A graph for describing events in TSR (v2, 2020-11-01)

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Abstract. We present a graphical approach for illustrating and understanding a chain of events within the theory of special relativity (TSR). The graph represents an alternative to the 'world line', and is related to the work of Minkowski. It provides the standard time & position–parameters; in particular the so-called 'proper time', which we here refer to as 'internal time'. The object (event) that we follow has a velocity being piecewise constant. The travelling twin provides an example.

Key words: Event graph, travelling twin, dimensions of time, embedded clock.

1 Introduction

In previous works, Hokstad (2018a, 2018b, 2019) we introduced a time vector within the theory of special relativity TSR. Here we build on this work; presenting a graph to describe the *chain of events* of a moving object. The present work also elaborates on the findings of Minkowski, *e.g.* see Petkov (2012).

The velocity of the moving object must be piecewise constant. Further, we imagine this object to have a built in ('embedded') clock showing the 'internal time'. This is often denoted 'proper time', and is known to be independent of the reference frame (RF) of the observer.

We present a graph to illustrate the development of both this embedded clock reading and the standard space and time parameters of the event chain. The 'travelling twin' example is used to illustrate the approach; actually, applying two different RFs ('perspectives').

The graph may describe an object with a piecewise constant velocity. We apply just one space coordinate, but a generalization to three space coordinates is simple.

We point to the essential distinction between [1)] the embedded clock reading, which is independent of the chosen RF, and [2)] the two other parameters, time and position, of the chosen RF.

2 Basic notation

The standard situation in TSR usually involves two RFs, moving relative to each other at relative speed, v. Here we start out with a single one-dimensional RF, K; this has synchronized, stationary clocks located at virtually any position on the RF. Further, there is an object moving at speed, v relative to K, along the x-axis. The object starts out from the origin of K, when the local clock at this position reads 0. So, all events are given relative to this 'point of initiation'.

Further, (we imagine that) the moving object brings with it a clock. When this passed the origin at time 0, also this was synchronized with the clock on K located here. Now we apply the following notation

- τ = Clock reading ('internal time') of the moving object/clock.
- x_τ = Position of the moving object on K, at the instant when its embedded clock reads τ.
- t_τ = Clock reading ('external time') of the clock located at the position x_τ on K, when the moving clock arrives here and reads τ.

Now an 'event' is simply given by the moving clock reading, τ, together with the corresponding parameters (t_τ, x_τ) on K. Thus, (t_τ, x_τ) for $\tau \geq 0$ defines the chain of events defined by the moving object having velocity v relative to K:

$$x_\tau = vt_\tau$$

Further, we introduce c = speed of light. According to the *time dilation* in TSR we have:

$$\tau = \sqrt{t_\tau^2 - (x_\tau/c)^2} = t_\tau\sqrt{1-(v/c)^2}$$

Here τ, t_τ and x_τ/c are the fundamental parameters of an event. Note that all three represent time; in particular, x_τ/c equals the time required for light to go the distance x_τ. Now, we introduce the following time vector[1]

$$\overrightarrow{t_\tau} = \begin{pmatrix} \tau \\ x_\tau/c \end{pmatrix} = \begin{pmatrix} v/c \\ \sqrt{1-(v/c)^2} \end{pmatrix} t_\tau \tag{1}$$

Its absolute value equals

$$\left|\overrightarrow{t_\tau}\right| = t_\tau = \sqrt{\tau^2 + (x_\tau/c)^2}$$

Thus, the time vector comprises the information of the three 'fundamental time parameters'.

As it is the event chain that here is the focus of our interest, we consider the clock reading of the embedded clock, τ, as the 'basic' parameter. It is well-known that this is independent of which RF we have chosen for performing the observations. We refer to τ as the *internal* time of the event.

In contrast, the (t_τ, x_τ) parameters, will of course depend on the RF we have chosen for performing the observations. We refer to t_τ as the *external* time.

Finally, we point out the strong link to Minkowski's space-time, (Minkowski, 1909). Note that Minkowski refers to our τ as 'proper time', and our t_τ as the 'coordinate time', (*cf.* Petkov, 2012).

3 A graphical approach

Now we are ready to introduce the graphical approach related to the situation described.

3.1 Basic case: Constant velocity and a single space coordinate

Recall that we presently restrict to $x_\tau = vt_\tau$. Then Fig. 1 provides simultaneous values of the three basic time parameters, τ, x_τ/c and t_τ.

