An Event Graph in Special Relativity

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Abstract. We present a graph to illustrate a chain of events within the theory of special relativity. This spacetime diagram provides both the so-called *proper time* and *calendar time,* in addition to one space parameter. All parameters of the graph are given in time units, and we also relate the graph to a time vector in two dimensions. The graph depicts objects (events) having a piecewise constant velocity. By change in the velocity it is required to 'restart' the process by providing new initial conditions. The approach is exemplified *e.g.* by versions of the 'traveling twin paradox'.

Keywords: Spacetime diagram, event graph, time vector, internal clock, traveling twin, μ-mesons.

1 Introduction

Previously we have introduced a time vector in the theory of special relativity (TSR); *cf.* Hokstad (2018). Here we build on this work and present a graph to describe the movements of an object with piecewise constant velocity. In its simplest form, this was also suggested by Epstein (1987).

This 'event graph' is a spacetime diagram, analogous *e.g.* to the 'worldline' of Minkowski; *cf.* Petkov (2012). However, we use the so-called 'proper time' and the position as the two main parameters of the diagram; also providing the 'calendar time'. Starting out from the time vector, we give all these parameters in time units.

We apply the approach to versions of the 'traveling twin' example. Also the case of the μ-mesons is used as an illustration,

The current version, (v4), is an update to agree with the presentation given in Hokstad (2023).

2 Basic notation and the time vector

We start out from the following basic situation: There is a Reference Frame (RF), denoted *K*, having one space coordinate, *x*. This has synchronized, stationary clocks located at virtually any position. Further, there is an object moving at velocity, *w* relative to *K,* along the *x*-axis. The process is initiated by an object starting out (say from the origin of *K*), when the clock at this position reads $t = 0$.

Also (we imagine that) this moving object brings with it a clock, and when the object passes the origin at time 0, this clock is synchronized with the clock on *K* at this position.

We introduce three fundamental parameters related to the movement of this object. First:

τ = Clock reading of the clock following the moving object. We would say that this is the 'internal time' of the object/event, but usually, this is referred to as the 'proper time'.

Further, we have two parameters (t, x), which specify an event on the chosen RF, *K*:

x = Position of the moving object relative to *K,* (when the passing clock reads τ);

t = Clock reading of the clock permanently located at position x on *K*, when the moving clock reads τ; (this t is usually referred to as 'calendar time').

It is well-known that the proper time, τ is independent of which RF we choose as the basis for our observations. In contrast, the (t, x) parameters, will of course depend on which RF we have chosen for describing the observations. Further,

$w = x/\,t$, *i.e.* the velocity of the object relative to the RF,

c = speed of light,

Then - according to the so-called *time dilation* in TSR, we have (*e.g.* Mermin (2005)) that:

$$\tau = \sqrt{t^2 - (x/c)^2} = t\sqrt{1 - (w/c)^2} \tag{1}$$

We proceed to introduce a time vector related to an event (*t, x*); (Hokstad, 2018):

$$\vec{t} = \binom{\tau}{x/c} = \binom{w/c}{\sqrt{1-(w/c)^2}} t \tag{2}$$

The absolute value of this vector equals

$$\left|\vec{t}\right| = t = \sqrt{\tau^2 + (x/c)^2} \tag{3}$$

We note that all three parameters τ, t and x/c represent time. In particular, x/c equals the time required for light to traverse the distance *x* from the origin to the location of the event. We can also see *t* as the event's total 'distance' (in time) from the 'initiating event'., where *eq.* (3) represents a decomposition of this *t* into its two components, τ and *x/c.*

This time vector, (2), actually comprises the information of all the above fundamental parameters of an event. Further, we may also define an angle, $\varphi_w \in [-\pi/2,\ \pi/2]$ by

$$\sin\varphi_w = \frac{w}{c} = \frac{x/c}{t} \tag{4}$$

3 A simple event graph

We now introduce a graphical approach to describe the series of events, defined by this moving object.

