A New numerical method for multi-roots finding with the R software

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Abstract

The most basic problem in numerical analysis (methods) is the root finding problem. In this paper we interesting in new numerical method which based on the Modified Bisection Method(MBM) referred to by Tanakan, (2013, [3]) superficially and didn't know her as a numerical method for finding the roots of a function. Hence in this study we define her as a new numerical method with error bound and the number iterations necessary. Finally we present a new MBM for multi-roots with the R software.

Keywords: Nonlinear equation , Linear interpolation , Bisection Method , Modified Bisection Algorithm.

1. Introduction

The process of finding the root of function is involves finding the value of x for which $f(x)=0$. If the function equals zero, x is the root of the function. The simplest root-finding algorithm is the bisection method is also called the interval halving method. It works when $f(x)$ is a continuous function and it requires previous knowledge of two initial guesses α and β such that $f(\alpha)$ and $f(\beta)$ have opposite signs, then find the midpoint of $[\alpha;\beta]$, and then decide whether the root lies on $[\alpha;(\alpha+\beta)/2]$ or $[\alpha;(\alpha+\beta);\beta]$ repeat until the interval is sufficiently small. However, the bisection method is reliable, but it converges slowly, gaining one bit of accuracy with each iteration. In this paper we interesting in new numerical method based on MBM and like a linear interpolation method is also known as the method of *regula falsi* (false position) see [1], but with other way.

Tanakan, (2013, [3]) in his paper proposed algorithm base in the bisection method and the value of x_s as we defined in (1) and (2) for solve nonlinear equation, but his study did not define the value of x_{s+1} in term of x_s he defined in term of the midpoint of $[\alpha;\beta]$, and bending the bisection method and the value of x_s , he named modified bisection algorithm [for more detail see Tanakan, (2013, [3])].Firstly we present the value of x_s and define a new MBM with her algorithm and the error bound with the iterations necessary for finding the root of function. In section (2) we programming the new MBM for multi-roots finding with the R software

By the intermediate value theorem implies that a number x^* exists in $[\alpha;\beta]$ with $f(x^*)=0$ if $f(\alpha)f(\beta)<0$, mean that $f(\alpha)f(\beta)<0$, which means that one of them is above the x-axis and the other one below the x-axis.

Let $f(x)$ be a continuous function and defined on interval $[\alpha;\beta]$ with $f(\alpha)f(\beta)<0$

Firstly, we set $\alpha=\alpha_1$ and $\beta=\beta_1$. For an integer $s\geq 1$. Then, we can find the equation of straight line from the points $(\alpha_s^*, f(\alpha_s^*))$ and $(\beta_s^*, f(\beta_s^*))$ as $y=ax+b$ where

$$a=\frac{f(\beta_s^*)-f(\alpha_s^*)}{\beta_s^*-\alpha_s^*} \text{ or } a=\frac{f(\beta_s^*)-f(\alpha_s^*)}{\beta_s^*-\alpha_s^*}$$

and

$$b=f(\beta_s^*)-a\beta_s^* \text{ or } b=f(\alpha_s^*)-a\alpha_s^*$$

Hence, the x -intercept of the straight line is at a point $x_s=\frac{-b}{a}$ That is

$$x_s=\beta_s^*-f(\beta_s^*)\frac{\beta_s^*-\alpha_s^*}{f(\beta_s^*)-f(\alpha_s^*)} \quad (1)$$

or

$$x_s=\alpha_s^*-f(\alpha_s^*)\frac{\beta_s^*-\alpha_s^*}{f(\beta_s^*)-f(\alpha_s^*)} \quad (2)$$

It's clearly that

$$x_1=\alpha-f(\alpha)\frac{\beta-\alpha}{f(\beta)-f(\alpha)} \quad (3)$$

or

$$x_1 = \beta - f(\beta)\frac{\beta-\alpha}{f(\beta)-f(\alpha)} \quad (4)$$

By the intermediate value theorem when f is a continuous function and monotone on $[\alpha;\beta]$,we have for $s\geq 1$. $f(\alpha)<f(x_s)<f(\beta)$ (or $f(\alpha)>f(x_s)>f(\beta)$), then we can defined two cases.

