text{per}\begin{pmatrix} C(y^{(1)},y)\text{Diag}(\lambda^{(1)}) \\ \dots \\ C(y^{(4)},y)\text{Diag}(\lambda^{(4)}) \end{pmatrix}$$ via a process analogical to the one described in the above proof of Theorem III.31: first we receive, likewise,

$$\text{per}\left(\begin{pmatrix} C^{\star 2}\big(y^{(1)},z\big) \\ C^{\star 3}\big(y^{(2)},z\big) \\ C^{\star 4}\big(y^{(3)},z\big) \\ C^{\star 5}\big(y^{(4)},z\big) \\ C^{\star 5}(y,z) \end{pmatrix}\text{Diag}(\lambda)\right) = \text{coef}_{\omega^d}\text{per}(C\begin{pmatrix} C^{\star 2}\big(y^{(1)},\check{z}\big) \\ C^{\star(2\vec{1}_4^T,3)}\big(y^{(2)},\check{z}\big) \\ C^{\star(2\vec{1}_4^T,3\vec{1}_4^T,4)}\big(y^{(3)},\check{z}\big) \\ C^{\star(2\vec{1}_4^T,3\vec{1}_4^T,4\vec{1}_4^T,5)}\big(y^{(4)},\check{z}\big) \\ C^{\star(2\vec{1}_4^T,3\vec{1}_4^T,4\vec{1}_4^T,5)}(y,\check{z}) \end{pmatrix}\text{Diag}(\check{\lambda})) =$$

$$= \text{coef}_{\omega^d}\lim_{\varepsilon\to 0}\frac{\text{per}(C^{\star 2}(\begin{pmatrix} y^{(1)} \\ y^{(2)}\ddot{+}\varepsilon\alpha \\ y^{(3)}\ddot{+}\varepsilon\beta \\ y^{(4)}\ddot{+}\varepsilon\gamma \\ y\ddot{+}\varepsilon\gamma \end{pmatrix},\check{z})\text{Diag}(\check{\lambda}))}{\text{per}^{n_2}((\varepsilon\alpha^T)^{\star(0\vec{1}_4^T,1)})\text{per}^{n_3}((\varepsilon\beta^T)^{\star(0\vec{1}_4^T,\vec{1}_4^T,2)})\text{per}^{n_3+m}((\varepsilon\gamma^T)^{\star(0\vec{1}_4^T,\vec{1}_4^T,2\vec{1}_4^T,3)})}$$

where α,β,γ are arbitrary 5-, 9-, 14-vectors correspondingly, $\check{z} = \begin{pmatrix} z \\ \hat{z} \end{pmatrix}$, $\check{\lambda} = \begin{pmatrix} \lambda \\ \hat{\lambda} \end{pmatrix}$, $d = 4n_2+8n_3+16n_4+16m$,

for q =1,2,3,4 $\sum_{j=1}^{\dim(\hat{z})}\dfrac{\hat{\lambda}_j^r}{(y_i^{(q)}-\hat{z}_j)^s} = \begin{bmatrix} \omega, r = 2, s = 3+q \\ 0\,, \text{else} \end{bmatrix}$ for $i = 1,\dots,n_q$, $\dim(y^{(q)}) = n_q$,

$$\sum_{j=1}^{\dim(\hat{z})} \frac{\hat{\lambda}_j^r}{(y_k - \hat{z}_j)^s} = \begin{bmatrix} \omega & \text{if } r = 2, s = 3 + q \\ & 0 \text{ , else} \end{bmatrix} \quad \text{for k = 1 ,...,m, dim(y) = m;}$$

and then we get

$$\text{per}\begin{pmatrix} C(y^{(1)}, y)\text{Diag}(\lambda^{(1)}) \\ ... \\ C(y^{(4)}, y)\text{Diag}(\lambda^{(4)}) \end{pmatrix} = \text{coef}_{\omega^{m - \Sigma_{q=1}^4 n_q}} \text{per}(\begin{pmatrix} C^{\star 2}(y^{(1)}, z) \\ C^{\star 3}(y^{(2)}, z) \\ C^{\star 4}(y^{(3)}, z) \\ C^{\star 5}(y^{(4)}, z) \\ C^{\star 5}(y, z) \end{pmatrix} \text{Diag}(\lambda))$$

where

for q = 1,2,3,4 $\sum_{j=1}^{\dim(z)} \frac{\lambda^r}{(y_i^{(q)} - z_j)^s} = 0$ for $i = 1, ..., n_q$,

$$\sum_{j=1}^{\dim(z)} \frac{\lambda^r}{(y_k - z_j)^s} = \begin{bmatrix} \lambda_k^{(q)} & \text{if } r = 2, \ s = q \\ \omega & \text{if } r = 1, \ s = 5 \\ & 0 \text{ , else} \end{bmatrix} \quad \text{for k = 1, ..., m}$$