3.1 Specification

Fig. 1 provides a graphical illustration of the four parameters: τ*, x/c,* t and *w,* related to a specific event. The absolute value (length) of the time vector, $\vec{t}$, provides t. The slope of the line provides φ_w and, thus, also the velocity, *w*. Now the two components (abscissa and ordinate) of the time vector are given as see *eqs*. (1) - (4):

$$\tau = t\sqrt{1 - (w/c)^2} = t\cos\varphi_w$$

$$\frac{x}{c} = \frac{wt}{c} = t\sin\varphi_w$$

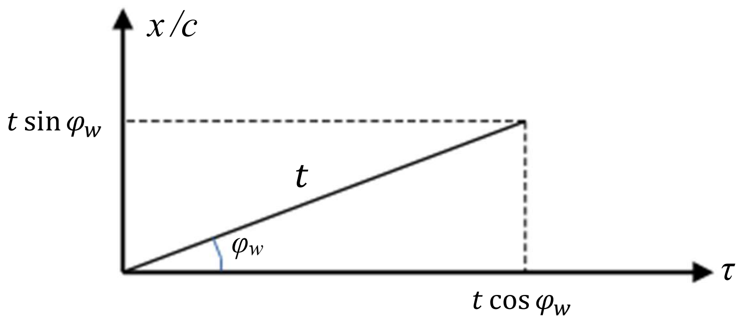


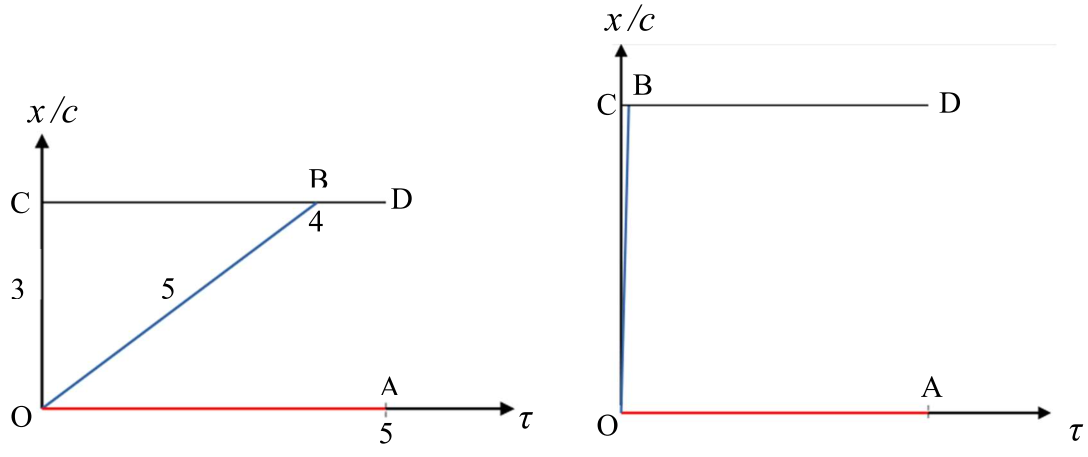
Figure 1. An 'event graph': A chain of events, $(t,x)_{\tau\geq 0}$ on a specific RF; describing an object of constant velocity, *w* relative to this RF.

Thus, by specifying the object's velocity, *w* (relative to our RF) and its proper time*,* τ, the graph will directly give the corresponding change of the event parameters, (t,x) relative to the chosen RF.

Now, in addition to illustrate a specific time vector, this line also illustrates the whole chain of events, representing the object's movement from time 0. We denote this an '*event graph*'.

3.2 Examples

We illustrate the above graph with a couple of standard examples.



a) Travelling twin (three event graphs) b) My-mesons (three event graphs)

Figure 2. Simple examples of the event graph.

Example 1. *The traveling twin.*

We apply a standard numerical example, as given for instance in Mermin (2005). Here the travelling twin leaves the earth in a rocket of velocity, $w = 0.6c$. This gives $\sqrt{1-(w/c)^2} = 0.8$. He travels to a 'star' located 3 light years from the earth. This means that $x/c = 3$ years. Further, the RF of the earth has a synchronized clock located at the star, and by the arrival of the twin to the star, this clock will now read,

$$t = x/w = 3c/0.6c = 5 \text{ years.}$$

At this instant the travelling twin's clock will - according to the TSR, (see equation (1)) – read,

$$\tau = t\sqrt{1-(w/c)^2} \;=\; 5\cdot 0.8 = 4 \text{ years.}$$

Thus, by the arrival of the rocket to the star we have $\tau = 4$ years, $x/c = 3$ years, $t = 5$ years. Or, expressed by the time vector related to this even, (see the blue line OB of Fig. 2. A)):

$$\overrightarrow{t_{OB}} = \begin{pmatrix} \tau \\ x/c \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \text{ with } \left|\vec{t}\right| = t = 5.$$

Here O represents the twins' start of the travel, and *B* the event of his arrival to the star. Thus, the line OB also represents the twins' travel *to* the star.