In the figure we have also introduced $\varphi_v \in (-\pi/2,\ \pi/2)$ given by

$$\sin\varphi_v = \frac{v}{c} = \frac{x_\tau/c}{t_\tau} \tag{2}$$

Thus

$$x_\tau/c = t_\tau \sin\varphi_v$$

$$\tau = t_\tau\sqrt{1-(v/c)^2} = t_\tau\cos\varphi_v \tag{3}$$

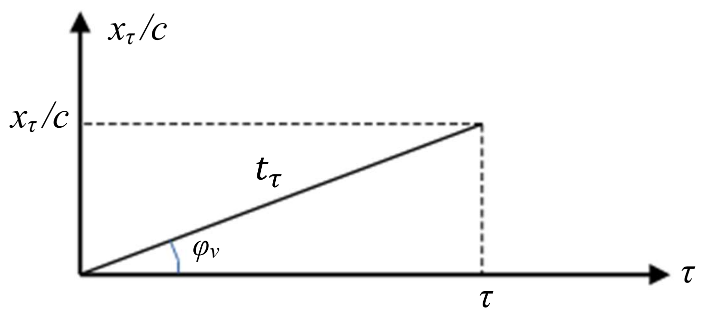

Figure 1. On a specific RF there is a chain of events, $(t_\tau, x_\tau)_{\tau\geq 0}$; describing an object at a constant velocity, *v*. The abscissa, τ, gives the reading of the object's 'embedded clock' (=internal time).

[1] This time vector differs slightly from the one considered in previous works, (which considered questions related to the Lorentz transformation).

Now by specifying the object's velocity, v relative to our RF, and its 'internal time', τ, we can from the graph directly read the corresponding parameters, (t_τ, x_τ) on the chosen RF.

Thus, the figure illustrates the time vector $\overrightarrow{t_\tau}$, with its two coordinates (τ, x_τ/c), and its length t_τ. The graph describes how the time vector evolves on *K*; following an object of velocity v and internal time τ.

3.2 Piecewise constant velocity

The approach so far is based on having a constant velocity, v. But we can similarly illustrate the chain of events for an object having a *piecewise constant* v, see the example given in Fig. 2. In this figure we start with a constant v >0. After a certain time the object is brought to rest (that is $v = \varphi_v = 0$), giving a graph parallel to the abscissa, τ. Next, the object approaches the origin; *i.e.* having a negative v. Thereafter the 'object', (or now rather 'event'), has velocity, $v = c$, ($\varphi_v = \pi/2$), and the line is orthogonal to the τ-axis, *etc.*

But the question is whether such abrupt changes of velocity actually is compatible with the framework of the TSR. Strictly speaking, obviously not! The TSR applies only for a chain of events having constant v. Therefore, at the instants where the velocity, v changes, we must also imagine a change of the embedded clock: A clock with the new velocity is passing the previous clock at exactly the right moment, and is then synchronized with the old embedded clock. Whether this is physically feasible is not really the question. We consider these 'embedded clocks' as being imagined, and the 'internal time' as the time reading of these clocks *if* they were present. However, we should of course be fully aware of this restriction. If v is *not* constant, we essentially refer to a thought experiment; *cf.* the case of travelling twin, which we discuss in the next chapter.

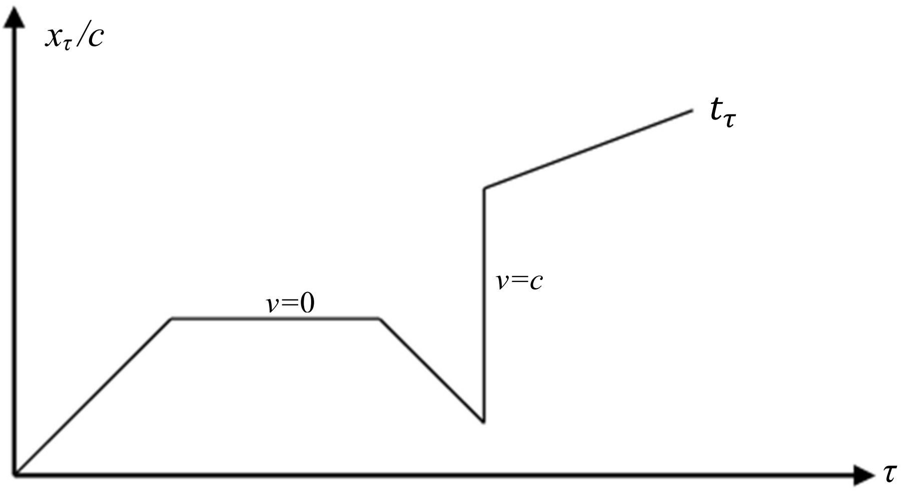

Figure 2. A graph illustrating an object's movement, when the velocity, *v* is piecewise constant. The length of the graph (the 'time line') equals the elapsed time, t_τ on the RF.