For the case1, if $f(\alpha)$ and $f(x_1)$ have opposite signs implies that $f(x_1)$and $f(\beta)$ have same signs so the work remains on the interval $[\alpha;x_1]$ and for all $x_s\in[\alpha,x_1]$ as defined in the relationship (2) we get $f(\alpha)f(x_s)<0$ then x_s in (2) becomes defined for $x_0=\beta$ by

$$x_{s+1} = \alpha - f(\alpha)\frac{x_s-\alpha}{f(x_s)-f(\alpha)} \quad (5)$$

Finally, we choose the new subinterval for the next iteration $[\alpha;x^*_{s+1}]=[\alpha;x_{s+1}]$

We stay in the same process until we find a solution or a approximate value for it as we define in theorem1 .

And the same thing we find in the other case $x_s\in[x_1;\beta]$ for $x_0=\alpha$ with

$$x_{s+1} = \beta - f(\beta)\frac{\beta-x_s}{f(\beta)-f(x_s)} \quad (6)$$

and we choose the new subinterval for the next iteration $[x^*_{s+1};\beta]=[x_{s+1};\beta]$

1.1 Algorithm for a new Modified Bisection Method

The steps to apply the bisection method to find the root of the equation $f(x)=0$ are

- Step1:Choose α , β for $\alpha<\beta$where $f(\alpha)f(\beta)<0$. and tolerance 1×10^{-7}
- Step2: Compute

$$x_1 = \alpha - f(\alpha)\frac{\beta-\alpha}{f(\beta)-f(\alpha)}.$$

- Step3: If $f(\alpha)f(x_1)<0$ for $n\geq1$, compute

$$x_{n+1} = \alpha - f(\alpha)\frac{x_n-\alpha}{f(x_n)-f(\alpha)}$$

ELSE

$$x_{n+1} = \beta - f(\beta)\frac{\beta-x_n}{f(\beta)-f(x_n)}$$

- Step 4: If $|x_{n+1}-x^*|<1\times10^{-7}$, then stop program (i.e $x_{n+1}=x^*$) the zero is x_{n+1}.
  ELSE if $f(\alpha)f(x_1)<0, \alpha=\alpha; x^*_{n+1}=x_{n+1}$

  else

  $\beta=\beta; x^*_{n+1}=x_{n+1}$

and set $n=n+1$, GOTO Step3.

Next we present theorems for the error bound of a new MBM, but before we going to this theorems let us recall same definitions

We say that a function f defined on interval I to $\mathbb{R}$ admits a maximum in $\gamma\in I$ if for all $x\in I$ we have $f(x)\leq f(\gamma)$ and we say that $f(\gamma)$ is the global maximum of f .

If I has a distance, or a norm , we say that f admits a local maximum, or a relative maximum in $\gamma\in I$ if there is a neighborhood V of γ such that for all $x\in V$, we have $f(x)\leq f(\gamma)$.

Of course, by changing the inequalities of meaning we can define a global minimum and local minimum.

An extremum is maximum or a minimum. The search for extremum is related to the differential calculus. Thus if f is defined on interval I of $\mathbb{R}$ and if f is differentiable in γ, then $f'(\gamma)=0$. The study of the sign of f' on the neighborhood of γ often makes it possible to conclude as to existence of a maximum or minimum.

Theorem 1: *Let f be a continuous function and defined on $[\alpha;\beta]$ which $f(\alpha)f(\beta)<0$. A new Modified Bisection generate s a sequence $\{x_n\}_{n=1}^{\infty}$ approximating a zero x^* of f for*

$$\lambda = \frac{f(\gamma_2)}{f(\gamma_1)} \quad (7)$$

where $f(\gamma_2)$ is a maximum of f in $[\alpha;\beta]$ and $f(\gamma_1)$ is a minimum of f in $[\alpha;\beta]$ respectively

Case1, for $f(\alpha)f(x_1)<0$

$$|x_n - x^*| < \left|\frac{f(\alpha)}{f(\gamma_1)}\right|^n \frac{\beta-\alpha}{|\lambda-1|^n} \quad (8)$$

where x_1 defined in(3)

Case2, for $f(\beta)f(x_1)<0$

$$|x_n - x^*| < \left|\frac{f(\beta)}{f(\gamma_2)}\right|^n \frac{\beta-\alpha}{\left|\frac{1}{\lambda}-1\right|^n} \quad (9)$$

where x_1 defined in(4)

Theorem 2: *Let f be a continuous and monotone function defined on $[\alpha;\beta]$ with $f(\alpha)f(\beta)<0$. Then $f(\alpha)\leq f(x)\leq f(\beta)$ as an global extremum for all $x\in[\alpha;\beta]$ then the previous theorem becomes:*