In addition, we have in this figure inserted two other event graphs. First, the red line OA, illustrating the clock of the brother who remains on the earth, (thus, having $x/c \equiv 0$); and finally, the line CD, illustrating the (imagined) clock permanently situated on the star.

So, in this illustration, it is essential that all the three event graphs are related to the same RF, (that of the earth). In this perspective, all three event chains start 'simultaneously' with $t = 0$, (at events O and C, respectively).

So, in the perspective of this RF, it is fair to say that the figure gives a snapshot of three time vectors, $\overrightarrow{t_{OA}}$, $\overrightarrow{t_{OB}}$ and $\overrightarrow{t_{CD}}$ representing three 'simultaneous' (chains of) events.

Example 2. *The My-mesons.*
For instance, Mermin (2005) refers to this standard example: The μ-mesons are produced by cosmic rays in the upper atmosphere. When 'at rest' they have a lifetime of about 2 microseconds. So if their internal clocks ran at a rate independent of their speed, about half of them would be gone after they had traveled 2.000 feet. Yet about half of the μ-mesons produced in the upper atmosphere (about 100.000 feet up) manage to make it all the way down to the ground. This is because they travel at speed so close to the speed of light that they can survive for 50 times as long as when they are stationary.

Describing this phenomenon in the perspective of (the RF of) the earth, the situation is quite similar to that of *Example* 1; and is illustrated in Fig. 2 b). The creation of the μ-mesons in the upper atmosphere is event O, ($x/c = \tau = 0$). The blue line OB represents the movements of the my-mesons, and by the arrival to the earth (*B*), the τ value of the my-mesons equals (*cf.* CB) $\tau = 2 \cdot 10^{-6}$ sec.

Further, the line *CD* is the event graph of the clock located on the earth, having the duration $t \approx 2 \cdot 10^{-6} \cdot 50 = 10^{-4}$ sec. This equals the distance *OB*. Further, the distance traveled by the μ-mesons, (OC), equals $x/c \approx 10^{-4}$ sec.

This case is interesting, as there are different ways to describe/interpret these results. Mermin (2005) says: "The atomic particles can go much further because their internal clocks that govern when they decay are running much more slowly in the frame in which they rush along at speed close to *c*. This is a real effect. .... ". The 'internal clock' referred here is obviously the imagined clock, following the mesons, and reading the 'proper time', τ. And no doubt, the lifetime given by this τ-value is much less than the life time, *t* , observed by the clocks of the earth's RF.

However, this τ-value is truly an inner clock, not affected by the velocity, (the chosen RF). So rather than describing this as a true physical phenomenon, I suggest it should be seen as an 'observational phenomenon' (caused by the choice of 'observational RF); also see Serret (2018).

Example 3. *Equivalent events. The Lorentz Transformation.*
Now consider a specific event, and illustrate how this is specified by different RFs, moving relative to each other. Thus, we illustrate the Lorentz Transformation, involving an arbitrary number of RFs.

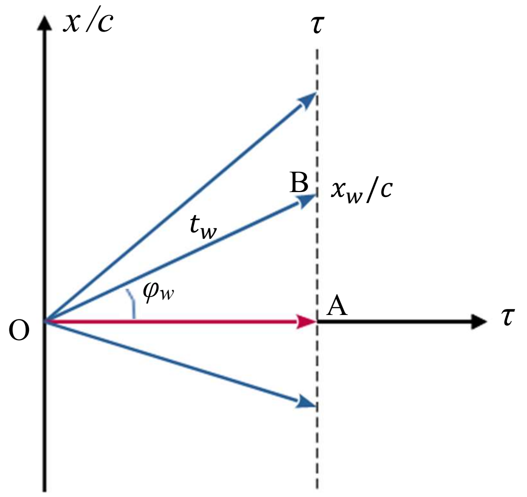


Figure 3. Diagram illustrating the Lorentz transformation (for any number of RFs).