The expression $\text{per}\begin{pmatrix} C(y^{(1)}, y)\text{Diag}(\lambda^{(1)}) \\ ... \\ C(y^{(4)}, y)\text{Diag}(\lambda^{(4)}) \end{pmatrix}$ generates, also by the technique shown

in Theorem III.31's proof while replacing p by p-1 (i.e. through considering its partial

case $\text{per}\begin{pmatrix} C(v, y)\text{Diag}(\lambda^{(1)}) \\ ... \\ C(v, y)\text{Diag}(\lambda^{(4)}) \end{pmatrix}^{\star(1,1,1,1,2)} = \lim_{\varepsilon \to 0} \frac{\text{per}\begin{pmatrix} C(v \ddot{\mp} \beta, y)\text{Diag}(\lambda^{(1)}) \\ ... \\ C(v \ddot{\mp} \beta, y)\text{Diag}(\lambda^{(4)}) \end{pmatrix}}{\text{per}^{4\dim(v)}((\varepsilon\beta^T)^{\star(0,0,0,0,1)})}$, where β is an

arbitrary 5-vector, and putting all its 5,0,0,0,6-, 0,5,0,0,6-, 0,0,5,0,6- and 0,0,0,5,6-
row-weights equal to ω and, for i = 1,...,dim(v), its i-th 1,1,1,1,8-row-weight equal to
$\frac{\alpha_i}{-3!} \omega^4$ for computing the coefficient at $\omega^{4\dim(v)}$, where ω is a formal variable), the

sum $\sum_I (\prod_{i \in I} \alpha_i) \text{per}\begin{pmatrix} C(v_I, y)\text{Diag}(\lambda^{(1)}) \\ ... \\ C(v_I, y)\text{Diag}(\lambda^{(4)}) \end{pmatrix}^{\star(\vec{1}_4^T)}$. This sum generates, via its

denominator-values' grouping into infinitesimal-close pairs and applying the limit

technique, the sum $\sum_I (\prod_{t \in I} \frac{\alpha_i \partial}{\partial v_t}) \text{per}\begin{pmatrix} C(v_I, y)\text{Diag}(\lambda^{(1)}) \\ ... \\ C(v_I, y)\text{Diag}(\lambda^{(4)}) \end{pmatrix}^{\star(\vec{1}_4^T)}$ where we can put all

the prolonged 1,1,1,1,5-row-weights equal to a formal variable for computing its

maximal degree power and receive $\sum_I(\prod_{t\in I}\alpha_i)\,\mathrm{per}\begin{pmatrix}C(v_I,y)\mathrm{Diag}(\lambda^{(1)})\\ \dots \\ C(v_I,y)\mathrm{Diag}(\lambda^{(4)})\end{pmatrix}^{\star(\overrightarrow{1_3^T})}$. This

sum, in turn, generates (also via its denominator-values' infinitesimal-close pairing and applying the limit technique, like it was done for the valence $\overrightarrow{1_4^T}$) the partial

derivative $(\prod_{t=1}^{\dim(v)}\frac{\partial}{\partial v_t})\mathrm{per}\begin{pmatrix}C(v,y)\mathrm{Diag}(\lambda^{(1)})\\ \dots \\ C(v,y)\mathrm{Diag}(\lambda^{(4)})\end{pmatrix}^{\star(\overrightarrow{1_3^T})}$. The latter expression is the

sum of Cauchy-like permanents where for each of the denominator-values its **multiple valence** (for $\lambda^{(1)},\lambda^{(2)},\lambda^{(3)},\lambda^{(4)}$) is one of the following four:

$((1,1,2),(1,1,1),(1,1,1),(1,1,1))$,

$((1,1,1),(1,1,2),(1,1,1),(1,1,1))$,

$((1,1,1),(1,1,1),(1,1,2),(1,1,1))$,

$((1,1,1),(1,1,1),(1,1,1),(1,1,2))$,

what allows to receive the multiple valence $((1,1,2),\emptyset,\emptyset,\emptyset)$ (via taking, as a formal variable, the prolonged 0,1,0,0,1-, 0,0,1,0,1- and 0,0,0,1,1-row-weights and computing the coefficient at its maximal degree power) that, in turn, generates either $((1),\emptyset,\emptyset,\emptyset)$ or $((2),\emptyset,\emptyset,\emptyset)$ via taking, as a formal variable, either the prolonged 2,0,0,0,3-row-weight or the prolonged 1,0,0,0,1-row-weight correspondingly (while choosing one of these two options for each denominator-value) and computing the coefficient at its maximal degree power. Thus we obtain

the expression $\mathrm{per}\begin{pmatrix}C^{\star\gamma_1}(v_1,y)\mathrm{Diag}(\lambda^{(1)})\\ \dots \\ C^{\star\gamma_m}(v_m,y)\mathrm{Diag}(\lambda^{(4)})\end{pmatrix}$ for the left denominator-value

spectrum $\{v_1,\dots,v_m\}$ whose valences are either (1) or (2) and this expression polynomial-time generates, according to the scheme of Theorem III.31's proof we

already referred earlier in the present proof, $\mathrm{per}\begin{pmatrix}C(y^{(1)},y)\mathrm{Diag}(\lambda^{(1)})\\ \dots \\ C(y^{(p)},y)\mathrm{Diag}(\lambda^{(5)})\end{pmatrix}$ that is, as it

also was shown in the referred proof, $\#_5P$-complete.

We hence can conclude that the above theorem implies once more, independently of Theorem III.14, the $\#_5 P$-completeness of the trace-determinant $\det_{tr}\left(\tilde{C}(x) + \right.$

$\mathrm{Diag}(g), \left\{G^{(i)}M\right\}_n\right)$ (where $M = \begin{pmatrix} 0_{2\times 2} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0_{2\times 2} \end{pmatrix}$) for arbitrary symmetric

$G^{(1)}, \dots, G^{(n)}$ and absence-weights and, eventually, the permanent's polynomial-time computability in characteristic 5.

Definition:

Let $x_1, \dots, x_n$ be independent variables, $A = A(x) = \{a_{i,j}(x_i, x_j)\}_{n\times n}$ be an $n\times n$-matrix such that for k = 1,…,n its k-th row and column are functions in the variable x_k and for k = 1,…,n $\alpha_k = \{\alpha_{k,u}\}_{\dim(\alpha_k)}$, $\beta_k = \{\beta_{k,v}\}_{\dim(\beta_k)}$ be non-decreasing sequences (optionally empty) of natural numbers of lengths $\dim(\alpha_k)$ and $\dim(\beta_k)$ correspondingly.

Then we define the $\pmb{\alpha, \beta}$-**valence power** of A as the $(\sum_{i=1}^n \dim(\alpha_i)) \times (\sum_{j=1}^n \dim(\beta_j))$-matrix $A^{\langle\alpha,\beta\rangle} :=$

$$\left\{\left\{\frac{1}{(\alpha_{i,u}-1)!(\beta_{j,v}-1)!}\frac{\partial^{\alpha_{i,u}-1}}{\partial x_i^{\alpha_{i,u}-1}}\frac{\partial^{\beta_{j,v}-1}}{\partial x_j^{\beta_{j,v}-1}}a_{i,j}(x_i, x_j)\right\}_{\substack{u=1,\dots,\dim(\alpha_i) \\ v=1,\dots,\dim(\beta_j)}}\right\}_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$$

and we'll call α_k and β_k the row and column (or left and right) valences of x_k (or just the k-th row and column valences) correspondingly, while the pair $\mathrm{val}(x_k) := (\alpha_k, \beta_k)$ will be called the valence of x_k, the vectors $\alpha = \{\alpha_k\}_n$ and $\beta = \{\beta_k\}_n$ – the left and right valence-vectors correspondingly, and the vector $\{(\alpha_k, \beta_k)\}_n$ – the valence-vector.

This definition implies that all the Cauchy-like matrices we considered in the present article's chapter III are in fact either the α, β-valence powers of a Cauchy-wave matrix for some left and right valence-vectors α, β or can be expressed through them via the operations of the left- and right-multiplication by diagonal matrices and vector-composition.