In Fig. 2 the length of the graph (the 'time line') of an event chain $(t_\tau, x_\tau)_{\tau\geq 0}$ up to a certain τ-value still equals the time, t_τ observed on the relevant RF. But now this length is not necessarily equal to $|\overrightarrow{t_\tau}|$. As illustrated in Fig. 2: If the v is just piecewise constant, the time, t_τ (=length of the graph) is not entirely given by the *current* $\overrightarrow{t_\tau}$, as it will also depend on the past history. However, the absolute value, $|\overrightarrow{t_\tau}|$, now represents the *minimum* attainable value of t_τ, corresponding to having a constant velocity all the way, (as in Fig. 1).

3.3 Three space coordinates

In the approach described so far, we have applied a single space coordinate, x_τ, only. However, we could easily introduce three space coordinates (x_τ, y_τ, z_τ) with corresponding velocity coordinates, (v_x, v_y, v_z). Then, having the 'external' time, t_τ at the position (x_τ, y_τ, z_τ) gives

$$(x_\tau,\ y_\tau,\ z_\tau) = t_\tau\,(v_x,\ v_y,\ v_z).$$

At this time t_τ the spatial distance from the origin equals

$$l_\tau = \sqrt{x_\tau^2 + y_\tau^2 + z_\tau^2} = t_\tau\sqrt{v_x^2 + v_y^2 + v_z^2}$$

Here the absolute value of the velocity vector, $\vec{v} = (v_x,\ v_y,\ v_z)'$, equals $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$. Now

$$\frac{l_\tau}{c} = \frac{v}{c}t_\tau$$

gives an analogy to *eq.* (2) of Section 3. Thus, we use the results of Section 3, just replacing x_τ by l_τ. The time line is still defined by φ_v, now having

$$\sin\varphi_v = v/c = \quad l_\tau/ct_\tau\ .$$

At any time, t_τ on the RF, we may decompose l_τ into its components ($x_\tau,\ y_\tau,\ z_\tau$), according to the direction of the velocity-vector.

4 Example: the travelling twin

The travelling twin paradox is frequently discussed, *e.g.* see Schuler and Robert (2014), and is very well suited to illustrate the graphical approach of the previous Section.

We first illustrate our graphical approach from the perspective of the earthbound twin, *i.e.* choosing his RF, denoted K_1, for describing his twin brother's travel from the earth to the 'star' and back again. Next, we apply the perspective of the RF, K_2, of the travelling twin on his travel *from* the earth. However, the perspective of the third relevant RF - the one following the travelling twin on his *return* travel - will not be elaborated here.

For both the chosen perspectives (RFs), we follow *two* chains of events; one for each twin brother.

We apply the numerical example of *e.g.* Mermin (2005). Thus, the velocity of the travelling twin's rocket with respect to K_1 equals $v = 0.6c$, giving $\sqrt{1-(v/c)^2} = 0.8$. The distance from the earth to the 'star' is 3 light years, *i.e.* this distance equals x_τ=3c. By the arrival to the star the clock on K_1 (which we must assume is) located at the 'star', will read $x_\tau/v = 3c/0.6c = 5$ years. At this instant the clock of the travelling twin here reads $\tau = t_\tau\sqrt{1-(v/c)^2} = 5 \cdot 0.8 = 4$ years. To summarize; at the arrival of the rocket to the star we have (relative to K_1):

$$\tau = 4, \qquad x_4/\mathrm{c} = 3, \qquad t_4 = 5.$$

4.1 The perspective of the earthbound twin (K_1)

Fig. 3 presents two graphs, illustrating the chain of events for the travelling and earthbound twin, respectively. Both graphs see the events from the perspective of K_1.