We start by specifying an event having an 'internal clock' reading τ. For simplicity, we let this clock be located at the origin of an RF, *K*, having velocity 0. Further we have, for any *w,* an RF, K_w, which has velocity *w* relative to *K*. Furter, let (t_w, x_w) be the time and space coordinates on K_w for the specified event. Thus $x_w = w\, t_w$, and according to standard theory, (see *eq*. (1)):

$$\tau = \sqrt{t_w^2 - (x_w/c)^2} = t_w\sqrt{1-(w/c)^2}$$

This relation is valid for all *w*, and is illustrated in Fig. 3. Here OA represents the time vector of the 'internal clock'; (located at the origin of *K,* having velocity 0 and $t = \tau$). Further, the blue vectors represent the time vectors of the same event, relative to some arbitrary RFs. So, OB exemplifies the time vector for this event, as specified on an RF moving with velocity *w* relative to *K*.

So given the τ of the event – we just determine φ_w from (*eq.* (4)), that is $\sin\varphi_w = w/c$, and then directly observe x_w and t_w from the figure. So, this is similar to Fig. 1, In Fig 3, however, *w* is the velocity of the RF relative to the 'internal clock' of the event, and corresponds to -*w* in Fig. 1.

Note that if we have two events (t_{w_i}, x_{w_i}), *i* = 1, 2, corresponding to this τ, it follows that

$$\tau = t_{w_i}\sqrt{1-(w_i\,/c)^2} = t\cos\varphi_w\,,\ i = 1, 2$$

$$\frac{x_{w_i}}{c} = \frac{w_i t_{w_i}}{c} = t\sin\varphi_w,\ i = 1, 2$$

We note that this gives the following relation between (t_{w_1}, x_{w_1}) and (t_{w_2}, x_{w_2})

$$t_{w_2} = \frac{\sqrt{1-(w_1\,/c)^2}}{\sqrt{1-(w_2\,/c)^2}}t_{w_1}$$

$$x_{w_2} = \frac{w_2\sqrt{1-(w_1\,/c)^2}}{w_1\sqrt{1-(w_2\,/c)^2}}x_{w_1}$$

So, this represents an alternative formulation of the LT. Now the velocity, *v* between the two RFs are given by a standard formula in TSR, *e.g.* see Mermin (2005):

$$v = \frac{w_1 - w_2}{1 - w_1 w_2/c^2}$$

4 Generalizations

We now look at a couple of generalizations of the above approach.

4.1 An event graph of piecewise constant velocity

The event graph of Ch. 3 is based on having an event of constant velocity, *w*. But we can similarly illustrate a chain of events for an object having a *piecewise* constant *w*, see the example given in Fig. 4. In this figure we start with a constant velocity, $w > 0$, see the line OA. After a certain time, the object is brought to rest; that is, the velocity is changed to $w = \varphi_w = 0$. When such a change occurs we may assume that a clock with the new velocity is passing the previous clock, and there is an update, providing new initial conditions: The new clock is synchronized with the local clock at the relevant position. Whether this is physically feasible is really not the question. We just specify what the relevant clocks would shall *if* they are present.

Back to Fig. 4: At event A the local clock reads a value equal to the length of OA. BC represents the time vector of this local clock and has the same length. The update, following the change of velocity, means that the event graph continues from event C.

Following Fig. 4, the object then remains at rest (*w* =0) for a period, giving the line (CD) parallel to the abscissa, τ. Next, the object approaches the origin; *i.e.* have a negative *w,* (see line DE). (No update is required at event D, as the object has been at the same location in the period CD.) However, 'arriving' at E, a new update is required; imposing a new start at event G; (here FG has the same length/duration as DE).

Thereafter the 'object', (or now rather 'event'), has velocity, $w = c$, ($\varphi_w = \pi/2$), and the line (GH) is orthogonal to the τ-axis. A new update occurs at H; with HI having the same length as GH; *etc*. The total graph is given by the red vectors.

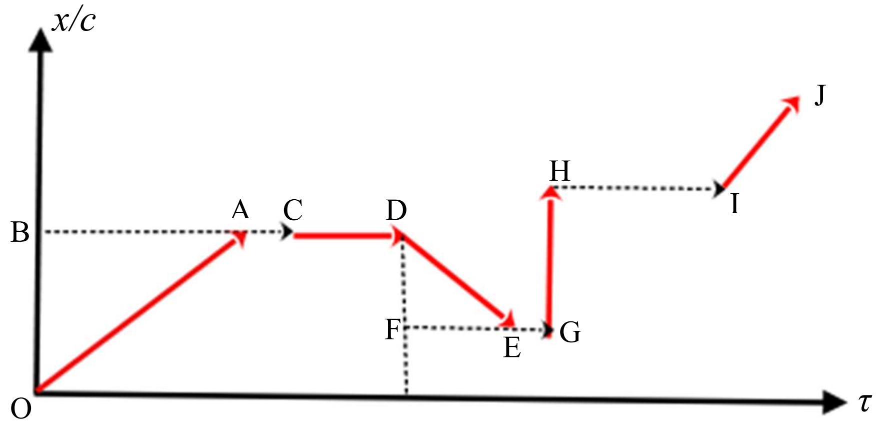


Figure 4. An event graph illustrating an object's movement, when the velocity, *w* is piecewise constant. The total length of the (red) graph equals the elapsed (calendar) time, *t* on the RF.

