FREE QUANTUM GROUPS AND RELATED TOPICS

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ABSTRACT. The unitary group U_N has a free analogue U_N^+, and the study of the closed subgroups $G \subset U_N^+$ is a problem of general interest. We review here the general theory of U_N^+ and its subgroups, with all the needed preliminaries included. We discuss as well a number of more advanced topics, selected for their beauty, and potential importance.

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INTRODUCTION

One important discovery, going back to the beginning of the 20th century, is that at the subatomic level the "coordinates" of the various moving objects (particles) do not necessarily commute. In fact, at this level, our ambient space $\mathbb{R}^3$ gets replaced with something not commutative, and infinite dimensional - typically a space of infinite matrices.

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Understanding why is it so, and working out all the details, remains an open problem, belonging of course to physics. However, mathematically speaking, the problem makes sense as well. To be more precise, the challenge is that of developing a theory of "noncommutative geometry", as nice and beautiful as the classical geometry. With a bit of luck, such a theory could be exactly what the physicists are looking for.

The quantum groups belong to this circle of ideas. They are meant to play the role of "symmetry groups" in this hypothetical noncommutative geometry theory.

There is no simple way of introducing the quantum groups. Indeed, these objects are of "quantum" nature, in the sense that, as for the elementary particles, their coordinates do not necessarily commute. This is not much of an issue in the long run, after getting used to the "think quantum" philosophy, but in order to get started, some sort of algebraic geometry formalism is definitely needed. We will use here the operator algebra one.

As a basic, central example of a quantum group, we have the free analogue U_N^+ of the unitary group U_N. This quantum group appears as follows:

$$C(U_N^+)=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^*=u^{-1},u^t=\bar{u}^{-1}\right)$$

To be more precise, on the right we have a certain universal algebra, constructed with generators and relations. Our claim is that if we call this algebra $C(U_N^+)$, then U_N^+ is a compact quantum group, which can be thought of as being a "free analogue" of U_N.

Our first task is that of explaining the definition of the universal algebra on the right. The details here are as follows:

(1) Consider an abstract square matrix $u=(u_{ij})$. Assuming that the entries u_{ij} live in some complex algebra having an involution $*$, we can form the adjoint matrix, according to the formula $u^*=(u_{ji}^*)$. With this convention, our first condition, $u^*=u^{-1}$, is a shorthand for the usual unitarity condition $uu^*=u^*u=1$.

(2) We recall that for the usual matrices $U\in M_N(\mathbb{C})$ the transpose of a unitary matrix is unitary too, and we have $U^*=U^{-1}\implies U^t=\bar{U}^{-1}$. However, this latter implication fails for the abstract matrices $u=(u_{ij})$ that we are interested in, and this is why we have to impose the condition $u^t=\bar{u}^{-1}$ as well.

(3) With these observations in hand, we can consider the universal complex algebra with involution generated by N^2 abstract variables $(u_{ij})_{i,j=1,\ldots,N}$, subject to the $4N^2$ relations coming from the matrix equalities $uu^*=u^*u=u^t\bar{u}=\bar{u}u^t=1$, which make our unitarity conditions $u^*=u^{-1},u^t=\bar{u}^{-1}$ hold.

(4) Finally, we would like to have a norm on our algebra, having the C^*-algebra property $||aa^*||=||a||^2$. In order to do so, we can simply consider the abstract biggest C^*-norm on our algebra, and complete with respect to this norm. We obtain in this way the universal C^*-algebra that we are interested in.

Summarizing, we have constructed so far a certain universal C^*-algebra, say A. Now observe that we have an obvious arrow $A \to C(U_N)$, mapping the abstract variables u_{ij} to the coordinates of the unitary matrices, $U \to U_{ij}$. With a bit more work, one can prove that we have in fact $A_{comm} = C(U_N)$, where the quotient $A \to A_{comm}$ is obtained by assuming that the variables u_{ij}, u_{ij}^* commute. Thus, our algebra A appears as a "liberation" of $C(U_N)$, so it is natural to denote it by $C(U_N^+)$, with the abstract symbol U_N^+ standing for some kind of "quantum space", obtained by enlarging U_N.

In order to further build now on this construction, observe that the algebra $C(U_N^+)$ has a comultiplication, a counit and an antipode map, constructed by using the universal property of $C(U_N^+)$, according to the following formulae:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}^*$$

There is an obvious similarity here with the group operations for the usual unitary matrices, namely $(UV)_{ij} = \sum_k U_{ik} V_{kj}$, $(1_N)_{ij} = \delta_{ij}$, $(U^{-1})_{ij} = U_{ji}^*$. In fact, Δ, ε, S can be thought of as coming from a multiplication map $m : U_N^+ \times U_N^+ \to U_N^+$, a unit map $u : \{.\} \to U_N^+$, and an inverse map $i : U_N^+ \to U_N^+$. Thus our quantum space U_N^+ is a compact quantum group, which appears by definition as a free version of U_N.

All this might seem of course a bit mysterious, but will be explained in great detail, in what follows. We will first review the operator algebra theory, and the space/algebra correspondence $X \leftrightarrow C(X)$ coming from it. Then we will discuss the Hopf algebra formalism, which amounts in replacing the structure maps m, u, i of a group or quantum group G by the corresponding maps Δ, ε, S at the level of the associated function algebra $C(G)$. With these ingredients in hand, we will be able to talk then about compact quantum groups in general, and about the above quantum group U_N^+ in particular.

More generally, we will be interested here in the closed quantum subgroups $G \subset U_N^+$, the main examples of such quantum groups being:

(1) The compact Lie groups, $G \subset U_N$.
(2) The duals $G = \widehat{\Gamma}$ of the finitely generated groups $\Gamma =< g_1, \ldots, g_N >$.
(3) Deformations of the compact Lie groups, with parameter $q = -1$.
(4) Liberations, half-liberations, quantum permutation groups, and more.

Once again, all this remains to be explained. Let us mention, however, that the key examples are the compact Lie groups, which appear first in the above list. It is known, indeed, that any such group G appears as a subgroup of a unitary group, so we have embeddings $G \subset U_N \subset U_N^+$, which make G a quantum group in our sense. Thus, by getting back now to the general case, we can think of the closed quantum subgroups $G \subset U_N^+$ that we are interested in as being the "compact quantum Lie groups".

We will present here the main tools for dealing with such objects, and we will discuss as well a number of more advanced topics. The general idea is that such quantum groups do not have a Lie algebra, or much differential geometric structure, but one can study them via representation theory, with a mix of algebraic and probability techniques.

Regarding the possible applications of all this, the problem is open. The closed subgroups $G \subset U_N^+$ are potentially related to many things, and can normally be of help in connection with a number of questions in quantum physics. This remains to be seen.

The present text is organized in four parts, as follows:

(1) Sections 1-3 are an introduction to the closed subgroups $G \subset U_N^+$, with the main examples $(O_N, O_N^*, O_N^+, U_N, U_N^*, U_N^+)$ explained in detail.
(2) Sections 4-6 contain basic theory, with the main examples, their bistochastic versions (B_N, B_N^+, C_N, C_N^+) and their twists $(\bar{O}_N, \bar{O}_N^*, \bar{U}_N, \bar{U}_N^*)$ worked out.
(3) Sections 7-9 are concerned with quantum permutations (S_N, S_N^+), quantum reflections $(H_N, H_N^*, H_N^+, K_N, K_N^*, K_N^+)$, and other related quantum groups.
(4) Sections 10-12 deal with some further topics, which are in need of more development: toral subgroups, homogeneous spaces, and modelling questions.

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1. Operator algebras

In order to introduce the quantum groups, we will use the space/algebra correspondence coming from operator algebra theory. Here by "operator" we mean bounded linear operator $T : H \to H$ on a Hilbert space, and our starting point will be:

Definition 1.1. *A Hilbert space is a complex vector space H given with a scalar product $< x, y >$, satisfying the following conditions:*

1. *$< x, y >$ is linear in x, and antilinear in y.*
2. *$\overline{< x, y >} =< y, x >$, for any x, y.*
3. *$< x, x >> 0$, for any $x \neq 0$.*
4. *H is complete with respect to the norm $||x|| = \sqrt{< x, x >}$.*

Here the fact that $||.||$ is indeed a norm comes from the Cauchy-Schwarz inequality, $| < x, y > | \leq ||x|| \cdot ||y||$, which can be established by using the fact that the degree 2 polynomial $f(t) = ||x + ty||^2$ being positive, its discriminant must be negative.

As a basic example, we have $H = \mathbb{C}^N$, which scalar product $< x, y >= \sum_i x_i \bar{y}_i$. Another example is $H = l^2(\mathbb{N})$, the space of sequences $x = (x_i)$ satisfying $\sum_i |x_i|^2 < \infty$, with similar scalar product. In fact, given a measured space X, we have as example $H = L^2(X)$, with $< f, g >= \int_X f(x)\overline{g(x)}dx$. Observe that with $X = \{1, \ldots, N\}$ and $X = \mathbb{N}$, with the counting measure, we obtain the spaces $H = \mathbb{C}^N$ and $H = l^2(\mathbb{N})$.

Given a Hilbert space H, any algebraic basis $\{f_i\}_{i\in I}$ can be turned into an orthonormal basis $\{e_i\}_{i\in I}$, by using the Gram-Schmidt procedure. Thus, we have $H \simeq l^2(I)$. When I is countable, H is called separable. As a basic example, $H = L^2[0, 1]$ is separable, because we can use the basis $f_i = x^i$ with $i \in \mathbb{N}$, coming from the Weierstrass theorem.

Let us get now into the study of operators. We first have:

Proposition 1.2. *Let H be a Hilbert space, with orthonormal basis $\{e_i\}_{i\in I}$. The algebra $\mathcal{L}(H)$ of linear operators $T : H \to H$ embeds then into the matrix algebra $M_I(\mathbb{C})$, with T corresponding to the matrix $M_{ij} =< Te_j, e_i >$. In particular:*

1. *In the finite dimensional case, where $\dim(H) = N < \infty$, we obtain in this way a usual matrix algebra, $\mathcal{L}(H) \simeq M_N(\mathbb{C})$.*
2. *In the separable infinite dimensional case, where $I \simeq \mathbb{N}$, we obtain in this way a subalgebra of the infinite matrices, $\mathcal{L}(H) \subset M_\infty(\mathbb{C})$.*

Proof. The correspondence $T \to M$ in the statement is indeed linear, and its kernel is $\{0\}$. As for the last two assertions, these are clear as well. □

The above result is something quite theoretical, because for basic spaces like $L^2[0, 1]$, which do not have a simple orthonormal basis, the embedding $\mathcal{L}(H) \subset M_\infty(\mathbb{C})$ that we obtain is not very useful. Thus, while the operators $T : H \to H$ are basically some infinite matrices, it is better to think of these operators as being objects on their own.

In what follows we will be interested in the operators $T : H \to H$ which are bounded. Regarding such operators, we have the following result:

Theorem 1.3. *Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that $||T|| = \sup_{||x||\leq 1} ||Tx||$ is finite, form a complex algebra with unit, denoted $B(H)$. This algebra has the following properties:*

1. *$B(H)$ is complete with respect to $||.||$, and so we have a Banach algebra.*
2. *$B(H)$ has an involution $T \to T^*$, given by $< Tx, y >=< x, T^*y >$.*

In addition, the norm and the involution are related by the formula $||TT^*|| = ||T||^2$.

Proof. The fact that we have indeed an algebra follows from:

$$||S+T|| \leq ||S|| + ||T|| \quad , \quad ||\lambda T|| = |\lambda| \cdot ||T|| \quad , \quad ||ST|| \leq ||S|| \cdot ||T||$$

Regarding now (1), if $\{T_n\} \subset B(H)$ is Cauchy then $\{T_nx\}$ is Cauchy for any $x \in H$, so we can define the limit $T = \lim_{n\to\infty} T_n$ by setting $Tx = \lim_{n\to\infty} T_nx$.

As for (2), here the existence of T^* comes from the fact that $\varphi(x) =< Tx, y >$ being a linear map $H \to \mathbb{C}$, we must have $\varphi(x) =< x, T^*y >$, for a certain vector $T^*y \in H$. Moreover, since this vector is unique, T^* is unique too, and we have as well:

$$(S+T)^* = S^* + T^* \quad , \quad (\lambda T)^* = \bar{\lambda} T^* \quad , \quad (ST)^* = T^*S^* \quad , \quad (T^*)^* = T$$

Observe also that we have indeed $T^* \in B(H)$, because:

$$||T|| = \sup_{||x||=1} \sup_{||y||=1} < Tx, y >= \sup_{||y||=1} \sup_{||x||=1} < x, T^*y >= ||T^*||$$

Regarding the last assertion, we have $||TT^*|| \leq ||T|| \cdot ||T^*|| = ||T||^2$. Also, we have:

$$||T||^2 = \sup_{||x||=1} | < Tx, Tx > | = \sup_{||x||=1} | < x, T^*Tx > | \leq ||T^*T||$$

By replacing $T \to T^*$ we obtain from this $||T||^2 \leq ||TT^*||$, and we are done. $\square$

In view of Proposition 1.2 above we have an embedding $B(H) \subset M_I(\mathbb{C})$, with the subalgebra $B(H)$ consisting of the $I \times I$ complex matrices satisfying a certain technical boundedness condition. Moreover, in this picture the adjoint operation $T \to T^*$ takes a very simple form, namely $(M^*)_{ij} = \overline{M}_{ji}$ at the level of the associated matrices.

We will be interested here in the algebras of operators, rather than in the operators themselves. The axioms here, coming from Theorem 1.3, are as follows:

Definition 1.4. *A unital C^*-algebra is a complex algebra with unit A, having:*

1. *A norm $a \to ||a||$, making it a Banach algebra (the Cauchy sequences converge).*
2. *An involution $a \to a^*$, which satisfies $||aa^*|| = ||a||^2$, for any $a \in A$.*

We know from Theorem 1.3 that the full operator algebra $B(H)$ is a C^*-algebra, for any Hilbert space H. In particular, any usual matrix algebra $M_N(\mathbb{C})$ is a C^*-algebra. Observe that at $N = 1$ our C^*-algebra is $A = \mathbb{C}$, with norm $z \to |z|$ and involution $z \to \bar{z}$, and with the condition $||aa^*|| = ||a||^2$ corresponding to the formula $|z\bar{z}| = |z|^2$.