Case1, for $f(\alpha)f(x_1)<0$

$$|x_n - x^*| < \frac{\beta-\alpha}{|\lambda-1|^n} \quad (8)$$

where x_1 defined in(3)

Case2, for $f(\beta)f(x_1)<0$

$$\left|x_n - x^*\right| < \frac{\beta - \alpha}{\left|\frac{1}{\lambda} - 1\right|^n} \quad (9)$$

with

$$\lambda = \frac{f(\beta)}{f(\alpha)} \quad (7)$$

In next study we interesting in theorem2 Going back to the previous theorem2, with first case we noted that if $\lambda = 3$ or $\lambda = -1$ we have that

$$\left|x_n - x^*\right| < \frac{\beta - \alpha}{2^n}$$

which is the error bound of the bisection method, so a new MBM tied to the value of the real number $\lambda \neq 0$ and if $\lambda \in]-1;0[\cup]0;3[$ the new MBM is slowly than the bisection method and that's what we have to calculate the value of λ before working with a new MBM. However if $\lambda \leq -1$ and $\lambda \geq 3$ the new MBM is better than the Bisection method. Since for the second case if $\lambda = 1/3$ or $\lambda = -1$ we have that $\left|x_n - x^*\right|$ becomes the error bound of the bisection method and if $\lambda \in]-1;0[\cup]0;1/3[$ with $\lambda \neq 0$ the new MBM is slowly than the bisection method and the contrary if $\lambda \leq -1$ and $\lambda \geq 1/3$ the new MBM has better convergence than bisection method. To see how many iterations will be necessary, for $f(\alpha)f(x_1) < 0$ suppose we want to have

$$\left|x_n - x^*\right| < \eta_1$$

This will be satisfied if

$$\frac{\beta - \alpha}{|\lambda - 1|^n} < \eta_1$$

Taking logarithms of both sides, we can solve this to give

$$n \geq \frac{\ln\left(\frac{\beta - \alpha}{\eta_1}\right)}{\ln|\lambda - 1|} \quad (10)$$

and for the case where $f(\beta)f(x_1) < 0$

$$n \geq \frac{\ln\left(\frac{\beta - \alpha}{\eta_1}\right)}{\ln\left|\frac{1}{\lambda} - 1\right|} \quad (11)$$

2. Algorithm for a new Modified Bisection Method for multi-roots:

In this section we provide an algorithm that relies on the method of a new MBM which finding the roots of function. This algorithm is with the R software version i386 3.3.3.

The R is an integrated suite of software facilities for data manipulation, calculation and graphical display. R is very much a vehicle for newly developing methods of interactive data analysis. It has developed rapidly, and has been extended by a large collection of packages, many people use R as a statistics system. There are about 25 packages supplied with R (called \standard" and \recommended" packages) and many more are available through the CRAN family of Internet sites (via http://CRAN.R-project.org) and elsewhere.

We can state an alternative a new MBM algorithm for multi-roots as the following ,

- Step1: Divide the interval into several intervals by $h = (b - a)/t$ where $t \in \mathbb{N}$ is the number of intervals for example $t = 10$.
- Step2: For $i = 1,\ldots,10$, and tolerance 1×10^{-7} ; and $\alpha_1 = \alpha_i$, $\beta_1 = \beta_{i+1}$ where $f(\alpha_i)f(\beta_{i+1}) < 0$.
- Step3: Compute
$$x_1 = \alpha_i - f(\alpha_i)\frac{\beta_{i+1} - \alpha_i}{f(\beta_{i+1}) - f(\alpha_i)}.$$
- Step4: If $f(\alpha_i)f(x_1) < 0$ for $n \geq 1$, compute
$$x_{n+1} = \alpha_i - f(\alpha_i)\frac{x_n - \alpha_i}{f(x_n) - f(\alpha_i)}$$
ELSE
$$x_{n+1} = \beta_{i+1} - f(\beta_{i+1})\frac{\beta_{i+1} - x_n}{f(\beta_{i+1}) - f(x_n)}$$
- Step 5: If $\left|x_{n+1} - x^*\right| < 1 \times 10^{-7}$, then stop program (i.e $x_{n+1} = x^*$) the zero is x_{n+1}.
ELSE if $f(\alpha_i)f(x_1) < 0, \alpha_i = \alpha_i ; x^*_{n+1} = x_{n+1}$
else
$$\beta_{i+1} = \beta_{i+1} ; x^*_{n+1} = x_{n+1}$$

and set $n = n + 1$, GOTO Step3.