Also in this generalized diagram, the length of the graph for events $(t,\ x)_{\tau\geq 0}$ up to a certain τ-value will equal the calendar time, t of the end event; in the given example of Fig.4 we have that this time equals

$$t = \left|\overrightarrow{t_{OA}}\right| + \left|\overrightarrow{t_{CD}}\right| + \left|\overrightarrow{t_{DE}}\right| + \left|\overrightarrow{t_{GH}}\right| + \left|\overrightarrow{t_{IJ}}\right|.$$

Observe that the same value is obtained from BC + Cd + FG +HI + …

4.2 Extension to three space coordinates

In the approach, described so far, there is a single space coordinate, x, only. We could, however, introduce three space coordinates $(x, y\ , z)$ with corresponding velocity coordinates, $(w_x,\ w_y,\ w_z)$. For simplicity we just consider the case of having a constant velocity throughout; now with a velocity vector

$$\vec{w} = (w_x,\ w_y,\ w_z)^{\mathrm{T}}$$

So, when we have the calendar time, t at the position $(x, y\ , z)$ on *K*, it follows that

$$(x,\ y,\ z)^{\mathrm{T}} = t \cdot (w_x,\ w_y,\ w_z)^{\mathrm{T}}$$

At this instant the spatial distance from the origin of *K* equals

$$l = \sqrt{x^2\ + y^2\ + z^2}\ = t \cdot \sqrt{w_x^2 + w_y^2 + w_z^2}$$

The absolute value of the velocity vector, $\vec{w} = (w_x,\ w_y,\ w_z)^{\mathrm{T}}$ equals $w = \sqrt{w_x^2 + w_y^2 + w_z^2}$. Thus, $w=l/t$.

Now we can apply the results of Section 3.1, by just replacing x by l. In analogy with *eq.* (4) the event graph is still defined by φ_w, now having

$$\sin\varphi_w = w/c = \quad l/ct.$$

Finally, we note that this graphical approach is somewhat similar to Minkowski's spacetime diagram with three space coordinates, *x, y, z*, and an imaginary time coordinate, *t.* The coordinates of his space are $(x, y, z, i{\cdot}ct)$. As opposed to this, we have transformed the spatial parameters to time, and we have τ rather than t as the time coordinate. In his paper, Minkowski (1909) also introduced the spacetime distance, given as $\sqrt{c^2t^2 - x^2 - y^2 - z^2}$ in his four-dimensional space, while we in the current approach refer to $t = \sqrt{\tau^2 + (x/c)^2}$ as the total 'distance in time' (from an initiating event).

Example: the travelling twin

We now discuss Example 1 of Section 3.2 in more detail. This travelling twin paradox, goes back to Langevin (1911), also discussed for instance by Debs and Redhead (1996) and Schuler and Robert (2014). We now exemplify the generalised graph of Section 4.1, by considering the entire travel and apply the same numerical values as in Example 1,

5.1 The perspective of the earth-based RF, *K*

Fig. 5 presents two graphs, illustrating the total chain of events, both that of the travelling twin, and also that of the twin remaining on the earth. Both graphs describe the events in the perspective of the earth's RF, *K*. As seen in Example 1 the travelling twin's travel *to* the star is described by the blue line, OB at an angle φ_w, where $\sin\varphi_w = w/c = 0.6$.