More generally, any closed $*$-subalgebra $A \subset B(H)$ is a C^*-algebra. The celebrated Gelfand-Naimark-Segal (GNS) theorem states that any C^*-algebra appears in fact in this way. This is something non-trivial, and we will be back to it later on.

For the moment, we are interested in developing the theory of C^*-algebras, without reference to operators, or Hilbert spaces. Our first task will be that of understanding the structure of the commutative C^*-algebras. As a first observation, we have:

Proposition 1.5. *If X is an abstract compact space, the algebra $C(X)$ of continuous functions $f : X \to \mathbb{C}$ is a C^*-algebra, with structure as follows:*

(1) *The norm is the usual sup norm, $||f|| = \sup_{x \in X} |f(x)|$.*
(2) *The involution is the usual involution, $f^*(x) = \overline{f(x)}$.*

This algebra is commutative, in the sense that $fg = gf$, for any $f, g \in C(X)$.

Proof. Almost everything here is trivial. Observe also that we have indeed:

$$||ff^*|| = \sup_{x \in X} |f(x)\overline{f(x)}| = \sup_{x \in X} |f(x)|^2 = ||f||^2$$

Finally, we have $fg = gf$, since $f(x)g(x) = g(x)f(x)$ for any $x \in X$. $\square$

Our claim now (the Gelfand theorem) is that any commutative C^*-algebra appears in this way. This is a non-trivial result, which requires a number of preliminaries.

We will need some basic spectral theory. Let us begin with:

Definition 1.6. *The spectrum of an element $a \in A$ is the set*

$$\sigma(a) = \left\{ \lambda \in \mathbb{C} \big| a - \lambda \notin A^{-1} \right\}$$

where $A^{-1} \subset A$ is the set of invertible elements.

As a basic example, the spectrum of a usual matrix $M \in M_N(\mathbb{C})$ is the collection of its eigenvalues. Also, the spectrum of a continuous function $f \in C(X)$ is its image. In the case of the trivial algebra $A = \mathbb{C}$, the spectrum of an element is the element itself.

As a first, basic result regarding spectra, we have:

Proposition 1.7. *We have the following formula, valid for any $a, b \in A$:*

$$\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$$

Moreover, there are examples where $\sigma(ab) \neq \sigma(ba)$.

Proof. We first prove that $1 \notin \sigma(ab) \implies 1 \notin \sigma(ba)$. Assume indeed that $1 - ab$ is invertible, with inverse $c = (1 - ab)^{-1}$. We have $abc = cab = c - 1$, and we obtain:

$$\begin{aligned}(1 + bca)(1 - ba) &= 1 + bca - ba - bcaba \\ &= 1 + bca - ba - bca + ba \\ &= 1\end{aligned}$$

A similar computation shows that we have $(1-ba)(1+bca)=1$, and we conclude that $1-ba$ is invertible, with inverse $1+bca$, which proves our claim.

By multiplying by scalars, we deduce from this that we have $\lambda\notin\sigma(ab)\implies\lambda\notin\sigma(ba)$ for any $\lambda\in\mathbb C-\{0\}$, which leads to the conclusion $\sigma(ab)\cup\{0\}=\sigma(ba)\cup\{0\}$.

Regarding now the last claim, let us first recall that for usual matrices $a,b\in M_N(\mathbb C)$ we have $0\in\sigma(ab)\iff 0\in\sigma(ba)$, because ab is invertible if any only if ba is.

However, this latter fact fails for general operators on Hilbert spaces. As a basic example, we can take a,b to be the shift $S(e_i)=e_{i+1}$ on the space $l^2(\mathbb N)$, and its adjoint. Indeed, we have $S^*S=1$, and SS^* being the projection onto $e_0^\perp$, it is not invertible. □

Given an element $a\in A$, and a rational function $f=P/Q$ having poles outside $\sigma(a)$, we can construct the element $f(a)=P(a)Q(a)^{-1}$. For simplicity, we write:

$$f(a)=\frac{P(a)}{Q(a)}$$

With this convention, we have the following result:

Proposition 1.8. *We have the "rational functional calculus" formula*

$$\sigma(f(a))=f(\sigma(a))$$

valid for any rational function $f\in\mathbb C(X)$ having poles outside $\sigma(a)$.

Proof. In order to prove this result, we can proceed in two steps, as follows:

(1) Assume first that we are in the polynomial case, $f\in\mathbb C[X]$. We pick $\lambda\in\mathbb C$, and we write $f(X)-\lambda=c(X-r_1)\ldots(X-r_n)$. We have then, as desired:

$$\begin{aligned}\lambda\notin\sigma(f(a))&\iff f(a)-\lambda\in A^{-1}\\&\iff c(a-r_1)\ldots(a-r_n)\in A^{-1}\\&\iff a-r_1,\ldots,a-r_n\in A^{-1}\\&\iff r_1,\ldots,r_n\notin\sigma(a)\\&\iff \lambda\notin f(\sigma(a))\end{aligned}$$

(2) Assume now that we are in the general case, $f\in\mathbb C(X)$. We pick $\lambda\in\mathbb C$, we write $f=P/Q$, and we set $F=P-\lambda Q$. By using (1), we obtain:

$$\begin{aligned}\lambda\in\sigma(f(a))&\iff F(a)\notin A^{-1}\\&\iff 0\in\sigma(F(a))\\&\iff 0\in F(\sigma(a))\\&\iff \exists\mu\in\sigma(a),F(\mu)=0\\&\iff \lambda\in f(\sigma(a))\end{aligned}$$

Thus, we have obtained the formula in the statement. □

Given an element $a \in A$, its spectral radius $\rho(a)$ is the radius of the smallest disk centered at 0 containing $\sigma(a)$. We have the following key result:

Proposition 1.9. *Let A be a C^*-algebra.*

(1) *The spectrum of a norm one element is in the unit disk.*
(2) *The spectrum of a unitary element ($a^* = a^{-1}$) is on the unit circle.*
(3) *The spectrum of a self-adjoint element ($a = a^*$) consists of real numbers.*
(4) *The spectral radius of a normal element ($aa^* = a^*a$) is equal to its norm.*

Proof. We use the various results established above.

(1) This comes from the following formula, valid when $||a|| < 1$:

$$\frac{1}{1-a} = 1 + a + a^2 + \ldots$$

(2) This follows by using Proposition 1.8, with $f(z) = z^{-1}$. Indeed, we have:

$$\sigma(a)^{-1} = \sigma(a^{-1}) = \sigma(a^*) = \overline{\sigma(a)}$$

Now since $\lambda^{-1} = \bar{\lambda}$ characterizes the elements $\lambda \in \mathbb{T}$, this gives the result.

(3) This follows by using (2), and Proposition 1.8, with $f(z) = (z + it)/(z - it)$, with $t \in \mathbb{R}$. Indeed, for $t >> 0$ the element $f(a)$ is well-defined, and we have:

$$\left(\frac{a+it}{a-it}\right)^* = \frac{a-it}{a+it} = \left(\frac{a+it}{a-it}\right)^{-1}$$

Thus $f(a)$ is a unitary, and by using (2) its spectrum is contained in $\mathbb{T}$. We conclude that we have $f(\sigma(a)) = \sigma(f(a)) \subset \mathbb{T}$, and so $\sigma(a) \subset f^{-1}(\mathbb{T}) = \mathbb{R}$, as desired.

(4) We already know from (1) that we have $\rho(a) \leq ||a||$. For the converse, if we fix a number $\rho > \rho(a)$, we have the following computation:

$$\int_{|z|=\rho} \frac{z^n}{z-a}\,dz = \sum_{k=0}^{\infty} \left(\int_{|z|=\rho} z^{n-k-1} dz\right) a^k = a^{n-1}$$

By applying the norm and taking n-th roots we obtain from this:

$$\rho \geq \lim_{n \to \infty} ||a^n||^{1/n}$$

In the case $a = a^*$ we have $||a^n|| = ||a||^n$ for any exponent of the form $n = 2^k$, and by taking n-th roots we get $\rho \geq ||a||$. This gives the missing inequality $\rho(a) \geq ||a||$.

In the general case $aa^* = a^*a$ we have $a^n(a^n)^* = (aa^*)^n$, and we get $\rho(a)^2 = \rho(aa^*)$. Now since aa^* is self-adjoint, we get $\rho(aa^*) = ||a||^2$, and we are done. $\square$

Summarizing, we have so far a collection of results regarding the spectra of the elements in C^*-algebras, which are similar to the results regarding the eigenvalues of the usual matrices. We are now in position of proving a key result, from [61], namely:

Theorem 1.10 (Gelfand). *Any commutative C^*-algebra is the form $C(X)$, with its "spectrum" $X = Spec(A)$ appearing as the space of characters $\chi : A \to \mathbb{C}$.*

Proof. Given a commutative C^*-algebra A, we can define indeed X to be the set of characters $\chi : A \to \mathbb{C}$, with the topology making continuous all the evaluation maps $ev_a : \chi \to \chi(a)$. Then X is a compact space, and $a \to ev_a$ is a morphism of algebras $ev : A \to C(X)$. We first prove that ev is involutive. We use the following formula:

$$a = \frac{a + a^*}{2} - i \cdot \frac{i(a - a^*)}{2}$$

Thus it is enough to prove the equality $ev_{a^*} = ev_a^*$ for self-adjoint elements a. But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, because $ev_a(\chi) = \chi(a)$ is an element of $\sigma(a)$, contained in $\mathbb{R}$.

Since A is commutative, each element is normal, so ev is isometric:

$$||ev_a|| = \rho(a) = ||a||$$

It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

As a first consequence of the Gelfand theorem, we can extend Proposition 1.8 above in the case of the normal elements ($aa^* = a^*a$), in the following way:

Proposition 1.11. *Assume that $a \in A$ is normal, and let $f \in C(\sigma(a))$.*

1. *We can define $f(a) \in A$, with $f \to f(a)$ being a morphism of C^*-algebras.*
2. *We have the "continuous functional calculus" formula $\sigma(f(a)) = f(\sigma(a))$.*

Proof. Since our element a is normal, the C^*-algebra $B =< a >$ that is generates is commutative, and the Gelfand theorem gives an identification $B = C(X)$, with $X = Spec(B)$. The map $X \to \sigma(a)$ given by evaluation at a being bijective, we have an identification $X = \sigma(a)$. Thus we have $B = C(\sigma(a))$, and this gives all the assertions. □

We can as well develop the theory of positive elements, as follows:

Proposition 1.12. *For an element $a \in A$, the following are equivalent:*

1. *a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.*
2. *$a = b^2$, for some $b \in A$ satisfying $b = b^*$.*
3. *$a = cc^*$, for some $c \in A$.*

Proof. Regarding (1) $\implies$ (2), observe that $\sigma(a) \subset \mathbb{R}$ implies $a = a^*$. Thus the algebra $< a >$ is commutative, and by using the Gelfand theorem, we can set $b = \sqrt{a}$.

The implication (2) $\implies$ (3) is trivial, because we can set $c = b$. Observe that (2) $\implies$ (1) is clear too, because we have $\sigma(a) = \sigma(b^2) = \sigma(b)^2 \subset \mathbb{R}^2 = [0, \infty)$.

For (3) $\implies$ (1), we proceed by contradiction. By multiplying c by a suitable element of $< cc^* >$, we are led to the existence of an element $d \neq 0$ satisfying $-dd^* \geq 0$.

By writing now $d=x+iy$ with $x=x^*,y=y^*$ we have:

$$dd^*+d^*d=2(x^2+y^2)\geq 0$$

Thus $d^*d\geq 0$. But this contradicts the elementary fact that $\sigma(dd^*),\sigma(d^*d)$ must coincide outside $\{0\}$, coming from Proposition 1.7 above. $\square$

The Gelfand theorem has as well some important philosophical consequences. Indeed, in view of this theorem, we can formulate the following definition:

Definition 1.13. *Given an arbitrary C^*-algebra A, we write $A=C(X)$, and call X a noncommutative compact space. Equivalently, the category of the noncommutative compact spaces is the category of the C^*-algebras, with the arrows reversed.*

When A is commutative, the space X considered above exists indeed, as a Gelfand spectrum, $X=Spec(A)$. In general, X is something rather abstract, and our philosophy here will be that of studying of course A, but formulating our results in terms of X. For instance whenever we have a morphism $\Phi:A\to B$, we will write $A=C(X),B=C(Y)$, and rather speak of the corresponding morphism $\phi:Y\to X$. And so on.

We will see later on, after developing some more theory, that this formalism has its limitations, and needs a fix. For the moment, however, let us explore the possibilities that it opens up. As basic examples of such noncommutative spaces, we have:

Definition 1.14. *We have noncommutative spaces, constructed as follows,*

$$C(S^{N-1}_{\mathbb{R},+})=C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\sum_ix_i^2=1\right)$$

$$C(S^{N-1}_{\mathbb{C},+})=C^*\left(x_1,\ldots,x_N\Big|\sum_ix_ix_i^*=\sum_ix_i^*x_i=1\right)$$

called respectively the free real sphere, and the free complex sphere.

Here the C^* symbols on the right stand for "universal C^*-algebra generated by". The fact that such universal C^*-algebras exist indeed follows by considering the corresponding universal $*$-algebras, and then completing with respect to the biggest C^*-norm. Observe that this biggest C^*-norm exists indeed, because the quadratic conditions give:

$$||x_i||^2=||x_ix_i^*||\leq||\sum_ix_ix_i^*||=1$$

Given a noncommutative space X, its classical version is the space X_{class} obtained by dividing $C(X)$ by its commutator ideal, and then applying the Gelfand theorem:

$$C(X_{class})=C(X)/I\quad,\quad I=<[a,b]>$$

Observe that we have an embedding of noncommutative spaces $X_{class}\subset X$. In this situation, we also say that X appears as a "liberation" of X.

As a first result regarding the above free spheres, we have:

Proposition 1.15. *We have embeddings of noncommutative spaces, as follows,*

$$\begin{tikzcd} S^{N-1}_{\mathbb{C}} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\ S^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{R},+} \arrow[u] \end{tikzcd}$$

and the spaces on the right appear as liberations of the spaces of the left.