2.1 A new MBM for multi-roots With The R Software:

Next we present our algorithm for multi-roots with the R software as the following ,

```
ridhaRoot<-function(f,a,b,num=10,eps=1e-07)
 {h=abs(b-a)/num
  i=0
  j=0
  a1=b1=0
  while(i<=num)
  {  a1=a+i*h
    b1=a1+h
 if(f(a1)==0){print(a1)
           print(f(a1))}
 else if (f(b1)==0){print(b1)
                print(f(b1))}
```

```
else if (f(a1)*f(b1)<0){
repeat{  if(abs(b1-a1)<eps)   break
         m=(b1-a1)/(f(b1)-f(a1))
         d=b1-f(b1)*m
  if(f(a1)*f(d)<0)  b1<-d else  a1<-d  }
print(j+1)
j=j+1
print(d)
print(f(d))
}
i=i+1
}
if(j==0)
print("finding root is fail")
else print("finding root is successful")
}
```

Example1: we ga to finding the roots of $f(x)=x^2-2$ on the interval $[0;2]$ and $[-2;2]$

```
f <- function(x){x^2-2}
ridhaRoot(f,-2,2)
#########################################
> f <- function(x){x^2-2}
> ridhaRoot(f,-2,2)
[1] 1
[1] -1.414214
[1] 4.440892e-16
[1] 2
[1] 1.414214
[1] 4.440892e-16
[1] "finding root is successful"
#######################################
```

Example2: We go to finding the roots of $f(x)=x^3-2x+2$ on the interval $[-3;3]$

```
f <- function(x){x^3-2*x+2}
ridhaRoot(f,-3,3)
#########################################
> f <- function(x){x^3-2*x+2}
> ridhaRoot(f,-3,3)
[1] 1
[1] -1.769292
[1] 0
[1] "finding root is successful"
#####################################
```

3. Discussion

Hence, by the intermediate value theorem a modified bisection method, it works when f is a continuous function on $[\alpha;\beta]$ with the initial condition $f(\alpha)f(\beta)<0$ existing with each iteration. And the select the number λ is an important role in credible of a new MBM because if we have $|\lambda-1|<2$ with case1 or $\left|\frac{1}{\lambda}-1\right|<2$ with the case2 as we describe in theorem2, the new MBM become useless and the Bisection method is better than a new MBM with this situation. Unlike this situation the new MBM is better because we fund that have $|\lambda-1|>2$ with case1 or $\left|\frac{1}{\lambda}-1\right|>2$ with the case2 and faster convergent than bisection method. And another important point is the interval $[\alpha;\beta]$, increase on $f(\alpha)f(\beta)<0$supporting power the new MBM is selecting images $f(\alpha)$ and $f(\beta)$ as an global extremum and this is indicated by the theorem2 that if f is a continuous and monotone on $[\alpha;\beta]$. More than if the difference $|\lambda-1|$ with case1 or $\left|\frac{1}{\lambda}-1\right|$ with the case2 is too large the error bound will become too small and little of the iteration number. Finally, in practice we're out several important points of the application this new numerical method for roots finding of nonlinear equation called Kouider Method (KM) which based on the new MBM. The important point for KM is the choose interval by take the following points, $f(\alpha)f(\beta)<0$ with $f(\alpha)$ and $f(\beta)$ as an global extremum and more importantly it whenever the big different $|\lambda-1|$ with case1 or $\left|\frac{1}{\lambda}-1\right|$ with the case2 as we define in theorem2, the new MBM become a good.

4. Proof of the theorem1

Let$f$ is a continuous function on $[\alpha;\beta]$ $f(\alpha)$ and $f(\beta)$ have opposite signs i.e $f(\alpha)f(\beta)<0$, and from (3) or (4) we have $\alpha<x_1<\beta$. Then form the new MBM we get two cases as following:

Case1: $f(\alpha)f(x_1)<0$ then for any $n\geq 0$ $\alpha<x^*<x_n<x_1$, so we have $x_n-x^*<x_n-\alpha<x_1-\alpha<\beta-\alpha$. Since, we found the terms x_n by(5) with$x_0=\beta$ as

$$
\begin{aligned}
x_1 &= \alpha-f(\alpha)\frac{x_0-\alpha}{f(x_0)-f(\alpha)}\\
x_2 &= \alpha-f(\alpha)\frac{x_1-\alpha}{f(x_1)-f(\alpha)}\\
x_3 &= \alpha-f(\alpha)\frac{x_2-\alpha}{f(x_2)-f(\alpha)}\\
&\vdots\\
x_n &= \alpha-f(\alpha)\frac{x_{n-1}-\alpha}{f(x_{n-1})-f(\alpha)}
\end{aligned}
\qquad (5)
$$

by recurrence we can write x_n as

$$x_n = \alpha - f(\alpha)^n \frac{\beta-\alpha}{\prod_{j=0}^{n-1}\left(f(x_j)-f(\alpha)\right)} \qquad (12)$$

then for any $n \geq 0$ and (12) we find

$$x_n - \alpha = -f(\alpha)^n \frac{\beta-\alpha}{\prod_{j=0}^{n-1}\left(f(x_j)-f(\alpha)\right)}$$

Since $\left|x_n - x^*\right| < \left|x_n - \alpha\right|$ we have

$$\left|x_n - x^*\right| < \left|f(\alpha)\right|^n \frac{\beta-\alpha}{\prod_{j=0}^{n-1}\left|f(x_j)-f(\alpha)\right|} \qquad (13)$$

Suppose that $f(\gamma_1) \leq f(x) \leq f(\gamma_2)$ for all $x \in [\alpha;\beta]$ and $(\gamma_1,\gamma_2) \in [\alpha;\beta]^2$ we have for $j = \{0,1,\ldots,n-1\}$

$$\frac{1}{\left|f(x_j)-f(\alpha)\right|} < \frac{1}{\left|f(\gamma_2)-f(\gamma_1)\right|} \qquad (14)$$

to this end we can get

$$\left|x_n - x^*\right| < (\beta-\alpha)\left|\frac{f(\alpha)}{f(\gamma_2)-f(\gamma_1)}\right|^n$$

we potted

$$\lambda = \frac{f(\gamma_2)}{f(\gamma_1)}$$

we find

$$\left|x_n - x^*\right| < \left|\frac{f(\alpha)}{f(\gamma_1)}\right|^n \frac{(\beta-\alpha)}{\left|\lambda-1\right|^n}$$

and if $f(\gamma_1) = f(\alpha)$ we have

$$\left|x_n - x^*\right| < \frac{(\beta-\alpha)}{\left|\lambda-1\right|^n}$$

Case2: $f(\beta)f(x_1)<0$ then for any $n \geq 0$ $x_1 < x_n < x^* < \beta$, so we have $x_n - x^* < x_n - \beta < x_1 - \beta < \beta - \alpha$;

Since we found the terms x_n with (6) and $x_0 = \alpha$, by recurrence we can write x_n as

$$x_n = \beta - f(\beta)^n \frac{\beta-\alpha}{\prod_{j=0}^{n-1}\left(f(\beta)-f(x_j)\right)} \qquad (15)$$

then for any $n \geq 0$ and (12) we find

$$x_n - \beta = -f(\beta)^n \frac{\beta-\alpha}{\prod_{j=0}^{n-1}\left(f(\beta)-f(x_j)\right)}$$

Since $\left|x_n - x^*\right| < \left|x_n - \alpha\right|$ and we suppose that $f(\gamma_1) \leq f(x) \leq f(\gamma_2)$ for all $x \in [\alpha;\beta]$ and $(\gamma_1,\gamma_2) \in [\alpha;\beta]^2$ we have for $j = \{0,1,\ldots,n-1\}$

$$\frac{1}{\left|f(\beta)-f(x_j)\right|} < \frac{1}{\left|f(\gamma_2)-f(\gamma_1)\right|} \qquad (16)$$

to this end we can get

$$\left|x_n - x^*\right| < (\beta-\alpha)\left|\frac{f(\beta)}{f(\gamma_2)-f(\gamma_1)}\right|^n$$

we potted

$$\lambda = \frac{f(\gamma_2)}{f(\gamma_1)}$$

we find

$$\left|x_n - x^*\right| < \left|\frac{f(\beta)}{f(\gamma_2)}\right|^n \frac{(\beta-\alpha)}{\left|\frac{1}{\lambda}-1\right|^n}$$

and if $f(\gamma_2) = f(\beta)$ we have

$$\left|x_n - x^*\right| < \frac{(\beta-\alpha)}{\left|\lambda-1\right|^n}$$

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