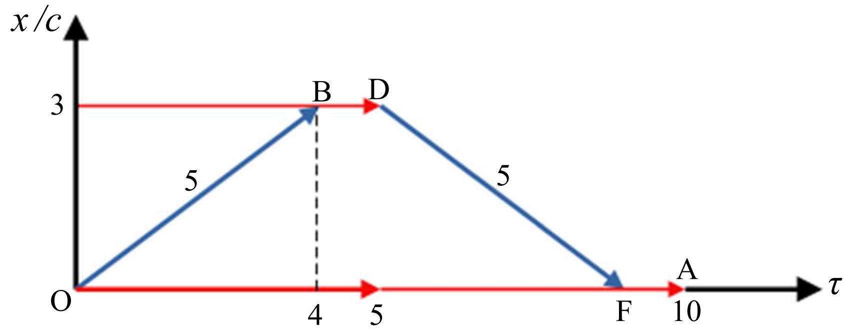


Figure 5. Event graphs of the twins in the perspective of the earth-based RF, *K*. The blue graphs describe the travelling twin's journey. The red line, OE follows the earthbound twin, having $x/c \equiv 0$.

At the event, B we found $\tau = 4,\ x/c = 3,\ t = 5$. Then the velocity abruptly changes to -*w.* An update is required; as the local clock reads $t = 5$; see event D. Now the return travel is given by DF and the blue graphs illustrate the results that by the return to the earth (where again $x/c = 0$), the returning clock has 'aged' a value $\tau = 8$ years; while it has elapsed a time $t = 5 + 5 = 10$ years on *K*, (OB + DF).

The red line, OA illustrates the very simple 'chain of events' related to the twin at the earth. He is permanently stationed at the origin, O of *K*. So, in his case we have $w \equiv 0$, and thus also $\varphi_w \equiv 0$, giving $x/c \equiv 0$ and $t \equiv \tau$, (all τ). As we just found by identifying the blue graph: by the return of the twin to the earth it has elapsed a time $t = 10$ years on *K,* and the red event graph ends with $t = \tau = 10$ years.

5.2 The perspective of the travelling twin K_T

As an exercise we now consider the same two chains of events from the perspective of the RF moving away from the earth at velocity, *w.* The rocket is located at the origin of this RF, denoted K_T, (with time, t_S and position, x_S). When the travelling twin arrives to the star, this K_T continues its journey away from the earth; (as we consider RFs of constant velocity, only).

Fig. 6 provides an illustration. Again, the red line refers to the event chain for the earthbound twin. This twin is moving along the negative x_T/c-axis of K_T at velocity -*w,* Thus, this graph has an angle φ_{-w} relative to the τ-axis, where $\sin\varphi_{-w} = -w/c = -0.6$. This line is valid for the entire event chain. So, at the instant when his internal clock reads $\tau = 10$ years, it follows that the parameters on K_T of this event equals x_T/c = -7.5 years and t_T = 12.5 years. Thus, the red line gives a full description of the 'journey' for the earthbound twin, as described in K_T.

To follow the travelling twin, we look at the blue line in Fig. 6. For the first four years he remains at the origin of K_T. When $\tau = 4$, (and $x_T = 0$, $t_T = 4$) there is an abrupt change in his velocity; (but no update is required, as his 'internal' clock is identical with the local clock of this RF). For $\tau \geq 4$ the blue line illustrates the *returning* twin (or rather clock). This shall have a velocity *w* relative to *K*. Further, the speed between K and K_T also equals w. So now we must apply the rule for adding velocities in TSR, *e.g.* see Mermin (2005). Thus, the returning clock it will have the following velocity relative to K_T:

$$u = \frac{-w-w}{1+(-w/c)^2} = -\frac{2w}{1+(w/c)^2} \tag{5}$$

Now inserting $w/c = 0.6$ here, we get $u/c = -15/17$. So, for the return travel ($\tau \geq 4$) the blue graph has an angle, φ_u given by $\sin\varphi_u = u/c = -15/17$, (and further, $\cos\varphi_u = \sqrt{1-(u/c)^2} = 8/17$. This fully describes the direction of the blue graph for $\tau \geq 4$.