Proof. The first assertion is clear. For the second one, we must establish the following isomorphisms, where C^*_{comm} stands for "universal commutative C^*-algebra":

$$C(S^{N-1}_{\mathbb{R}}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i = x_i^*, \sum_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C}}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^* = \sum_i x_i^*x_i = 1\right)$$

But these isomorphisms are both clear, by using the Gelfand theorem. □

We can enlarge our class of basic manifolds by introducing tori, as follows:

Proposition 1.16. *Given a closed subspace $S \subset S^{N-1}_{\mathbb{C},+}$, the subspace $T \subset S$ given by*

$$C(T) = C(S)\Big/\left\langle x_ix_i^* = x_i^*x_i = \frac{1}{N}\right\rangle$$

is called associated torus. In the real case, $S \subset S^{N-1}_{\mathbb{R},+}$, we also call T cube.

As a basic example here, for $S = S^{N-1}_{\mathbb{C}}$ the corresponding submanifold $T \subset S$ appears by imposing the relations $|x_i| = \frac{1}{\sqrt{N}}$ to the coordinates, so we obtain a torus:

$$S = S^{N-1}_{\mathbb{C}} \implies T = \left\{x \in \mathbb{C}^N\Big||x_i| = \frac{1}{\sqrt{N}}\right\}$$

As for the case of the real sphere, $S = S^{N-1}_{\mathbb{R}}$, here the submanifold $T \subset S$ appears by imposing the relations $x_i = \pm\frac{1}{\sqrt{N}}$ to the coordinates, so we obtain a cube:

$$S = S^{N-1}_{\mathbb{R}} \implies T = \left\{x \in \mathbb{R}^N\Big|x_i = \pm\frac{1}{\sqrt{N}}\right\}$$

Observe that we have a relation here with group theory, because the complex torus computed above is the group $\mathbb{T}^N$, and the cube is the finite group $\mathbb{Z}_2^N$.

In general now, in order to compute T, we can use the following simple fact:

Proposition 1.17. *When $S\subset S^{N-1}_{\mathbb C,+}$ is an algebraic manifold, in the sense that*

$$C(S)=C(S^{N-1}_{\mathbb C,+})\Big/\Big\langle f_i(x_1,\ldots,x_N)=0\Big\rangle$$

for certain noncommutative polynomials $f_i\in\mathbb C<x_1,\ldots,x_N>$, we have

$$C(T)=C^*\left(u_1,\ldots,u_N\Big|u_i^*=u_i^{-1},g_i(u_1,\ldots,u_N)=0\right)$$

with the poynomials g_i being given by $g_i(u_1,\ldots,u_N)=f_i(\sqrt{N}u_1,\ldots,\sqrt{N}u_N)$.

Proof. According to our definition of the torus $T\subset S$, the following variables must be unitaries, in the quotient algebra $C(S)\to C(T)$:

$$u_i=\frac{x_i}{\sqrt{N}}$$

Now if we assume that these elements are unitaries, the quadratic conditions $\sum_ix_ix_i^*=\sum_ix_i^*x_i=1$ are automatic. Thus, we obtain the space in the statement. □

Summarizing, we are led to the question of computing certain algebras generated by unitaries. In order to deal with this latter problem, let us start with:

Proposition 1.18. *Let Γ be a discrete group, and consider the complex group algebra $\mathbb C[\Gamma]$, with involution given by the fact that all group elements are unitaries:*

$$g^*=g^{-1}\quad,\quad\forall g\in\Gamma$$

The maximal C^*-seminorm on $\mathbb C[\Gamma]$ is then a C^*-norm, and the closure of $\mathbb C[\Gamma]$ with respect to this norm is a C^*-algebra, denoted $C^*(\Gamma)$.

Proof. In order to prove the result, we must find a $*$-algebra embedding $\mathbb C[\Gamma]\subset B(H)$, with H being a Hilbert space. For this purpose, consider the space $H=l^2(\Gamma)$, having $\{h\}_{h\in\Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi:\mathbb C[\Gamma]\subset B(H)\quad,\quad\pi(g)(h)=gh$$

Indeed, since $\pi(g)$ maps the basis $\{h\}_{h\in\Gamma}$ into itself, this operator is well-defined, bounded, and is an isometry. It is also clear from the formula $\pi(g)(h)=gh$ that $g\to\pi(g)$ is a morphism of algebras, and since this morphism maps the unitaries $g\in\Gamma$ into isometries, this is a morphism of $*$-algebras. Finally, the faithfulness of π is clear. □

In the abelian group case, we have the following result:

Proposition 1.19. *Given an abelian discrete group Γ, we have an isomorphism*

$$C^*(\Gamma)\simeq C(G)$$

where $G=\widehat{\Gamma}$ is its Pontrjagin dual, formed by the characters $\chi:\Gamma\to\mathbb T$.

Proof. Since Γ is abelian, the corresponding group algebra $A = C^*(\Gamma)$ is commutative. Thus, we can apply the Gelfand theorem, and we obtain $A = C(X)$, with $X = Spec(A)$. But the spectrum $X = Spec(A)$, consisting of the characters $\chi : C^*(\Gamma) \to \mathbb{C}$, can be identified with the Pontrjagin dual $G = \widehat{\Gamma}$, and this gives the result. $\square$

The above result suggests the following definition:

Definition 1.20. *Given a discrete group Γ, the noncommutative space G given by*

$$C(G) = C^*(\Gamma)$$

is called abstract dual of Γ, and is denoted $G = \widehat{\Gamma}$.

This notion should be taken in the general sense of Definition 1.13. The same warning as there applies, because there is a functoriality problem here, which needs a fix.

To be more precise, in the context of Proposition 1.18, we can see that the closure $C^*_{red}(\Gamma)$ of the group algebra $\mathbb{C}[\Gamma]$ in the regular representation is a C^*-algebra as well. We have a quotient map $C^*(\Gamma) \to C^*_{red}(\Gamma)$, and if this map is not an isomorphism, which is something that can happen, we are in trouble. We will be back to this, later on.

In short, our formalism so far is "mathematically correct, but physically wrong". So, taking now advantage of the freedom of thought offered by abstract mathematics, let us just ignore this, and go ahead. By getting back to the spheres, we have:

Theorem 1.21. *The tori of the basic spheres are all group duals, as follows,*

$$\begin{tikzcd}
\mathbb{T}^N \arrow[r] & \widehat{F_N} \\
\mathbb{Z}_2^N \arrow[u] \arrow[r] & \widehat{\mathbb{Z}_2^{*N}} \arrow[u]
\end{tikzcd}$$

where F_N is the free group on N generators, and $*$ is a group-theoretical free product.

Proof. By using the presentation result in Proposition 1.17 above, we obtain that the diagram formed by the algebras $C(T)$ is as follows:

$$\begin{tikzcd}
C^*(\mathbb{Z}^N) \arrow[d] & C^*(\mathbb{Z}^{*N}) \arrow[l] \arrow[d] \\
C^*(\mathbb{Z}_2^N) & C^*(\mathbb{Z}_2^{*N}) \arrow[l]
\end{tikzcd}$$

According to Definition 1.20, the corresponding noncommutative spaces are:

$$\begin{array}{ccc}\widehat{\mathbb{Z}^N} & \longrightarrow & \widehat{\mathbb{Z}^{*N}}\\ \uparrow & & \uparrow\\ \widehat{\mathbb{Z}_2^N} & \longrightarrow & \widehat{\mathbb{Z}_2^{*N}}\end{array}$$

Together with the Fourier transform identifications from Proposition 1.19 above, and with our free group convention $F_N = \mathbb{Z}^{*N}$, this gives the result. $\square$

As a conclusion to these considerations, the Gelfand theorem alone produces out of nothing, or at least out of some basic common sense, some potentially interesting mathematics. We will be back later on to these objects, on several occasions.

Let us review now the other fundamental result regarding the C^*-algebras, namely the representation theorem of Gelfand, Naimark and Segal, which states that any C^*-algebra appears as an algebra of operators, $A \subset B(H)$, over some Hilbert space H.

In the commutative case, the precise statement is as follows:

Proposition 1.22. *Let A be a commutative C^*-algebra, write $A = C(X)$, with X being a compact space, and let μ be a positive measure on X. We have then an embedding*

$$A \subset B(H)$$

where $H = L^2(X)$, with $f \in A$ corresponding to the operator $g \to fg$.

Proof. Given $f \in C(X)$, consider the operator $T_f(g) = fg$, on the Hilbert space $H = L^2(X)$. Observe that T_f is indeed well-defined, and bounded as well, because:

$$||fg||_2 = \sqrt{\int_X |f(x)|^2 |g(x)|^2 d\mu(x)} \leq ||f||_\infty ||g||_2$$

The application $f \to T_f$ being linear, involutive, continuous, and injective as well, we obtain in this way a C^*-algebra embedding $A \subset B(H)$, as claimed. $\square$

In general, the idea will be that of extending the above construction. In order to do so, we must first discuss the analogues of the positive measures.

In order to do so, we will use a functional analysis trick, coming from the Riesz theorem, which amounts in replacing the positive measures μ with the corresponding integration functionals. To be more precise, let us start with the following definition:

Definition 1.23. *Consider a linear map $\varphi : A \to \mathbb{C}$.*

(1) *φ is called positive when $a \geq 0 \implies \varphi(a) \geq 0$.*
(2) *φ is called faithful and positive when $a \geq 0, a \neq 0 \implies \varphi(a) > 0$.*

In the commutative case, $A = C(X)$, the positive linear forms appear as follows, with μ being positive, and strictly positive if we want φ to be faithful and positive:

$$\varphi(f) = \int_X f(x)d\mu(x)$$

In general, the positive linear forms can be thought of as being integration functionals with respect to some underlying "positive measures". We can use them as follows:

Proposition 1.24. *Let $\varphi : A \to \mathbb{C}$ be a positive linear form.*

(1) *$< a, b >= \varphi(ab^*)$ defines a generalized scalar product on A.*
(2) *By separating and completing we obtain a Hilbert space H.*
(3) *$\pi(a) : b \to ab$ defines a representation $\pi : A \to B(H)$.*
(4) *If φ is faithful in the above sense, then π is faithful.*

Proof. Almost everything here is straightforward, as follows:
(1) This is clear from definitions, and from Proposition 1.12.
(2) This is a standard procedure, which works for any scalar product.
(3) All the verifications here are standard algebraic computations.
(4) This follows indeed from $a \neq 0 \implies \pi(aa^*) \neq 0 \implies \pi(a) \neq 0$. □

In order to establish the GNS theorem, it remains to prove that any C^*-algebra has a faithful and positive linear form $\varphi : A \to \mathbb{C}$. This is something more technical:

Proposition 1.25. *Let A be a C^*-algebra.*

(1) *Any positive linear form $\varphi : A \to \mathbb{C}$ is continuous.*
(2) *A linear form φ is positive iff there is a norm one $h \in A_+$ such that $||\varphi|| = \varphi(h)$.*
(3) *For any $a \in A$ there exists a positive norm one form φ such that $\varphi(aa^*) = ||a||^2$.*
(4) *If A is separable there is a faithful positive form $\varphi : A \to \mathbb{C}$.*

Proof. The proof here, which is quite technical, inspired from the existence proof of the probability measures on abstract compact spaces, goes as follows:
(1) This follows from Proposition 1.24, via the following inequality:

$$|\varphi(a)| \leq ||\pi(a)||\varphi(1) \leq ||a||\varphi(1)$$

(2) In one sense we can take $h = 1$. Conversely, let $a \in A_+$, $||a|| \leq 1$. We have:

$$|\varphi(h) - \varphi(a)| \leq ||\varphi|| \cdot ||h - a|| \leq \varphi(h)1 = \varphi(h)$$

Thus we have $Re(\varphi(a)) \geq 0$, and it remains to prove that the following holds:

$$a = a^* \implies \varphi(a) \in \mathbb{R}$$

By using $1-h \geq 0$ we can apply the above to $a = 1-h$ and we obtain $Re(\varphi(1-h)) \geq 0$. We conclude that $Re(\varphi(1)) \geq Re(\varphi(h)) = ||\varphi||$, and so $\varphi(1) = ||\varphi||$.

Summing up, we can assume $h = 1$. Now observe that for any self-adjoint element a, and any $t \in \mathbb{R}$ we have the following inequality:

$$\begin{aligned} |\varphi(1+ita)|^2 &\leq ||\varphi||^2 \cdot ||1+ita||^2 \\ &= \varphi(1)^2 ||1+t^2a^2|| \\ &\leq \varphi(1)^2(1+t^2||a||^2) \end{aligned}$$

On the other hand with $\varphi(a) = x + iy$ we have:

$$\begin{aligned} |\varphi(1+ita)| &= |\varphi(1) - ty + itx| \\ &\geq (\varphi(1) - ty)^2 \end{aligned}$$

We therefore obtain that for any $t \in \mathbb{R}$ we have:

$$\varphi(1)^2(1+t^2||a||^2) \geq (\varphi(1) - ty)^2$$

Thus we have $y = 0$, and this finishes the proof of our remaining claim.

(3) Consider the linear subspace of A spanned by the element aa^*. We can define here a linear form by the formula $\varphi(\lambda aa^*) = \lambda ||a||^2$. This has norm one, and by Hahn-Banach we get a norm one extension to the whole A. The positivity of φ follows from (2).

(4) Let (a_n) be a dense sequence inside A. For any n we can construct as in (3) a positive form satisfying $\varphi_n(a_n a_n^*) = ||a_n||^2$, and then define φ in the following way:

$$\varphi = \sum_{n=1}^\infty \frac{\varphi_n}{2^n}$$

Let $a \in A$ be a nonzero element. Pick a_n close to a and consider the pair (H, π) associated to the pair (A, φ_n), as in Proposition 1.24. We have then:

$$\begin{aligned} \varphi_n(aa^*) &= ||\pi(a)1|| \\ &\geq ||\pi(a_n)1|| - ||a - a_n|| \\ &= ||a_n|| - ||a - a_n|| \\ &> 0 \end{aligned}$$

Thus $\varphi_n(aa^*) > 0$. It follows that we have $\varphi(aa^*) > 0$, and we are done. □

With these ingredients in hand, we can now state and prove:

Theorem 1.26 (GNS theorem)**.** *Let A be a C^*-algebra.*

1. *A appears as a closed $*$-subalgebra $A \subset B(H)$, for some Hilbert space H.*
2. *When A is separable (usually the case), H can be chosen to be separable.*
3. *When A is finite dimensional, H can be chosen to be finite dimensional.*

Proof. This result, from [62], follows indeed by combining the construction from Proposition 1.24 with the existence result from Proposition 1.25 above. □

The GNS theorem is something powerful and concrete, which perfectly complements the Gelfand theorem, and the resulting noncommutative compact space formalism.