Definition:

Let for k = 1,…,n $\alpha_k^{(1)}, \dots, \alpha_k^{(t_k)}, \beta_k^{(1)}, \dots, \beta_k^{(t_k)}$ be non-decreasing sequences of natural numbers, $d_k^{(1)}, \dots, d_k^{(t_k)}$ be elements of the ground field and A be an n×n-matrix. Then we define for variables $x_1, \dots, x_n$:

the formal sum $\mathfrak{v}_k := \mathrm{valsum}(x_k) := \sum_{r_k=1}^{t_k} d_k^{(r_k)} (\alpha_k^{(r_k)}, \beta_k^{(r_k)})$ as the **valence-sum** of x_k (or the k-th valence-sum) where $d_k^{(r_k)}$ will be called the summation-weight of the valence $(\alpha_k^{(r_k)}, \beta_k^{(r_k)})$

and

the expression $\mathrm{per}_{\mathfrak{v}}(A) := \mathrm{per}_{\mathfrak{v}_1,\dots,\mathfrak{v}_n}(A) :=$

$$:= \sum_{r_1=1}^{t_1} \cdots \sum_{r_n=1}^{t_n} d_1^{(r_1)} \cdots d_n^{(r_n)} \delta\left(\sum_{k=1}^{n}(\dim(\alpha_k^{(r_k)}) - \dim(\beta_k^{(r_k)}))\right) \mathrm{per}\left(A^{\langle\{\alpha_k^{(r_k)}\}_n, \{\beta_k^{(r_k)}\}_n\rangle}\right) $$

where for a real number m $\delta(m) = \begin{bmatrix} 0, m \neq 0 \\ 1, m = 0 \end{bmatrix}$ (and hence the summation is over all the possible n-tuples $r_1, \dots, r_n$ making the matrix $A^{\langle\{\alpha_k^{(r_k)}\}_n, \{\beta_k^{(r_k)}\}_n\rangle}$ square) as the **$\mathfrak{v}$-sum permanent** of A (or the valence-sum permanent of A on the vector $\mathfrak{v}$ that we'll call the valence-sum vector of this permanent).

According to the two latter definitions, we can state that $\mathrm{per}_{\mathfrak{v}}(\tilde{C}(\binom{y}{z}))$ is a polynomial in the row-weights $\sum_{j=1}^{\dim(z)} \frac{\lambda_j^r}{(y_i - z_j)^s}$ if for j = 1,…,dim(z) $\mathrm{valsum}(z_j) = \lambda_j(\emptyset, (1)) + (\emptyset, \emptyset)$. This fact provides the opportunity of taking those row-weights (due to their algebraic independence in characteristic p for all the pairs r,s that are not both divisible by p) equal to polynomials in a formal variable and receiving, upon computing the formal variable's maximal degree power's coefficient of $\mathrm{per}_{\mathfrak{v}}(\tilde{C}(\binom{y}{z}))$ (as this permanent would become, in such a case, a polynomial in this formal variable as well), $\mathrm{per}_{\hat{\mathfrak{v}}}(\tilde{C}(y))$ where the new valence-sum vector $\hat{\mathfrak{v}}$ is obtained from the subvector of $\mathfrak{v}$ corresponding to y via transforming, for i = 1,…,dim(y), the old valence-sum of y_i in the accordance with the polynomials in the formal variable the corresponding (i.e. having the denominator-value y_i) row-weights are equal to. We can call such a transformation *a prolongation derivative* and its partial cases were actually applied in the proofs of Theorems III.31 and III.32 for left-sided (having only empty right parts in all their valences) valence-sums. Taking into account the fact that $\mathrm{per}_{\mathfrak{v}}(A) = \mathrm{per}_{\mathfrak{v}^*}(A^T)$ where $\mathfrak{v}^*$ denotes the valence-sum vector where all the

left and right valences exchanged places in each involved valence, we receive an option to apply a prolongation derivative to any valence-sum on "both its sides".

Besides, grouping the denominator-values (i.e. the variables) into infinitesimal-close families generates, accordingly, a contraction of a family of valence-sums into a new valence-sum analogical to the contraction occurring in families of matrix-weights for the trace-determinant of a Cauchy-wave matrix via the open trace-determinant compression operator that was discussed earlier. Let's call such a contraction an infinitesimal-close contraction and we can also notice that quiet a number of partial cases of this contraction were applied in the proofs of Theorems III.31 and III.32.

If we speak about characteristic 3, we can notice that the above-proven polynomial-time computability of $\text{per}(C(x, z)\text{Diag}(\lambda))$ allows, in this characteristic, to polynomial-time compute the valence-sum permanent of a Cauchy-wave matrix where the valence-sum vector can contain any singleton (i.e. having just one valence in its sum) left-sided valence-sum $((1,1,\dots,k,k),\emptyset)$ for any natural k that is able to generate, via its various prolongation-derivatives, quiet a complex variety of valence-sums including a number of two-sided and non-singleton ones. However, as it was shown in Theorem III.31's proof, computing the valence-sum permanent of a Cauchy-wave matrix on a valence-sum vector whose entries are arbitrarily taken from a set including the singleton valence-sums $((1),\emptyset)$ and $((2),\emptyset)$ is $\#_3$-P-complete and this fact arises the question of determining the closure of the valence-sum set $\{((1,1,\dots,k,k),\emptyset), k \in \mathbb{N}\}$ by all the existing prolongation-derivative and infinitesimal-close contraction operators.