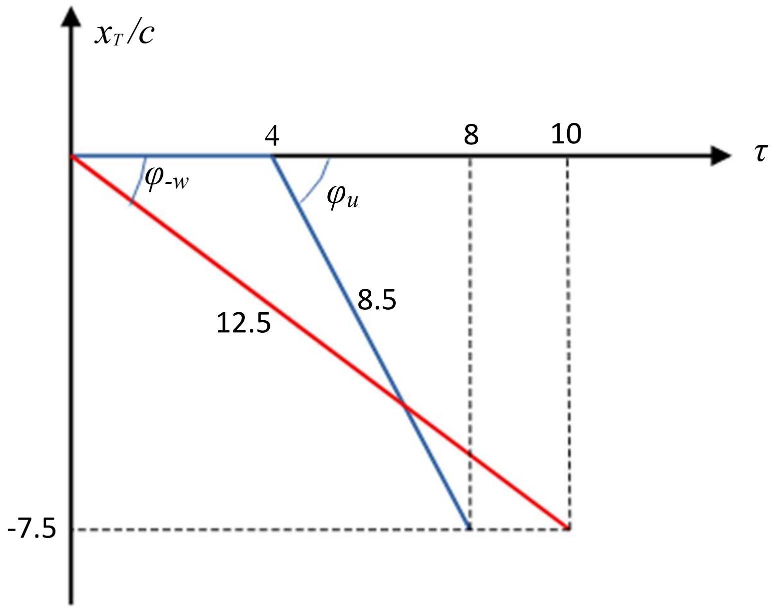


Figure 6. Event graphs of the two twins in the perspective of K_T, which is the RF of the travelling twin on his travel *away* from earth.

Further, we found that the two event chains end when x_T/c = -7.5 years on K_T. This is the location on K_T where the two twins are reunited. So, now we have fully determined also the blue graph. Fig. 6 demonstrates that when x_T/c = -7.5 years the returning clock reads $\tau = 8$ years, while that of the twin on the earth reads $\tau = 10$ years; (which we already know from Section 5.1, as τ does not depend on the RF). We also see that the clock reading on K_T when the blue graph terminates, equals $t_T = 4 + 8.5 = 12.5$ years; which of course is equal to the length of the red graph.

5.3 A symmetric approach

Finally, we consider an alternative journey, giving a symmetry between the two twins. Now we again use K as the RF, *cf.* Fig. 7. As in Section 5.1 the travelling twin (blue graph) travels to the star and arrives there after 4 years (according to his own clock); the clock of K reading 5 years. In the present example he remains there until the brother arrives. The other twin (red graph) first remains on the earth in 5 years. Then he starts out to the star, with the same velocity as the brother; thus, arriving there 4 years later, according to his own clock. But according to the stationary clocks on K it has passed 5 years. So, by the reunion, both twins are located at the position x/c = 3 light years, and the time on K equals t = 10 years. However, according to their own clocks both twins agree that it has passed 4 + 5 = 9 years.

Finally, we note that the first parts of Fig. 7 (of course) is identical to Fig. 5.

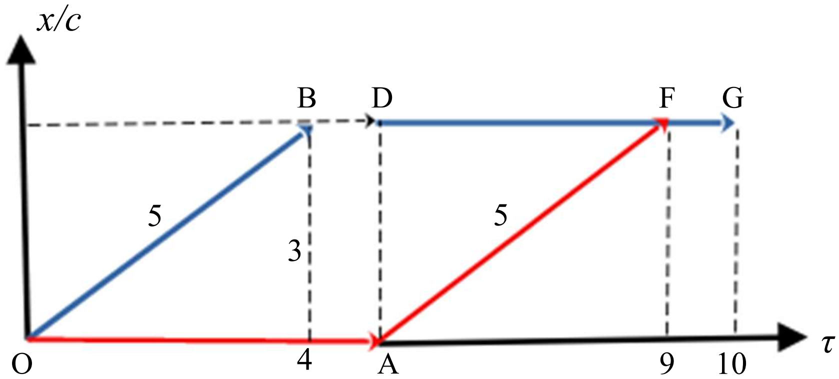


Figure 7 Event graphs for the two twins; meeting at the star.

5 Summary

We present a graphical approach to describe and investigate the movement of an object, as observed within a specific reference frame (RF). The event graph illustrates a total chain of events, provided the object has a piecewise constant velocity. It also illustrates all parameters of an event:

1. The proper time, τ of the moving object, *i.e.* the reading of its imagined 'internal clock'.
2. The calendar time, *t* as read on the chosen RF, (at the relevant location).
3. The position x - divided by the speed of light, *c*; (representing the time required for the light to traverse this distance).

The graph illustrates how the time, *t* is decomposed into its two components, τ and *x/c*. We also observe that the event (t, x) relative to the specified RF is derived from the proper time, τ, together with the object's relative velocity, *w,* (which we observe in the graph by the angle φ_w).. We note that the approach is somewhat related to the work of Minkowski, *cf.* his 'world line'.

As examples, we make extensive use of the travelling twin case; illustrating different 'perspectives', *i.e.* choice of RF, and also different strategies for the reunion of the twins.

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