As a first application, let us get back to the bad functoriality properties of the Gelfand correspondence. We can fix these issues by using the GNS theorem, as follows:

Definition 1.27. *The category of compact measured spaces (X,μ) is the category of the C^*-algebras with faithful traces (A,φ), with the arrows reversed. In the case where we have a C^*-algebra A with a non-faithful trace φ, we can still talk about the corresponding compact measured space (X,μ), by performing the GNS construction.*

Observe that this definition fixes the functoriality problem with Gelfand duality, at least for the group algebras. Indeed, in the context of the comments following Definition 1.20, consider an arbitrary intermediate C^*-algebra, as follows:

$$C^*(\Gamma)\to A\to C^*_{red}(\Gamma)$$

If we perform the GNS construction with respect to the canonical trace, we obtain the reduced group algebra $C^*_{red}(\Gamma)$. Thus, all these algebras A correspond to a unique noncommutative space in the above sense, which is the abstract group dual $\widehat{\Gamma}$.

Let us record a statement about this finding, as follows:

Proposition 1.28. *The category of group duals $\widehat{\Gamma}$ is a well-defined subcategory of the category of compact measured spaces, with each individual $\widehat{\Gamma}$ corresponding to the full group algebra $C^*(\Gamma)$, or the reduced group algebra $C^*_{red}(\Gamma)$, or any algebra in between.*

Proof. This is indeed more of an empty statement, coming from the above discussion. □

With this in hand, it is tempting to go even further, namely forgetting about the C^*-algebras, and trying to axiomatize instead the operator algebras of type $L^\infty(X)$. Such an axiomatization is possible, and the resulting class of operator algebras consists of a certain very special type of C^*-algebras, called "finite von Neumann algebras".

However, and here comes our point, doing so would be bad, and would lead to a weak theory, because many spaces such as the compact groups, or the compact homogeneous spaces, do not come with a measure by definition, but rather by theorem.

In short, our "fix" is not a very good fix, and if we want a really strong theory, we must invent something else. In order to do so, our idea will be that of restricting the attention to certain special classes of noncommutative algebraic manifolds, as follows:

Definition 1.29. *A real algebraic submanifold $X\subset S^{N-1}_{\mathbb{C},+}$ is a closed noncommutative space defined, at the level of the corresponding C^*-algebra, by a formula of type*

$$C(X)=C(S^{N-1}_{\mathbb{C},+})\Big/\Big\langle f_i(x_1,\ldots,x_N)=0\Big\rangle$$

for certain noncommutative polynomials $f_i\in\mathbb{C}<x_1,\ldots,x_N>$. We denote by $\mathcal{C}(X)$ the $*$-subalgebra of $C(X)$ generated by the coordinate functions $x_1,\ldots,x_N$.

Observe that any family of noncommutative polynomials $f_i \in \mathbb{C} < x_1, \ldots, x_N >$ produces such a manifold X, simply by defining an algebra $C(X)$ as above. Observe also that the use of the free complex sphere is essential in all this, because the quadratic condition $\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$ guarantees the fact that the universal C^*-norm is bounded.

We have already met such manifolds, in the context of the free spheres, free tori, and more generally in Proposition 1.17 above. Here is a list of examples:

Proposition 1.30. *The following are algebraic submanifolds $X \subset S^{N-1}_{\mathbb{C},+}$:*

1. *The spheres $S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{C}}, S^{N-1}_{\mathbb{R},+} \subset S^{N-1}_{\mathbb{C},+}$.*
2. *Any compact Lie group, $G \subset U_n$, when $N = n^2$.*
3. *The duals $\widehat{\Gamma}$ of finitely generated groups, $\Gamma =< g_1, \ldots, g_N >$.*

Proof. These facts are all well-known, the proof being as follows:

(1) This is true by definition of our various spheres.

(2) Given a closed subgroup $G \subset U_n$, we have indeed an embedding $G \subset S^{N-1}_{\mathbb{C}}$, with $N = n^2$, given in double indices by $x_{ij} = \frac{u_{ij}}{\sqrt{n}}$, that we can further compose with the standard embedding $S^{N-1}_{\mathbb{C}} \subset S^{N-1}_{\mathbb{C},+}$. As for the fact that we obtain indeed a real algebraic manifold, this is well-known, coming either from Lie theory or from Tannakian duality. We will be back to this fact later on, in a more general context.

(3) This follows from the fact that the variables $x_i = \frac{g_i}{\sqrt{N}}$ satisfy the quadratic relations $\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$, with the algebricity claim of the manifold being clear. $\square$

At the level of the general theory, we have the following version of the Gelfand theorem, which is something very useful, and that we will use many times in what follows:

Theorem 1.31. *When $X \subset S^{N-1}_{\mathbb{C},+}$ is an algebraic manifold, given by*

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle f_i(x_1, \ldots, x_N) = 0 \Big\rangle$$

for certain noncommutative polynomials $f_i \in \mathbb{C} < x_1, \ldots, x_N >$, we have

$$X_{class} = \left\{ x \in S^{N-1}_{\mathbb{C}} \Big| f_i(x_1, \ldots, x_N) = 0 \right\}$$

and X appears as a liberation of X_{class}.

Proof. This is something that already met, in the context of the free spheres. In general, the proof is similar, by using the Gelfand theorem. Indeed, if we denote by X'_{class} the manifold constructed in the statement, then we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X_{class}) \to C(X'_{class})$$

Conversely now, from $X \subset S^{N-1}_{\mathbb{C},+}$ we obtain $X_{class} \subset S^{N-1}_{\mathbb{C}}$, and since the relations defining X'_{class} are satisfied by X_{class}, we obtain an inclusion of subspaces $X_{class} \subset X'_{class}$.

Thus, at the level of algebras of continuous functions, we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X'_{class}) \to C(X_{class})$$

Thus, we have constructed a pair of inverse morphisms, and we are done. $\square$

With these results in hand, we are now ready for formulating our second "fix" for the functoriality issues of the Gelfand correspondence, as follows:

Definition 1.32. *The category of the real algebraic submanifolds $X \subset S^{N-1}_{\mathbb{C},+}$ is the category of the universal C^*-algebras of type*

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle f_i(x_1, \ldots, x_N) = 0 \Big\rangle$$

with $f_i \in \mathbb{C} < x_1, \ldots, x_N >$ being noncommutative polynomials, with the arrows $X \to Y$ being the $*$-algebra morphisms $\mathcal{C}(Y) \to \mathcal{C}(X)$, mapping coordinates to coordinates.

In other words, such algebraic manifolds are those from Definition 1.29 above, with the convention that we identify $X \simeq Y$ in the case where we have an isomorphism of $*$-algebras $\mathcal{C}(Y) \to \mathcal{C}(X)$, mapping coordinates to coordinates.

This fix works indeed for the group algebras, because we have:

Theorem 1.33. *The category of finitely generated groups $\Gamma =< g_1, \ldots, g_N >$, with the morphisms being the group morphisms mapping generators to generators, embeds contravariantly via $\Gamma \to \widehat{\Gamma}$ into the category of real algebraic submanifolds $X \subset S^{N-1}_{\mathbb{C},+}$.*

Proof. We know from Proposition 1.30 that, given a group $\Gamma =< g_1, \ldots, g_N >$, we have an embedding $\widehat{\Gamma} \subset S^{N-1}_{\mathbb{C},+}$ given by $x_i = \frac{g_i}{\sqrt{N}}$. Now since a morphism $C[\Gamma] \to C[\Lambda]$ mapping coordinates to coordinates means a morphism of groups $\Gamma \to \Lambda$ mapping generators to generators, our notion of isomorphism is indeed the correct one, as claimed. $\square$

We will see later on that Theorem 1.33 has various extensions to the quantum groups and quantum homogeneous spaces that we will be interested in, in what follows.

So, this will be our formalism, and operator algebra knowledge required. We should mention that our approach heavily relies on Woronowicz's philosophy in [98]. Also, part of the above has been folklore for a long time, with the details worked out in [14].

Getting back now to the main examples, namely the free analogues $S^{N-1}_{\mathbb{R},+}, S^{N-1}_{\mathbb{C},+}$ of the spheres $S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{C}}$, in order to further understand their structure, we would need to have as well free analogues O_N^+, U_N^+ of the orthogonal and unitary groups O_N, U_N. Constructing these free analogues will be our next purpose, in what follows.

2. Quantum groups

In order to construct liberations of O_N, U_N, and other "quantum groups", the idea is very simple, coming from the usual multiplicative formulae for unitary matrices:

$$(UV)_{ij} = \sum_k U_{ik} V_{kj} \quad , \quad (1_N)_{ij} = \delta_{ij} \quad , \quad (U^{-1})_{ij} = U_{ji}^*$$

A bit of Gelfand duality thinking, to be explained in detail in the proof of Proposition 2.2 below, leads from this to the following definition, due to Woronowicz [98]:

Definition 2.1. *A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u \in M_N(A)$ whose coefficients generate A, such that:*

1. *$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ defines a morphism of C^*-algebras $A \to A \otimes A$.*
2. *$\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism of C^*-algebras $A \to \mathbb{C}$.*
3. *$S(u_{ij}) = u_{ji}^*$ defines a morphism of C^*-algebras $A \to A^{opp}$.*

In this case, we write $A = C(G)$, and call G a compact matrix quantum group.

In this definition $A \otimes A$ is the universal C^*-algebraic completion of the usual algebraic tensor product of A with itself, and A^{opp} is the opposite C^*-algebra, with multiplication $a \cdot b = ba$. The reasons for using A^{opp} instead of A itself will become clear later on.

The above morphisms Δ, ε, S are called comultiplication, counit and antipode. Observe that if these morphisms exist, they are unique. This phenomenon is analogous to the fact that a closed set of unitary matrices $G \subset U_N$ is either a compact group, or not.

The motivating examples of Woronowicz algebras are as follows:

Proposition 2.2. *Given a closed subgroup $G \subset U_N$, the algebra $A = C(G)$, with the matrix formed by the standard coordinates $u_{ij}(g) = g_{ij}$, is a Woronowicz algebra, and:*

1. *For this algebra, the morphisms Δ, ε, S appear as functional analytic transposes of the multiplication, unit and inverse maps m, u, i of the group G.*
2. *This Woronowicz algebra is commutative, and conversely, any Woronowicz algebra which is commutative appears in this way.*

Proof. Since we have $G \subset U_N$, the matrix $u = (u_{ij})$ is unitary. Also, since the coordinate functions u_{ij} separate the points of G, by the Stone-Weierstrass theorem we obtain that the $*$-subalgebra $\mathcal{A} \subset C(G)$ generated by them is dense. Finally, the fact that we have morphisms Δ, ε, S as in Definition 2.1 follows from the proof of (1) below.

(1) We use the above formulae for unitary matrices. The fact that m^T satisfies the condition in Definition 2.1 (1) follows from the following computation, with $U, V \in G$:

$$m^T(u_{ij})(U \otimes V) = (UV)_{ij} = \sum_k U_{ik} V_{kj} = \sum_k (u_{ik} \otimes u_{kj})(U \otimes V)$$

Regarding now the morphism u^T, the verification of the condition in Definition 2.1 (2) is trivial, coming from the following equalities:

$$u^T(u_{ij}) = 1_{ij} = \delta_{ij}$$

Finally, the morphism i^T verifies the condition in Definition 2.1 (3) as well, because we have the following computation, valid for any $U \in G$:

$$i^T(u_{ij})(U) = (U^{-1})_{ij} = \bar{U}_{ji} = u_{ji}^*(U)$$

(2) This folllows from the Gelfand theorem. Indeed, we can write $A = C(G)$, with G being a certain compact space. By using the matrix of coordinates $u = (u_{ij})$ we obtain an embedding $G \subset U_N$, and then by using Δ, ε, S, it follows that the subspace $G \subset U_N$ that we have obtained is in fact a closed subgroup, and we are done. □

Let us go back now to the general setting of Definition 2.1. According to Proposition 2.2, and to the general C^*-algebra philosophy, the morphisms Δ, ε, S can be thought of as coming from a multiplication, unit map and inverse map, as follows:

$$m : G \times G \to G \quad , \quad u : \{.\} \to G \quad , \quad i : G \to G$$

Of course, since G does not exist as a concrete object, nor do m, u, i exist. However, we can use some inpiration from group theory in order to study Δ, ε, S, and G itself.

Here is a first result of this type, expressing in terms of Δ, ε, S the fact that the underlying maps m, u, i should satisfy the usual group theory axioms:

Proposition 2.3. *The comultiplication, counit and antipode have the following properties, on the dense $*$-subalgebra $\mathcal{A} \subset A$ generated by the variables u_{ij}:*

1. *Coassociativity: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.*
2. *Counitality: $(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id$.*
3. *Coinversality: $m(id \otimes S)\Delta = m(S \otimes id)\Delta = \varepsilon(.)1$.*

In addition, the square of the antipode is the identity, $S^2 = id$.

Proof. Observe first that the result holds in the case where A is commutative. Indeed, by using Proposition 2.2 we can write $\Delta = m^t, \varepsilon = u^t, S = i^T$, and the above 3 conditions come by transposition from the basic 3 conditions satisfied by m, u, i, namely:

$$m(m \times id) = m(id \times m)$$

$$m(id \times u) = m(u \times id) = id$$

$$m(id \otimes i)\delta = m(i \otimes id)\delta = 1$$

Here $\delta(g) = (g, g)$. Observe also that the last condition, $S^2 = id$, is satisfied as well, coming from the identity $i^2 = id$, which is a consequence of the group axioms.