The travelling twin's travel to the star is described by the blue line at an angle φ_v:

$$\sin\varphi_v = v/c = 0.6.$$

At the turning point we have, as pointed out above, $\tau = 4,\ x_4/\mathrm{c} = 3,\ t_4 = 5$. But at this turning point, the velocity abruptly changes to $-v$. As we pointed out in Section 3.2, this actually implies that within the TSR we must introduce a second rocket bringing a clock back to the earth; and by passing the star at the right moment, this returning clock is set to $\tau = 4$. (So strictly speaking it is not the travelling twin or *his* clock that is returning).

The blue graph illustrates the well-known results that by the return to the earth (where again $x_\tau/c = 0$), the returning clock shows $\tau = 8$ years. Then it has elapsed a time $t_8 = 5 + 5 = 10$ years on K_1, which equals the total length of the blue graph.

The red line illustrates the very simple 'chain of events' related to the earthbound twin. He is permanently stationed at the origin of K_1. So, for this event chain we have

$$\sin \varphi_v = 0.$$

Thus, $x_\tau/c \equiv 0$ and $t_\tau \equiv \tau$, (all τ). Based on the distance to the star and the travelling twin's velocities, we get that by the 'return of the twin', it has elapsed a time t_τ = 5+5 = 10 years on K_1. (Actually, we just found in the blue graph, that by the 'twin's return' the relevant clock on K_1 showed 10 years.) Anyway, this second red 'event chain' ends with $\tau = 10$ and $t_{10} = 10$ years.

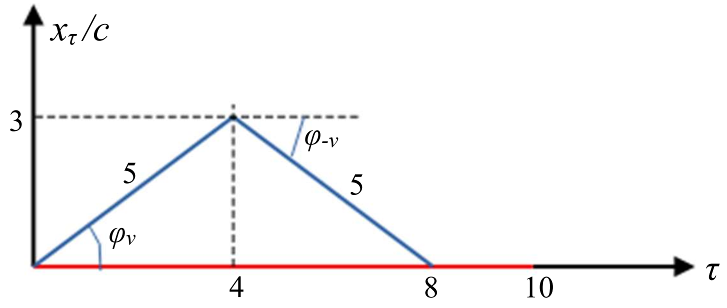

Figure 3. Graphs of two event chains in the perspective of the earthbound twin (K_1). The blue line describes the travelling twin's journey. The red line 'follows' the earthbound twin, having x_τ /c ≡ 0.

4.2 The perspective of the travelling twin (K_2)

As a useful exercise we now consider the same two chain of events in the perspective of the RF used by the travelling twin on his travel *away from the earth*. This is denoted K_2, and we will apply this for the entire journey, see Fig. 4**.**

Again, the red line refers to the event chain for the earthbound twin. This twin is moving along the negative x_τ/c-axis of K_2 at velocity -*v,* Thus, this graph has an angle φ_{-v} relative to the τ-axis:

$$\sin \varphi_{-v} = -v/c = -0.6.$$

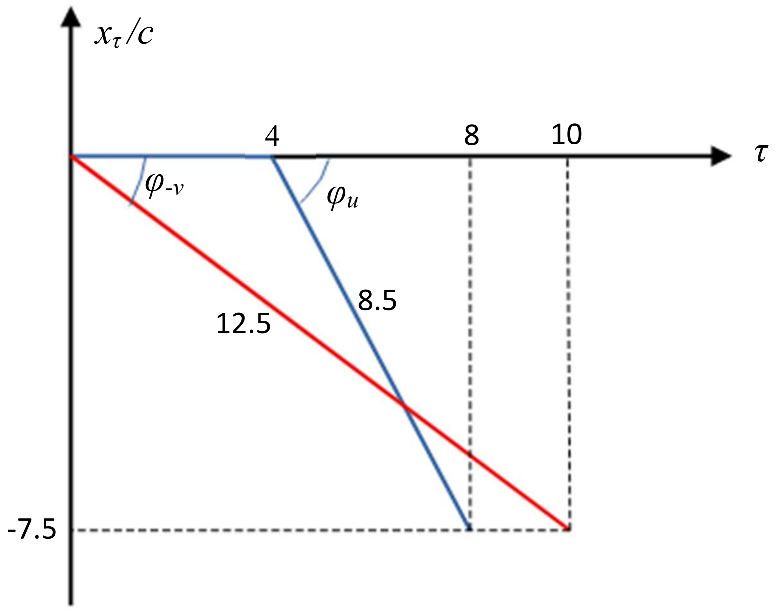

Figure 4. The graphs of the two event chains on K_2, (the RF of the travelling twin on his travel *away* from earth). The red line describes the earthbound twin's movement in K_2. The blue line describes the movement of the travelling twin in the same RF. At $\tau = 4$ years he starts his return travel.