References

1. *Allender, Eric; Gore, Vivec (1994), "A uniform circuit lower bound for the permanent", SIAM Journal on Computing, 23 (5): 1026–1049, doi:10.1137/s0097539792233907*
2. *Balasubramanian, K. (1980), Combinatorics and Diagonals of Matrices, Ph.D. Thesis, Department of Statistics, Loyola College, Madras, India, T073, Indian Statistical Institute, Calcutta*

3. *Bax, Eric (1998), Finite-difference Algorithms for Counting Problems, Ph.D. Dissertation, **223**, California Institute of Technology*
4. *Bax, Eric; Franklin, J. (1996), A finite-difference sieve to compute the permanent, Caltech-CS-TR-96-04, California Institute of Technology*
5. *Glynn, David G. (2010), "The permanent of a square matrix", European Journal of Combinatorics, **31** (7): 1887–1891, doi:10.1016/j.ejc.2010.01.010*
6. *Glynn, David G. (2013), "Permanent formulae from the Veronesean", Designs, Codes and Cryptography, **68** (1-3): 39–47, doi:10.1007/s10623-012-9618-1*
7. *Jerrum, M.; Sinclair, A.; Vigoda, E. (2001), "A polynomial-time approximation algorithm for the permanent of a matrix with non-negative entries", Proc. 33rd Symposium on Theory of Computing, pp. 712–721, doi:10.1145/380752.380877, ECCC TR00-079*
8. *Mark Jerrum; Leslie Valiant; Vijay Vazirani (1986), "Random generation of combinatorial structures from a uniform distribution", Theoretical Computer Science, **43**: 169–188, doi:10.1016/0304-3975(86)90174-X*
9. *Kogan, Grigoriy (1996), "Computing permanents over fields of characteristic 3: where and why it becomes difficult", 37th Annual Symposium on Foundations of Computer Science (FOCS '96)*
10. *van Lint, Jacobus Hendricus; Wilson, Richard Michale (2001), A Course in Combinatorics, ISBN 0-521-00601-5*
11. *Little, C. H. C. (1974), "An extension of Kasteleyn's method of enumerating the 1-factors of planar graphs", in Holton, D., Proc. 2nd Australian Conf. Combinatorial Mathematics, Lecture Notes in Mathematics, **403**, Springer-Verlag, pp. 63–72*
12. *Little, C. H. C. (1975), "A characterization of convertible (0, 1)-matrices", Journal of Combinatorial Theory, Series B, **18** (3): 187–208, doi:10.1016/0095-8956(75)90048-9*
13. *Marcus, M.; Minc, H. (1961), "On the relation between the determinant and the permanent", Illinois Journal of Mathematics, **5**: 376–381*
14. *Pólya, G. (1913), "Aufgabe 424", Arch. Math. Phys., **20** (3): 27*
15. *Reich, Simeon (1971), "Another solution of an old problem of pólya", American Mathematical Monthly, **78** (6): 649–650, JSTOR 2316574, doi:10.2307/2316574*
16. *Rempała, Grzegorz A.; Wesolowski, Jacek (2008), Symmetric Functionals on Random Matrices and Random Matchings Problems, p. 4, ISBN 0-387-75145-9*
17. *Ryser, Herbert John (1963), Combinatorial Mathematics, The Carus mathematical monographs, Mathematical Association of America*
18. *Vazirani, Vijay V. (1988), "NC algorithms for computing the number of perfect matchings in $K_{3,3}$-free graphs and related problems", Proc. 1st Scandinavian Workshop on Algorithm Theory (SWAT '88), Lecture Notes in Computer Science, **318**, Springer-Verlag, pp. 233–242, doi:10.1007/3-540-19487-8_27*
19. *Valiant, Leslie G. (1979), "The Complexity of Computing the Permanent", Theoretical Computer Science, Elsevier, **8** (2): 189–201, doi:10.1016/0304-3975(79)90044-6*
20. *"Permanent", CRC Concise Encyclopedia of Mathematics, Chapman & Hall/CRC, 2002*
21. *Greg Cohen (2010). "A new algebraic technique for polynomial-time computing the number modulo 2 of Hamiltonian decompositions and similar partitions of a graph's edge set" https://arxiv.org/abs/1005.2281*