In the general case now, the proof goes as follows:

(1) This follows from the following computations:

$$(\Delta \otimes id)\Delta(u_{ij}) = \sum_l \Delta(u_{il}) \otimes u_{lj} = \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj}$$

$$(id \otimes \Delta)\Delta(u_{ij}) = \sum_k u_{ik} \otimes \Delta(u_{kj}) = \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj}$$

(2) The proof here is quite similar, as follows:

$$(id \otimes \varepsilon)\Delta(u_{ij}) = \sum_k u_{ik} \otimes \varepsilon(u_{kj}) = u_{ij}$$

$$(\varepsilon \otimes id)\Delta(u_{ij}) = \sum_k \varepsilon(u_{ik}) \otimes u_{kj} = u_{ij}$$

(3) By using the fact that the matrix $u = (u_{ij})$ is unitary, we obtain:

$$m(id \otimes S)\Delta(u_{ij}) = \sum_k u_{ik} S(u_{kj}) = \sum_k u_{ik} u_{jk}^* = (uu^*)_{ij} = \delta_{ij}$$

$$m(S \otimes id)\Delta(u_{ij}) = \sum_k S(u_{ik}) u_{kj} = \sum_k u_{ki}^* u_{kj} = (u^*u)_{ij} = \delta_{ij}$$

Finally, the formula $S^2 = id$ holds as well on the generators, and we are done. □

Observe that, by continuity, the formulae (1) and (2) above hold on the whole algebra A. However, regarding the formula (3), the maps appearing there do not have an obvious extension to A, and this is why we have to formulate Proposition 2.3 as it is.

By getting back now to Proposition 2.2, this statement becomes even more interesting when coupled with the well-known fact that the closed subgroups $G \subset U_N$ are exactly the compact Lie groups. This is something non-trivial, but we can nevertheless formulate:

Definition 2.4. *Let $A = C(G)$ be a Woronowicz algebra, with coordinates u_{ij}.*

(1) *The dense $*$-subalgebra $\mathcal{A} =< u_{ij} >$ is denoted $C^\infty(G)$.*
(2) *We also call G a compact quantum Lie group.*

This definition is of course something quite philosophical, because we have not discussed here the above-mentioned result about the compact Lie groups, $G \subset U_N$, nor looked for quantum extensions of it. In addition, all this might seem to suggest that the compact matrix quantum groups have some kind of Lie type differential geometry structure, and this is not the case. For instance, we will see later on that certain noncommutative tori, such as the free group dual $\widehat{F}_N$, which is a well-known space in mathematics, notoriously having no interesting differential geometry, are compact matrix quantum groups.

As a last general comment, Definition 2.1 as formulated is not exactly the one used by Woronowicz in [98]. The formalism there is more general, negating the condition $S^2 = id$, in order to cover some extra examples, which were of interest at the time of [98].

Let us discuss now another class of basic examples, namely the group duals. The result here, which among others clarifies the use of A^{opp} instead of A, is as follows:

Proposition 2.5. *Given a finitely generated discrete group $\Gamma =< g_1,\ldots,g_N >$, the group algebra $A = C^*(\Gamma)$, together with the diagonal matrix formed by the standard generators, $u = diag(g_1,\ldots,g_N)$, is a Woronowicz algebra. Moreover:*

1. *For this algebra, the maps Δ,ε,S are given by the formulae $\Delta(g) = g\otimes g$, $\varepsilon(g) = 1$ and $S(g) = g^{-1}$, for any group element $g \in \Gamma$.*
2. *This Woronowicz algebra is cocommutative, in the sense that we have $\Sigma\Delta = \Delta$, where $\Sigma(a \otimes b) = b \otimes a$ is the flip map.*

Proof. Since the involution on $C^*(\Gamma)$ is by definition given by $g^* = g^{-1}$ for any group element $g \in \Gamma$, all these group elements are unitaries. In particular the standard generators $g_1,\ldots,g_N$ are unitaries, and so must be the diagonal matrix formed by them:

$$u = \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_N \end{pmatrix}$$

Also, since $g_1,\ldots,g_N$ generate Γ, these elements generate the group algebra $C^*(\Gamma)$ as well, in the algebraic sense. Finally, the fact that we have indeed maps Δ,ε,S as in Definition 2.1 follows from the explicit formulae in the proof of (1) below.

(1) This is clear from definitions, because the maps Δ,ε,S can be defined by the formulae in the statement, by using the universality property of $C^*(\Gamma)$. To be more precise, consider the following map, which is a unitary representation:

$$\Gamma \to C^*(\Gamma)\otimes C^*(\Gamma) \quad , \quad g \to g\otimes g$$

By the universal property of $C^*(\Gamma)$ this representation extends then into a morphism $\Delta : C^*(\Gamma) \to C^*(\Gamma)\otimes C^*(\Gamma)$, given by the same formula, $\Delta(g) = g \otimes g$.

The situation for ε is similar, because this comes from the trivial representation:

$$\Gamma \to \{1\} \quad , \quad g \to 1$$

Finally, the antipode S comes from the following unitary representation:

$$\Gamma \to C^*(\Gamma)^{opp} \quad , \quad g \to g^{-1}$$

Observe that we have to use the opposite algebra, for this map to be multiplicative.

(2) This property is clear from the formula $\Delta(g) = g \otimes g$, and from the fact that the group elements $g \in \Gamma$ span the whole group algebra $C^*(\Gamma)$. $\square$

We will see later on that any cocommutative Woronowicz algebra appears in fact as above, up to some standard equivalence relation for such algebras.

In the abelian group case we have a more precise result, as follows:

Proposition 2.6. *Assume that Γ as above is abelian, and let $G = \widehat{\Gamma}$ be its Pontrjagin dual, formed by the characters $\chi : \Gamma \to \mathbb{T}$. The canonical isomorphism*

$$C^*(\Gamma) \simeq C(G)$$

transforms then the comultiplication, counit and antipode of $C^*(\Gamma)$, given by

$$\Delta(g) = g \otimes g \quad , \quad \varepsilon(g) = 1 \quad , \quad S(g) = g^{-1}$$

into the comultiplication, counit and antipode of $C(G)$, given by:

$$\Delta\varphi(g,h) = \varphi(gh) \quad , \quad \varepsilon(\varphi) = \varphi(1) \quad , \quad S\varphi(g) = \varphi(g^{-1})$$

Thus, the identification $G = \widehat{\Gamma}$ is a compact quantum group isomorphism.

Proof. Assume indeed that $\Gamma =< g_1, \ldots, g_N >$ is abelian. Our claim is that with $G = \widehat{\Gamma}$ we have a group embedding $G \subset U_N$, constructed as follows:

$$\chi \to \begin{pmatrix} \chi(g_1) & & \\ & \ddots & \\ & & \chi(g_N) \end{pmatrix}$$

Indeed, this formula defines a unitary group representation, whose kernel is $\{1\}$.

Summarizing, we have two Woronowicz algebras to be compared, namely $C(G)$, constructed as in Proposition 2.2, and $C^*(\Gamma)$, constructed as in Proposition 2.5.

We already know from Proposition 1.19 above that the underlying C^*-algebras are isomorphic. Now since the morphisms Δ, ε, S agree on the standard generators $g_1, \ldots, g_N$, they agree everywhere, and we are led to the conclusions in the statement. $\square$

As a conclusion to all this, we can supplement Definitions 2.1 and 2.4 with:

Definition 2.7. *Given a Woronowicz algebra $A = C(G)$, we write as well*

$$A = C^*(\Gamma)$$

and call $\Gamma = \widehat{G}$ a finitely generated discrete quantum group.

We should mention that there is a slight problem with the functoriality here, because for certain groups like the free group $\Gamma = F_N$, the group $*$-algebra $\mathbb{C}[\Gamma]$ is known to have several C^*-algebraic completions, which are all Woronowicz algebras. In order to fix this problem, we will use the general philosophy from section 1 above. Let us start with:

Proposition 2.8. *Given a Woronowicz algebra $A = C(G)$, we have an embedding*

$$G \subset S^{N^2-1}_{\mathbb{C},+}$$

given in double indices by $x_{ij} = \frac{u_{ij}}{\sqrt{N}}$, where u_{ij} are the standard coordinates of G.

Proof. This is something that we already know for the classical groups, and for the group duals as well, from Proposition 1.30 above. In general, the proof is similar, coming from the fact that the matrices $u, \bar{u}$ are both unitaries, with the unitarity of $\bar{u}$ coming from the unitarity of u, by applying the antipode. We will be back later on to this latter fact, with full details, in the context of representation theory, in section 3 below. $\square$

As explained in the proof of Proposition 1.30, in the classical group case we obtain in this way an algebraic manifold, but this is not trivial. The situation in general is similar, and we will discuss this after developing some appropriate tools, in section 4 below.

However, in waiting for all this to be complete, we can nevertheless take some inspiration from Definition 1.32 above, and formulate the following definition:

Definition 2.9. *Given two Woronowicz algebras (A,u) and (B,v), we write $A \simeq B$, and indentify as well the corresponding compact and discrete quantum groups, when we have an isomorphism of $*$-algebras $\mathcal{A} \simeq \mathcal{B}$ mapping standard coordinates to standard coordinates.*

In view of the various results and comments from section 1, the functoriality problem for the compact and discrete quantum groups is therefore fixed. All this needs of course a bit more discussion, and we will do this later on, once we will have more tools.

Let us get now into a much more interesting question, namely the construction of examples. We can construct examples by using various operations. First, we have:

Proposition 2.10. *Given two compact quantum groups G, H, so is their product $G \times H$, constructed according to the following formula:*

$$C(G \times H) = C(G) \otimes C(H)$$

Equivalently, at the level of the associated discrete duals Γ, Λ, we can set

$$C^*(\Gamma \times \Lambda) = C^*(\Gamma) \otimes C^*(\Lambda)$$

and we obtain the same equality of Woronowicz algebras as above.

Proof. Assume indeed that we have two Woronowicz algebras, (A,u) and (B,v). Our claim is that the following construction produces a Woronowicz algebra:

$$C = A \otimes B \quad , \quad w = diag(u,v)$$

Indeed, the matrix w is unitary, and its coefficients generate C. As for the existence of the maps Δ, ε, S, this follows from the functoriality properties of $\otimes$, which is here, as usual, the universal C^*-algebraic completion of the algebraic tensor product.

With this claim in hand, the first assertion is clear. As for the second assertion, let us recall that when G, H are classical and abelian, we have the following formula:

$$\widehat{G \times H} = \widehat{G} \times \widehat{H}$$

Thus, our second assertion is simply a reformulation of the first assertion, with the $\times$ symbol used there being justified by this well-known group theory formula. $\square$

As a consequence of the above result, we can make products of groups and group duals, and we obtain in this way quantum groups which are not groups, nor group duals.

Here is now a more subtle construction, due to Wang [91]:

Proposition 2.11. *Given two compact quantum groups G,H, so is their dual free product $G\,\hat{*}\,H$, constructed according to the following formula:*

$$C(G\,\hat{*}\,H)=C(G)*C(H)$$

Equivalently, at the level of the associated discrete duals Γ,Λ, we can set

$$C^*(\Gamma*\Lambda)=C^*(\Gamma)*C^*(\Lambda)$$

and we obtain the same equality of Woronowicz algebras as above.

Proof. The proof here is identical with the proof of Proposition 2.10, by replacing everywhere the tensor product $\otimes$ with the free product $*$, with this latter product being by definition the universal C^*-algebraic completion of the algebraic free product. □

Here is another construction, which once again, has no classical counterpart:

Proposition 2.12. *Given a compact quantum group G, so is its free complexification $\widetilde{G}$, constructed according to the following formula, where $z=id\in C(\mathbb{T})$:*

$$C(\widetilde{G})\subset C(\mathbb{T})*C(G)\quad,\quad\tilde{u}=zu$$

Equivalently, at the level of the associated discrete dual Γ, we can set

$$C^*(\widetilde{\Gamma})\subset C^*(\mathbb{Z})*C^*(\Gamma)\quad,\quad\tilde{u}=zu$$

where $z=1\in\mathbb{Z}$, and we obtain the same Woronowicz algebra as above.

Proof. This follows from Proposition 2.11. Indeed, we know from there that $C(\mathbb{T})*C(G)$ is a Woronowicz algebra, with matrix of coordinates $w=diag(z,u)$. Now, let us try to replace this matrix with the matrix $\tilde{u}=zu$. This matrix is unitary, and we have:

$$\Delta(\tilde{u}_{ij})=(z\otimes z)\sum_k u_{ik}\otimes u_{kj}=\sum_k\tilde{u}_{ik}\otimes\tilde{u}_{kj}$$

Similarly, in what regards the counit, we have the following formula:

$$\varepsilon(\tilde{u}_{ij})=1\cdot\delta_{ij}=\delta_{ij}$$

Finally, recalling that S takes values in the opposite algebra, we have as well:

$$S(\tilde{u}_{ij})=u_{ji}^*\cdot\bar{z}=\tilde{u}_{ji}^*$$

Summarizing, the conditions in Definition 2.1 are satisfied, except for the fact that the entries of $\tilde{u}=zu$ do not generate the whole algebra $C(\mathbb{T})*C(G)$. We conclude that if we let $C(\widetilde{G})\subset C(\mathbb{T})*C(G)$ be the subalgebra generated by the entries of $\tilde{u}=zu$, as in the statement, then the conditions in Definition 2.1 are satisfied, as desired. □

Another standard operation is that of taking subgroups:

Proposition 2.13. *Let G be compact quantum group, and let $I \subset C(G)$ be a closed $*$-ideal satisfying the following condition:*

$$\Delta(I) \subset C(G) \otimes I + I \otimes C(G)$$

We have then a closed quantum subgroup $H \subset G$, constructed as follows:

$$C(H) = C(G)/I$$

At the dual level we obtain a quotient of discrete quantum groups, $\widehat{\Gamma} \to \widehat{\Lambda}$.