This line is valid for the entire event chain. So, at the instant when his internal clock reads $\tau = 10$ years, it follows that in K_2 the parameters of this event equals x_{10}/c = -7.5 years and t_{10} = 12.5 years. Thus, the red line gives a full description for the 'journey' of the earthbound twin as described in K_2.

To follow the travelling twin, we look at the blue line in Fig. 4. For the first four years he remains at the origin of K_2, (as he is stationary with respect to K_2).

When $\tau = 4$ (and $t_4 = 4$ in K_2) there will – as discussed above - be an abrupt change in his velocity also with respect to K_2. The blue line for $\tau \geq 4$, illustrates the *returning* clock. This has a speed v relative to K_1. Further, the speed between K_1 and K_2 also equals v. Thus, applying the rule for 'adding' velocities in TSR, the velocity we observe in K_2 of the returning clock becomes:

$$u = -\frac{2v}{1+(v/c)^2}$$

Inserting v/c = 0.6 here, we get $u/c = -15/17$. So the blue graph for the return travel ($\tau \geq 4$) has an angle, φ_u given by

$$\sin\varphi_u = u/c = -15/17$$

and so also $\cos\varphi_u = \sqrt{1-(u/c)^2} = 8/17$, and $\tan\varphi_u = -15/8$.

Above we found that the chains of events on K_2 ends when x_τ/c = -7.5 years, (and the two twins are 'reunited'). Fig. 4 demonstrates that when x_τ/c = -7.5 years the 'internal time' of the returning clock reads $\tau = 8$ years. And also for the blue graph, the clock reading on K_2 equals $t_8 = 4 + 8.5 = 12.5$ years when the twins are 'reunited'.

We finally summarize main observations obtained from Figs. 3 and 4. They illustrate *two* chains of events - the red and the blue trajectory - which start and end up at the same location in space. Further, we apply two different perspectives (RFs), K_1 and K_2. Now:

- The observed 'internal time', τ, by the return to the earth, will for both event chains be independent of the perspective:
  - The clock of the travelling twin reads $\tau = 8$ years in *both* RFs, (*cf.* blue graphs)
  - The clock of the earthbound twin reads $\tau = 10$ years in *both* RFs, (*cf.* red graphs).
- For the parameters x_τ/c and t_τ, however, it is 'opposite'. Since the two event chains end at the same position, their final x_τ/c and t_τ values of a specific RF will be identical:
  - For K_1 we get $x_\tau/c = 0$, and total external time, t_τ, = 10 years for both event chains, (Fig. 3).
  - For K_2 we get x_τ/c = -7.5 years, and total external time, t_τ, = 12.5 years for both event chains, (Fig. 4).

5 Summary and Conclusions

We present a graphical approach to describe and investigate chains of events of a moving object, as observed from a specific reference frame (RF). The object's velocity is constant. However, we also treat the case of having an event chain of *piecewise* constant velocity; the travelling twin is the most typical example of this. However, it should be stressed that this is actually outside the framework of the TSR; so this essentially refers to a thought experiment.

The graph ('time line') illustrates the three essential time parameters for describing an event. First, the internal time, τ of the moving object, *i.e.* the reading of its 'embedded' clock. Next, the clock reading ('external time'), t_τ as read on the chosen RF, (at the relevant location). Finally, the position, x_τ - divided by the speed of light, c - which represents the time required for the light to traverse this distance.

We note that the approach is related to work of Minkowski, *cf.* his 'proper time' and 'world line'.

For a constant velocity, v, the graph illustrates how the 'external time', t_τ of an RF is decomposed into its two orthogonal components, τ and x_τ/c. In the piecewise linear case, however, the time t_τ is not entirely given by these components, as it will also depend on the history.

We stress the essential distinction between - on one side - the internal time, τ, which value is independent of the chosen RF, and - on the other side - the two parameters, t_τ and x_τ, which depend on the chosen RF, and - on the other side. We actually derive t_τ and x_τ from τ and the velocity, *v* relative to the RF where we perform observations. Since it is the phenomenon as such, that should of main interest - irrespective of the RF - I would rather see τ as the primary, more fundamental, of the three time parameters. This view might affect the way we talk about/interpret time within the STR, and could be relevant *e.g.* when we discuss the case of the μ-mesons, *cf.* discussion of (Hokstad, 2019).

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