Proof. This follows indeed from the above conditions on I, which are designed precisely as for Δ, ε, S to factorize through the quotient. As for the last assertion, this is just a reformulation, coming from the functoriality properties of the Pontrjagin duality. □

In order to discuss now the quotient operation, let us agree to call "corepresentation" of a Woronowicz algebra A any unitary matrix $w \in M_n(\mathcal{A})$ satisfying:

$$\Delta(w_{ij}) = \sum_k w_{ik} \otimes w_{kj} \quad , \quad \varepsilon(w_{ij}) = \delta_{ij} \quad , \quad S(w_{ij}) = w_{ji}^*$$

We will study in detail such corepresentations in section 3 below. For the moment, we just need their definition, in order to formulate the following result:

Proposition 2.14. *Let G be a compact quantum group, and $w = (w_{ij})$ be a corepresentation of $C(G)$. We have then a quotient quantum group $G \to H$, given by:*

$$C(H) = < w_{ij} >$$

At the dual level we obtain a discrete quantum subgroup, $\widehat{\Lambda} \subset \widehat{\Gamma}$.

Proof. Here the first assertion follows from definitions, and the second assertion is just a reformulation, coming from the functoriality properties of the Pontrjagin duality. □

Finally, here is one more construction, which appears as a particular case of the quotient construction in Proposition 2.14, and which will be of importance in what follows:

Theorem 2.15. *Given a compact quantum group G, with fundamental corepresentation denoted $u = (u_{ij})$, the $N^2 \times N^2$ matrix given in double index notation by*

$$w_{ia,jb} = u_{ij} u_{ab}^*$$

is a corepresentation in the above sense, and we have the following results:

1. *The corresponding quotient $G \to PG$ is a compact quantum group.*
2. *Via the standard embedding $G \subset S^{N^2-1}_{\mathbb{C},+}$, this is the projective version.*
3. *In the classical group case, $G \subset U_N$, we have $PG = G/(G \cap \mathbb{T}^N)$.*
4. *In the group dual case, with $\Gamma = < g_i >$, we have $\widehat{P\Gamma} = < g_i g_j^{-1} >$.*

Proof. The fact that w is indeed a corepresentation is routine, and follows as well from the general properties of such corepresentations, to be discussed in section 3 below.

(1) This follows from Proposition 2.14 above.

(2) Observe first that, since the matrix $u = (u_{ij})$ is biunitary, we have indeed an embedding $G \subset S^{N^2-1}_{\mathbb{C},+}$ as in the statement, given in double index notation by:

$$x_{ij} = \frac{u_{ij}}{\sqrt{N}}$$

Now with this formula in hand, the assertion is clear from definitions.

(3) This follows from the elementary fact that, via Gelfand duality, w is the matrix of coefficients of the adjoint representation of G, whose kernel is the subgroup $G \cap \mathbb{T}^N$, where $\mathbb{T}^N \subset U_N$ denotes the subgroup formed by the diagonal matrices.

(4) This is something trivial, which follows from definitions. $\square$

At the level of the really "new" examples now, we first have, following [91]:

Proposition 2.16. *The following universal algebras are Woronowicz algebras,*

$$\begin{aligned} C(O_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u = \bar{u}, u^t = u^{-1}\right) \\ C(U_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^* = u^{-1}, u^t = \bar{u}^{-1}\right) \end{aligned}$$

so the underlying noncommutative spaces O_N^+, U_N^+ are compact quantum groups.

Proof. This follows from the elementary fact that if a matrix u is orthogonal or biunitary, then so must be the following matrices:

$$u^\Delta_{ij} = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u^\varepsilon_{ij} = \delta_{ij} \quad , \quad u^S_{ij} = u^*_{ji}$$

Consider indeed the matrix $U = u^\Delta$. This matrix is unitary, because:

$$\begin{aligned} (UU^*)_{ij} &= \sum_k U_{ik}U^*_{jk} = \sum_{klm} u_{il}u^*_{jm} \otimes u_{lk}u^*_{mk} = \sum_{lm} u_{il}u^*_{jm} \otimes \delta_{lm} = \delta_{ij} \\ (U^*U)_{ij} &= \sum_k U^*_{ki}U_{kj} = \sum_{klm} u^*_{kl}u_{km} \otimes u^*_{li}u_{mj} = \sum_{lm} \delta_{lm} \otimes u^*_{li}u_{mj} = \delta_{ij} \end{aligned}$$

The verification of the unitarity of $\bar{U}$ is similar, as follows:

$$\begin{aligned} (\bar{U}U^t)_{ij} &= \sum_k U^*_{ik}U_{jk} = \sum_{klm} u^*_{il}u_{jm} \otimes u^*_{lk}u_{mk} = \sum_{lm} u^*_{il}u_{jm} \otimes \delta_{lm} = \delta_{ij} \\ (U^t\bar{U})_{ij} &= \sum_k U_{ki}U^*_{kj} = \sum_{klm} u_{kl}u^*_{km} \otimes u_{li}u^*_{mj} = \sum_{lm} \delta_{lm} \otimes u_{li}u^*_{mj} = \delta_{ij} \end{aligned}$$

Regarding now the matrix $u^\varepsilon = 1_N$, this is clearly biunitary. Finally, regarding the matrix u^S, there is nothing to prove here either, because its unitarity its clear too.

Finally, observe that if u is real, then so are the above matrices $u^\Delta, u^\varepsilon, u^S$.

Thus, we can indeed define morphisms Δ, ε, S as in Definition 2.1, by using the universal properties of $C(O_N^+)$, $C(U_N^+)$, and this gives the result. $\square$

Let us study now the above quantum groups, with the techniques that we have. As a first observation, we have embeddings of compact quantum groups, as follows:

$$\begin{array}{ccc} U_N & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^+ \end{array}$$

The basic properties of O_N^+, U_N^+ can be summarized as follows:

Theorem 2.17. *The quantum groups O_N^+, U_N^+ have the following properties:*

(1) *The closed subgroups $G \subset U_N^+$ are exactly the $N \times N$ compact quantum groups. As for the closed subgroups $G \subset O_N^+$, these are those satisfying $u = \bar{u}$.*
(2) *We have liberation embeddings $O_N \subset O_N^+$ and $U_N \subset U_N^+$, obtained by dividing the algebras $C(O_N^+), C(U_N^+)$ by their respective commutator ideals.*
(3) *We have as well embeddings $\widehat{L}_N \subset O_N^+$ and $\widehat{F}_N \subset U_N^+$, where L_N is the free product of N copies of $\mathbb{Z}_2$, and where F_N is the free group on N generators.*

Proof. All these assertions are elementary, as follows:

(1) This is clear from definitions, with the remark that, in the context of Definition 2.1 above, the formula $S(u_{ij}) = u_{ji}^*$ shows that the matrix $\bar{u}$ must be unitary too.

(2) This follows from the Gelfand theorem. To be more precise, this shows that we have presentation results for $C(O_N), C(U_N)$, similar to those in Proposition 2.16, but with the commutativity between the standard coordinates and their adjoints added:

$$\begin{aligned} C(O_N) &= C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big| u = \bar{u}, u^t = u^{-1}\right) \\ C(U_N) &= C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big| u^* = u^{-1}, u^t = \bar{u}^{-1}\right) \end{aligned}$$

Thus, we are led to the conclusion in the statement.

(3) This follows from (1) and from Proposition 2.5 above, with the remark that with $u = diag(g_1, \ldots, g_N)$, the condition $u = \bar{u}$ is equivalent to $g_i^2 = 1$, for any i. $\square$

As an interesting philosophical conclusion, if we denote by L_N^+, F_N^+ the discrete quantum groups which are dual to O_N^+, U_N^+, then we have embeddings as follows:

$$L_N \subset L_N^+ \quad , \quad F_N \subset F_N^+$$

Thus F_N^+ is a kind of "free free group", and L_N^+ is its real counterpart.

The last assertion in Theorem 2.17, making a connection with the noncommutative geometry considerations from section 1 above, suggests the following construction:

Proposition 2.18. *Given a closed subgroup $G \subset U_N^+$, consider its "diagonal torus", which is the closed subgroup $T \subset G$ constructed as follows:*

$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

This torus is then a group dual, $T = \widehat{\Lambda}$, where $\Lambda =< g_1, \ldots, g_N >$ is the discrete group generated by the elements $g_i = u_{ii}$, which are unitaries inside $C(T)$.

Proof. Since u is unitary, its diagonal entries $g_i = u_{ii}$ are unitaries inside $C(T)$. Moreover, from $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

It follows that we have $C(T) = C^*(\Lambda)$, modulo identifying as usual the C^*-completions of the various group algebras, and so that we have $T = \widehat{\Lambda}$, as claimed. □

With this notion in hand, Theorem 2.17 (3) tells us that the diagonal tori of O_N^+, U_N^+ are the group duals $\widehat{L}_N, \widehat{F}_N$. There is an obvious relation here with the noncommutative geometry considerations from section 1 above, that we will analyse later on.

Here is now a more subtle result on O_N^+, U_N^+, having no classical counterpart:

Proposition 2.19. *Consider the quantum groups O_N^+, U_N^+, with the corresponding fundamental corepresentations denoted v, u, and let $z = id \in C(\mathbb{T})$.*

(1) *We have a morphism $C(U_N^+) \to C(\mathbb{T}) * C(O_N^+)$, given by $u = zv$.*
(2) *In other words, we have a quantum group embedding $\widetilde{O_N^+} \subset U_N^+$.*
(3) *This embedding is an isomorphism at the level of the diagonal tori.*

Proof. The first two assertions follow from Proposition 2.12 above, or simply from the fact that $u = zv$ is biunitary. As for the third assertion, the idea here is that we have a similar model for the free group F_N, which is well-known to be faithful, $F_N \subset \mathbb{Z} * L_N$. □

We will be back to the above morphism later on, with a proof of its faithfulness, after performing a suitable GNS construction, with respect to the Haar functionals.

As a conclusion here, modulo some results which are still to be worked out, the relation between O_N^+, U_N^+ is in fact simpler than the one between O_N, U_N, which appears by complexification at the Lie algebra level. We will see later on that, from many other points of view, the quantum groups O_N^+, U_N^+ are in fact "simpler" than O_N, U_N.

Let us construct now some more examples of compact quantum groups. As a simple construction here, we can introduce some intermediate liberations, as follows:

Proposition 2.20. *We have intermediate quantum groups as follows,*

$$\begin{array}{ccccc} U_N & \longrightarrow & U_N^* & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \end{array}$$

with $*$ standing for the fact that u_{ij}, u_{ij}^* must satisfy the relations $abc = cba$.

Proof. This is elementary, by using the fact that if the entries of $u = (u_{ij})$ half-commute, then so do the entries of the following matrices:

$$u_{ij}^\Delta = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u_{ij}^\varepsilon = \delta_{ij} \quad , \quad u_{ij}^S = u_{ji}^*$$

Thus, we have indeed morphisms Δ, ε, S, as in Definition 2.1. See [35], [37]. □

In the same spirit, we have as well intermediate spheres as follows, with the symbol $*$ standing for the fact that x_i, x_i^* must satisfy the relations $abc = cba$:

$$\begin{array}{ccccc} S^{N-1}_{\mathbb{C}} & \longrightarrow & S^{N-1}_{\mathbb{C},*} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},+} \end{array}$$

These constructions might seem quite anecdotical, but they are not. We will see later on that, under very strong combinatorial axioms, the half-commutation relations $abc = cba$ are the only possible relaxations of the commutation relations $ab = ba$.

At the level of the diagonal tori, we have the following result:

Theorem 2.21. *The tori of the basic spheres and quantum groups are as follows,*

$$\begin{array}{ccccc} \widehat{\mathbb{Z}^N} & \longrightarrow & \widehat{\mathbb{Z}^{\circ N}} & \longrightarrow & \widehat{\mathbb{Z}^{*N}} \\ \uparrow & & \uparrow & & \uparrow \\ \widehat{\mathbb{Z}_2^N} & \longrightarrow & \widehat{\mathbb{Z}_2^{\circ N}} & \longrightarrow & \widehat{\mathbb{Z}_2^{*N}} \end{array}$$

with $\circ$ standing for the half-classical product operation for groups.

Proof. The result on the left is well-known, the result on the right follows from Theorem 2.17 (3), and the middle result follows as well, by imposing the relations $abc = cba$. □

Let us discuss now the relation with the noncommutative spheres. Having the things started here is a bit tricky, and as a main source of inspiration, we have:

Proposition 2.22. *Given an algebraic manifold $X\subset S^{N-1}_\mathbb{C}$, the formula*

$$G(X)=\left\{U\in U_N\Big|U(X)=X\right\}$$

defines a compact group of unitary matrices (a.k.a. isometries), called affine isometry group of X. For the spheres $S^{N-1}_\mathbb{R},S^{N-1}_\mathbb{C}$ we obtain in this way the groups O_N,U_N.

Proof. The fact that $G(X)$ as defined above is indeed a group is clear, its compactness is clear as well, and finally the last assertion is clear as well. In fact, all this works for any closed subset $X\subset\mathbb{C}^N$, but we are not interested here in such general spaces. □

In the case of the spheres, $G(X)$ leaves invariant as well the Riemannian metric, simply because this metric is equivalent to the one inherited from $\mathbb{C}^N$, which is preserved by our isometries $U\in U_N$. Thus, we could have constructed as well $G(X)$ as being the group of metric isometries of X, with of course some extra care in relation with the complex structure, as for $X=S^{N-1}_\mathbb{C}$ to obtain $G(X)=U_N$ instead of $G(X)=O_{2N}$. However, in the noncommutative setting, all this becomes considerably more complicated.

We have the following quantum analogue of this construction:

Proposition 2.23. *Given an algebraic manifold $X\subset S^{N-1}_{\mathbb{C},+}$, the category of the closed subgroups $G\subset U_N^+$ acting affinely on X, in the sense that the formula*

$$\Phi(x_i)=\sum_a u_{ia}\otimes x_a$$

defines a morphism of C^*-algebras $\Phi:C(X)\to C(G)\otimes C(X)$, has a universal object, denoted $G^+(X)$, and called affine quantum isometry group of X.

Proof. Observe first that in the case where the above morphism Φ exists, this morphism is automatically a coaction, in the sense that it satisfies the following conditions:

$$(id\otimes\Phi)\Phi=(\Delta\otimes id)\Phi\quad,\quad(\varepsilon\otimes id)\Phi=id$$

In order to prove now the result, assume that $X\subset S^{N-1}_{\mathbb{C},+}$ comes as follows:

$$C(X)=C(S^{N-1}_{\mathbb{C},+})\Big/\Big\langle f_\alpha(x_1,\ldots,x_N)=0\Big\rangle$$

Our claim is that the universal quantum group $G=G^+(X)$ in the statement appears as follows, where $X_i=\sum_a u_{ia}\otimes x_a\in C(U_N^+)\otimes C(X)$:

$$C(G)=C(U_N^+)\Big/\Big\langle f_\alpha(X_1,\ldots,X_N)=0\Big\rangle$$

In order to prove this claim, we have to clarify how the relations $f_\alpha(X_1,\ldots,X_N)=0$ are interpreted inside $C(U_N^+)$, and then show that G is indeed a quantum group.

So, pick one of the defining polynomials, $f = f_\alpha$, and write it as follows:

$$f(x_1,\ldots,x_N)=\sum_r\sum_{i_1^r\ldots i_{s_r}^r}\lambda_r\cdot x_{i_1^r}\ldots x_{i_{s_r}^r}$$

With $X_i=\sum_a u_{ia}\otimes x_a$ as above, we have the following formula:

$$f(X_1,\ldots,X_N)=\sum_r\sum_{i_1^r\ldots i_{s_r}^r}\lambda_r\sum_{a_1^r\ldots a_{s_r}^r}u_{i_1^ra_1^r}\ldots u_{i_{s_r}^ra_{s_r}^r}\otimes x_{a_1^r}\ldots x_{a_{s_r}^r}$$

Since the variables on the right span a certain finite dimensional space, the relations $f(X_1,\ldots,X_N)=0$ correspond to certain relations between the variables u_{ij}. Thus, we have indeed a subspace $G\subset U_N^+$, with a universal map $\Phi:C(X)\to C(G)\otimes C(X)$.

In order to show now that G is a quantum group, consider the following elements:

$$u_{ij}^\Delta=\sum_k u_{ik}\otimes u_{kj}\quad,\quad u_{ij}^\varepsilon=\delta_{ij}\quad,\quad u_{ij}^S=u_{ji}^*$$

If we consider the associated elements $X_i^\gamma=\sum_a u_{ia}^\gamma\otimes x_a$, with $\gamma\in\{\Delta,\varepsilon,S\}$, then from the relations $f(X_1,\ldots,X_N)=0$ we deduce that we have:

$$f(X_1^\gamma,\ldots,X_N^\gamma)=(\gamma\otimes id)f(X_1,\ldots,X_N)=0$$

Thus we can map $u_{ij}\to u_{ij}^\gamma$ for any $\gamma\in\{\Delta,\varepsilon,S\}$, and we are done. $\square$

We can formulate a quantum isometry group result, from [4], as follows:

Theorem 2.24. *The quantum isometry groups of the basic spheres, namely*

$$\begin{array}{ccccc}
S^{N-1}_{\mathbb C}&\longrightarrow&S^{N-1}_{\mathbb C,*}&\longrightarrow&S^{N-1}_{\mathbb C,+}\\
\uparrow&&\uparrow&&\uparrow\\
S^{N-1}_{\mathbb R}&\longrightarrow&S^{N-1}_{\mathbb R,*}&\longrightarrow&S^{N-1}_{\mathbb R,+}
\end{array}$$

are the basic orthogonal and unitary quantum groups, namely

$$\begin{array}{ccccc}
U_N&\longrightarrow&U_N^*&\longrightarrow&U_N^+\\
\uparrow&&\uparrow&&\uparrow\\
O_N&\longrightarrow&O_N^*&\longrightarrow&O_N^+
\end{array}$$

modulo identifying, as usual, the various C^*-algebraic completions.

Proof. Let us first construct an action $U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$. We must prove here that the variables $X_i = \sum_a u_{ia} \otimes x_a$ satisfy the defining relations for $S^{N-1}_{\mathbb{C},+}$, namely $\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$. But this follows from the biunitarity of u, via the following computations:

$$\sum_i X_i X_i^* = \sum_{iab} u_{ia} u_{ib}^* \otimes x_a x_b^* = \sum_a 1 \otimes x_a x_a^* = 1 \otimes 1$$

$$\sum_i X_i^* X_i = \sum_{iab} u_{ia}^* u_{ib} \otimes x_a^* x_b = \sum_a 1 \otimes x_a^* x_a = 1 \otimes 1$$

Regarding now $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$, here we must check the extra relations $X_i = X_i^*$, and these are clear from $u_{ia} = u_{ia}^*$. Finally, regarding the remaining actions, the verifications are clear as well, because if the coordinates u_{ia} and x_a are subject to commutation relations of type $ab = ba$, or of type $abc = cba$, then so are the variables $X_i = \sum_a u_{ia} \otimes x_a$.

We must prove now that all these actions are universal:

$\underline{S^{N-1}_{\mathbb{R},+}, S^{N-1}_{\mathbb{C},+}}$. The universality of $U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$ is trivial by definition. As for the universality of $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$, this comes from the fact that $X_i = X_i^*$, with $X_i = \sum_a u_{ia} \otimes x_a$ as above, gives $u_{ia} = u_{ia}^*$. Thus $G \curvearrowright S^{N-1}_{\mathbb{R},+}$ implies $G \subset O_N^+$, as desired.

$\underline{S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{C}}}$. We use here a trick from [40]. Assuming first that we have an action $G \curvearrowright S^{N-1}_{\mathbb{C}}$, in terms of the projective coordinates $w_{ij,ab} = u_{ia} u_{jb}^*$ and $p_{ij} = x_i x_j^*$, the projective coaction map is given by $\Phi(p_{ij}) = \sum_{ab} w_{ij,ab} \otimes p_{ab}$, and we have:

$$\begin{aligned}
\Phi(p_{ij}) &= \sum_{a<b} (w_{ij,ab} + w_{ij,ba}) \otimes p_{ab} + \sum_a w_{ij,aa} \otimes p_{aa} \\
\Phi(p_{ji}) &= \sum_{a<b} (w_{ji,ab} + w_{ji,ba}) \otimes p_{ab} + \sum_a w_{ji,aa} \otimes p_{aa}
\end{aligned}$$

By comparing these two formulae, and then by using the linear independence of the variables $p_{ab} = x_a x_b^*$ for $a \leq b$, we conclude that we must have:

$$w_{ij,ab} + w_{ij,ba} = w_{ji,ab} + w_{ji,ba}$$

Let us apply now the antipode to this formula. For this purpose, observe first that we have $S(w_{ij,ab}) = S(u_{ia} u_{jb}^*) = S(u_{jb}^*) S(u_{ia}) = u_{bj} u_{ai}^* = w_{ba,ji}$. Thus by applying the antipode we obtain $w_{ba,ji} + w_{ab,ji} = w_{ba,ij} + w_{ab,ij}$, and by relabelling, we obtain:

$$w_{ji,ba} + w_{ij,ba} = w_{ji,ab} + w_{ij,ab}$$

Now by comparing with the original relation, we obtain $w_{ij,ab} = w_{ji,ba}$. But, with $w_{ij,ab} = u_{ia} u_{jb}^*$, this formula reads $u_{ia} u_{jb}^* = u_{jb}^* u_{ia}$. Thus $G \subset U_N$, as claimed.

Finally, the result for $O_N \curvearrowright S^{N-1}_{\mathbb{R}}$ follows from $U_N \curvearrowright S^{N-1}_{\mathbb{C}}$ and $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$.

$S^{N-1}_{\mathbb{R},*}, S^{N-1}_{\mathbb{C},*}$. Assume that we have an action $G \curvearrowright S^{N-1}_{\mathbb{C},*}$. From $\Phi(x_i) = \sum_a u_{ia} \otimes x_a$ we obtain then that we have $\Phi(p_{ij}) = \sum_{ab} u_{ia}u^*_{jb} \otimes p_{ab}$, with $p_{ab} = z_a\bar{z}_b$. We have:

$$\begin{aligned}
\Phi(p_{ij}p_{kl}) &= \sum_{abcd} u_{ia}u^*_{jb}u_{kc}u^*_{ld} \otimes p_{ab}p_{cd} \\
\Phi(p_{il}p_{kj}) &= \sum_{abcd} u_{ia}u^*_{ld}u_{kc}u^*_{jb} \otimes p_{ad}p_{cb}
\end{aligned}$$

The left terms being equal, and the last terms on the right being equal too, we deduce that, with $[a, b, c] = abc - cba$, we must have the following equality:

$$\sum_{abcd} u_{ia}[u^*_{jb}, u_{kc}, u^*_{ld}] \otimes p_{ab}p_{cd} = 0$$

Since the variables $p_{ab}p_{cd} = z_a\bar{z}_b z_c\bar{z}_d$ depend only on $|\{a, c\}|, |\{b, d\}| \in \{1, 2\}$, and this dependence produces the only relations between them, we are led to 4 equations:

(1) $u_{ia}[u^*_{jb}, u_{ka}, u^*_{lb}] = 0$, $\forall a, b$.

(2) $u_{ia}[u^*_{jb}, u_{ka}, u^*_{ld}] + u_{ia}[u^*_{jd}, u_{ka}, u^*_{lb}] = 0$, $\forall a$, $\forall b \neq d$.

(3) $u_{ia}[u^*_{jb}, u_{kc}, u^*_{lb}] + u_{ic}[u^*_{jb}, u_{ka}, u^*_{lb}] = 0$, $\forall a \neq c$, $\forall b$.

(4) $u_{ia}([u^*_{jb}, u_{kc}, u^*_{ld}]+[u^*_{jd}, u_{kc}, u^*_{lb}])+u_{ic}([u^*_{jb}, u_{ka}, u^*_{ld}]+[u^*_{jd}, u_{ka}, u^*_{lb}]) = 0, \forall a \neq c, \forall b \neq d$.

From (1,2) we conclude that (2) holds with no restriction on the indices. By multiplying now this formula to the left by u^*_{ia}, and then summing over i, we obtain:

$$[u^*_{jb}, u_{ka}, u^*_{ld}] + [u^*_{jd}, u_{ka}, u^*_{lb}] = 0$$

By applying now the antipode, then the involution, and finally by suitably relabelling all the indices, we successively obtain from this formula:

$$\begin{aligned}
& [u_{dl}, u^*_{ak}, u_{bj}] + [u_{bl}, u^*_{ak}, u_{dj}] = 0 \\
\implies & [u^*_{dl}, u_{ak}, u^*_{bj}] + [u^*_{bl}, u_{ak}, u^*_{dj}] = 0 \\
\implies & [u^*_{ld}, u_{ka}, u^*_{jb}] + [u^*_{jd}, u_{ka}, u^*_{lb}] = 0
\end{aligned}$$

Now by comparing with the original relation, above, we conclude that we have:

$$[u^*_{jb}, u_{ka}, u^*_{ld}] = [u^*_{jd}, u_{ka}, u^*_{lb}] = 0$$

Thus we have reached to the formulae defining U^*_N, and we are done.

Finally, in what regards the universality of $O^*_N \curvearrowright S^{N-1}_{\mathbb{R},*}$, this follows from the universality of $U^*_N \curvearrowright S^{N-1}_{\mathbb{C},*}$ and of $O^+_N \curvearrowright S^{N-1}_{\mathbb{R},+}$, and from $U^*_N \cap O^+_N = O^*_N$. $\square$

Summarizing, in respect to the noncommutative geometry questions raised in section 1 above, we have some advances. In order to further advance, we would need representation theory results, in the spirit of [95], for our quantum isometry groups.

3. Representation theory

In order to reach to some more advanced insight into the structure of the compact quantum groups, we can use representation theory. We will be interested here in the finite dimensional unitary smooth representations, which provide a good picture.

In algebraic terms, the definition that we need is as follows:

Definition 3.1. *A corepresentation of a Woronowicz algebra A, with dense $*$-subalgebra of smooth elements $\mathcal{A} \subset A$, is a unitary matrix $v \in M_n(\mathcal{A})$ satisfying:*

1. $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$.
2. $\varepsilon(v_{ij}) = \delta_{ij}$.
3. $S(v_{ij}) = v_{ji}^*$.

Observe that this is precisely the notion that we used in section 2, at various places.

As basic examples of such corepresentations we have the fundamental one $u = (u_{ij})$, its complex conjugate $\bar{u} = (u_{ij}^*)$, as well as the trivial corepresentation $1 = (1)$.

It is possible to combine these examples via various product constructions, as to obtain a whole family of corepresentations, and we will discuss this in a moment.

In the classical case, we recover in this way the usual representations of G:

Proposition 3.2. *Given a closed subgroup $G \subset U_N$, the corepresentations of the associated Woronowicz algebra $C(G)$ are in one-to-one correspondence, given by*

$$\pi(g) = \begin{pmatrix} v_{11}(g) & \dots & v_{1n}(g) \\ \vdots & & \vdots \\ v_{n1}(g) & \dots & v_{nn}(g) \end{pmatrix}$$

with the finite dimensional unitary smooth representations of G.

Proof. We first recall, from section 2 above, that any closed subgroup $G \subset U_N$ is a Lie group, and that with $A = C(G)$ we have $\mathcal{A} = C^\infty(G)$. Thus, the corepresentations that we are interested in are certain square matrices of the following type:

$$v \in M_n(C^\infty(G))$$

With this observation in hand, the fact that we have a correspondence $v \leftrightarrow \pi$ as in the statement is clear, by using the computations from the proof of Proposition 2.2. Alternatively, we can apply first Proposition 2.14, and then use Proposition 2.2. $\square$

In general now, we have the following operations on the corepresentations:

Proposition 3.3. *The corepresentations are subject to the following operations:*

1. *Making sums, $v + w = diag(v, w)$.*
2. *Making tensor products, $(v \otimes w)_{ia,jb} = v_{ij} w_{ab}$.*
3. *Taking conjugates, $(\bar{v})_{ij} = v_{ij}^*$.*

Proof. Observe that the result holds in the commutative case, where we obtain the usual operations on the representations of the corresponding group. In general now:

(1) Everything here is clear, as already mentioned in the proof of Proposition 2.10.

(2) First of all, the matrix $v\otimes w$ is unitary, because we have:

$$\sum_{jb}(v\otimes w)_{ia,jb}(v\otimes w)^*_{kc,jb}=\sum_{jb}v_{ij}w_{ab}w_{cb}^*v_{kj}^*=\delta_{ac}\sum_j v_{ij}v_{kj}^*=\delta_{ik}\delta_{ac}$$

$$\sum_{jb}(v\otimes w)^*_{jb,ia}(v\otimes w)_{jb,kc}=\sum_{jb}w_{ba}^*v_{ji}^*v_{jk}w_{bc}=\delta_{ik}\sum_b w_{ba}^*w_{bc}=\delta_{ik}\delta_{ac}$$

The comultiplicativity condition follows from the following computation:

$$\Delta((v\otimes w)_{ia,jb})=\sum_{kc}v_{ik}w_{ac}\otimes v_{kj}w_{cb}=\sum_{kc}(v\otimes w)_{ia,kc}\otimes(v\otimes w)_{kc,jb}$$

The proof of the counitality condition is similar, as follows:

$$\varepsilon((v\otimes w)_{ia,jb})=\delta_{ij}\delta_{ab}=\delta_{ia,jb}$$

As for the condition involving the antipode, this can be checked as follows:

$$S((v\otimes w)_{ia,jb})=w_{ba}^*v_{ji}^*=(v\otimes w)^*_{jb,ia}$$

(3) In order to check that $\bar{v}$ is unitary, we can use the antipode. Indeed, by regarding the antipode as an antimultiplicative map $S:A\to A$, we have:

$$(\bar{v}v^t)_{ij}=\sum_k v_{ik}^*v_{jk}=\sum_k S(v_{kj}^*v_{ki})=S((v^*v)_{ji})=\delta_{ij}$$

$$(v^t\bar{v})_{ij}=\sum_k v_{ki}v_{kj}^*=\sum_k S(v_{jk}v_{ik}^*)=S((vv^*)_{ji})=\delta_{ij}$$

As for the comultiplicativity axioms, these are all clear. $\square$

We have as well the following supplementary operation:

Proposition 3.4. *Given a corepresentation $v\in M_n(A)$, its spinned version*

$$w=UvU^*$$

is a corepresentation as well, for any unitary matrix $U\in U_n$.

Proof. The matrix w is unitary, and its comultiplicativity properties can be checked by doing some computations. Here is however another proof of this fact, using a useful trick. In the context of Definition 3.1, if we write $v\in M_n(\mathbb{C})\otimes A$, the axioms read:

$$(id\otimes\Delta)v=v_{12}v_{13}\quad,\quad(id\otimes\varepsilon)v=1\quad,\quad(id\otimes S)v=v^*$$

Here we use standard tensor calculus conventions. Now when spinning by a unitary the matrix that we obtain, with these conventions, is $w=U_1vU_1^*$, and we have:

$$(id\otimes\Delta)w=U_1v_{12}v_{13}U_1^*=U_1v_{12}U_1^*\cdot U_1v_{13}U_1^*=w_{12}w_{13}$$

The proof of the counitality condition is similar, as follows:

$$(id \otimes \varepsilon)w = U \cdot 1 \cdot U = 1$$

Finally, the last condition, involving the antipode, can be checked as follows:

$$(id \otimes S)w = U_1 v^* U_1^* = w^*$$

Thus, with usual notations, $w = UvU^*$ is a corepresentation, as claimed. □

As a philosophical comment, the above proof might suggest that the more abstract our notations and formalism, the easier our problems will become. This is wrong. Bases and indices are a blessing: they can be understood by undergraduate students, computers, fellow scientists, engineers, and of course also by yourself, when you're tired or so.

In addition, in the quantum group context, we will see later on, starting from section 5 below, that bases and indices can be turned into something very beautiful and powerful, allowing us to do some serious theory, well beyond the level of abstractions.

Back to work now, in the group dual case, we have the following result:

Proposition 3.5. *Assume $A = C^*(\Gamma)$, with $\Gamma =< g_1, \ldots, g_N >$ being a discrete group.*

1. *Any group element $h \in \Gamma$ is a 1-dimensional corepresentation of A, and the operations on corepresentations are the usual ones on group elements.*
2. *Any diagonal matrix of type $v = diag(h_1, \ldots, h_n)$, with $n \in \mathbb{N}$ arbitrary, and with $h_1, \ldots, h_n \in \Gamma$, is a corepresentation of A.*
3. *More generally, any matrix of type $w = Udiag(h_1, \ldots, h_n)U^*$ with $h_1, \ldots, h_n \in \Gamma$ and with $U \in U_n$, is a corepresentation of A.*

Proof. These assertions are all elementary, as follows:

(1) The first assertion is clear from definitions and from the comultiplication, counit and antipode formulae for the discrete group algebras, namely:

$$\Delta(h) = h \otimes h \quad , \quad \varepsilon(h) = 1 \quad , \quad S(h) = h^{-1}$$

The assertion on the operations is clear too, because we have:

$$(g) \otimes (h) = (gh) \quad , \quad \overline{(g)} = (g^{-1})$$

(2) This follows from (1) by performing sums, as in Proposition 3.3 above.

(3) This follows from (2) and from the fact that we can conjugate any corepresentation by a unitary matrix, as explained in Proposition 3.4 above. □

Observe that the class of corepresentations in (3) is stable under all the operations from Propositions 3.3 and 3.4. When Γ is abelian we can apply Proposition 3.2 with $G = \widehat{\Gamma}$, and after performing a number of identifications, we conclude that these are all the corepresentations of $C^*(\Gamma)$. We will see later on that this holds in fact for any Γ.

Let us go back now to the general case. Our next definition is:

Definition 3.6. *Given two corepresentations $v\in M_n(A),w\in M_m(A)$, we set*

$$Hom(v,w)=\left\{T\in M_{m\times n}(\mathbb C)\Big|Tv=wT\right\}$$

and we use the following conventions:

1. *We use the notations $Fix(v)=Hom(1,v)$, and $End(v)=Hom(v,v)$.*
2. *We write $v\sim w$ when $Hom(v,w)$ contains an invertible element.*
3. *We say that v is irreducible, and write $v\in Irr(G)$, when $End(v)=\mathbb C1$.*

In the classical case $A=C(G)$ we obtain the usual notions concerning the representations. Observe also that in the group dual case we have $g\sim h$ when $g=h$. Finally, observe that $v\sim w$ means that v,w are conjugated by an invertible matrix.

Here are a few basic results, regarding the Hom spaces:

Proposition 3.7. *We have the following results:*

1. *$T\in Hom(u,v),S\in Hom(v,w)\implies ST\in Hom(u,w)$.*
2. *$S\in Hom(p,q),T\in Hom(v,w)\implies S\otimes T\in Hom(p\otimes v,q\otimes w)$.*
3. *$T\in Hom(v,w)\implies T^*\in Hom(w,v)$.*

In other words, the Hom spaces form a tensor $*$-category.

Proof. The proofs are all elementary, as follows:

(1) By using our assumptions $Tu=vT$ and $Sv=Ws$ we obtain, as desired:

$$STu=SvT=wST$$

(2) Assume indeed that we have $Sp=qS$ and $Tv=wT$. With tensor product notations, as in the proof of Proposition 3.4 above, we have:

$$(S\otimes T)(p\otimes v)=S_1T_2p_{13}v_{23}=(Sp)_{13}(Tv)_{23}$$

$$(q\otimes w)(S\otimes T)=q_{13}w_{23}S_1T_2=(qS)_{13}(wT)_{23}$$

The quantities on the right being equal, this gives the result.

(3) By conjugating, and then using the unitarity of v,w, we obtain, as desired:

$$\begin{aligned}Tv=wT&\implies v^*T^*=T^*w^*\\&\implies vv^*T^*w=vT^*w^*w\\&\implies T^*w=vT^*\end{aligned}$$

Finally, the last assertion follows from definitions, and from the obvious fact that, in addition to (1,2,3) above, the Hom spaces are linear spaces, and contain the units. In short, this is just a theoretical remark, that will be used only later on. ☐

As a main consequence of the above result, the spaces $End(v)\subset M_n(\mathbb C)$ are unital subalgebras stable under the involution $*$, and so are C^*-algebras.

In order to exploit this fact, we will need a basic result, complementing the operator algebra theory presented in section 1 above, namely:

Proposition 3.8. *Let $B \subset M_n(\mathbb{C})$ be a C^*-algebra.*

(1) *The unit decomposes as $1 = p_1 + \ldots + p_k$, with $p_i \in B$ minimal projections.*
(2) *Each of the linear spaces $B_i = p_i B p_i$ is a non-unital $*$-subalgebra of B.*
(3) *We have a non-unital $*$-algebra sum decomposition $B = B_1 \oplus \ldots \oplus B_k$.*
(4) *We have unital $*$-algebra isomorphisms $B_i \simeq M_{r_i}(\mathbb{C})$, where $r_i = rank(p_i)$.*
(5) *Thus, we have a C^*-algebra isomorphism $B \simeq M_{r_1}(\mathbb{C}) \oplus \ldots \oplus M_{r_k}(\mathbb{C})$.*

In addition, the final conclusion holds for any finite dimensional C^*-algebra.

Proof. This is something well-known, with the proof of the assertions (1,2,3,4) in the statement being something elementary, and routine.

With these ingredients in hand, the final conclusion (5) follows.

As for the last assertion, this follows from (5) by using the GNS representation theorem, which provides us with an embedding $B \subset M_n(\mathbb{C})$, for some $n \in \mathbb{N}$. □

We can now formulate our first Peter-Weyl type theorem, from [98], as follows:

Theorem 3.9. *Let $v \in M_n(A)$ be a corepresentation, consider the C^*-algebra $B = End(v)$, and write its unit as $1 = p_1 + \ldots + p_k$, as above. We have then*

$$v = v_1 + \ldots + v_k$$

with each v_i being an irreducible corepresentation, obtained by restricting v to $Im(p_i)$.

Proof. This can be deduced from Proposition 3.8 above, as follows:

(1) We first associate to our corepresentation $v \in M_n(A)$ the corresponding coaction map $\Phi : \mathbb{C}^n \to A \otimes \mathbb{C}^n$, given by $\Phi(e_i) = \sum_j v_{ij} \otimes e_j$. We say that a linear subspace $V \subset \mathbb{C}^n$ is invariant if $\Phi(V) \subset A \otimes V$. In this case, the restriction map $\Phi_{|V} : V \to A \otimes V$ is a coaction map too, which must come from a subcorepresentation $w \subset v$.

(2) Consider now a projection $p \in End(v)$. From $pv = vp$ we obtain that the linear space $V = Im(p)$ is invariant under v, and so this space must come from a subcorepresentation $w \subset v$. It is routine to check that the operation $p \to w$ maps subprojections to subcorepresentations, and minimal projections to irreducible corepresentations.

(3) With these preliminaries in hand, let us decompose the algebra $End(v)$ as in Proposition 3.8 above, by using the decomposition $1 = p_1 + \ldots + p_k$ into minimal projections. If we denote by $v_i \subset v$ the subcorepresentation coming from the vector space $V_i = Im(p_i)$, then we obtain in this way a decomposition $v = v_1 + \ldots + v_k$, as in the statement. □

In order to formulate our second Peter-Weyl type theorem, we will need:

Definition 3.10. *We denote by $u^{\otimes k}$, with $k = \circ\bullet\bullet\circ\ldots$ being a colored integer, the various tensor products between $u, \bar{u}$, indexed according to the rules*

$$u^{\otimes\emptyset} = 1 \quad , \quad u^{\otimes\circ} = u \quad , \quad u^{\otimes\bullet} = \bar{u}$$

and multiplicativity, $u^{\otimes kl} = u^{\otimes k} \otimes u^{\otimes l}$, and call them Peter-Weyl corepresentations.

Here are a few examples of such corepresentations, namely those coming from the colored integers of length 2, to be often used in what follows:

$$u^{\otimes\circ\circ}=u\otimes u\quad,\quad u^{\otimes\bullet\bullet}=\bar{u}\otimes\bar{u}$$

$$u^{\otimes\circ\bullet}=u\otimes\bar{u}\quad,\quad u^{\otimes\bullet\circ}=\bar{u}\otimes u$$

There are some particular cases of interest, where simplifications appear:

Proposition 3.11. *The Peter-Weyl corepresentations $u^{\otimes k}$ are as follows:*

(1) *In the real case, $u=\bar{u}$, we can assume $k\in\mathbb{N}$.*
(2) *In the classical case, we can assume, up to equivalence, $k\in\mathbb{N}\times\mathbb{N}$.*

Proof. These assertions are both elementary, as follows:

(1) Here we have indeed the following formula, where $|k|\in\mathbb{N}$ is the length:

$$u^{\otimes k}=u^{\otimes|k|}$$

Thus the Peter-Weyl corepresentations are indexed by $\mathbb{N}$, as claimed.

(2) In the classical case, our claim is that we have equivalences $v\otimes w\sim w\otimes v$, implemented by the flip operator $\Sigma(a\otimes b)=b\otimes a$. Indeed, we have:

$$v\otimes w=v_{13}w_{23}=w_{23}v_{13}=\Sigma w_{13}v_{23}\Sigma=\Sigma(w\otimes v)\Sigma$$

In particular we have an equivalence $u\otimes\bar{u}\sim\bar{u}\otimes u$, and so the Peter-Weyl corepresentations follow to be the corepresentations of type $u^{\otimes k}\otimes\bar{u}^{\otimes l}$, with $k,l\in\mathbb{N}$. $\square$

Observe that, modulo equivalence, the conclusion in (1) extends to the case where we have $u\sim\bar{u}$. A similar discussion applies to (2), in the case $u\otimes\bar{u}\sim\bar{u}\otimes u$.

Here is the second Peter-Weyl theorem, also from [98], complementing Theorem 3.9:

Theorem 3.12. *Each irreducible corepresentation of a Woronowicz algebra A appears inside a tensor product of the fundamental corepresentation u and its adjoint $\bar{u}$.*

Proof. Given an arbitrary corepresentation $v\in M_n(A)$, consider its space of coefficients, $C(v)=span(v_{ij})$. It is routine to check that the construction $v\to C(v)$ is functorial, in the sense that it maps subcorepresentations into subspaces.

By definition of the Peter-Weyl corepresentations, we have:

$$\mathcal{A}=\sum_{k\in\mathbb{N}*\mathbb{N}}C(u^{\otimes k})$$

Now given a corepresentation $v\in M_n(A)$, the corresponding coefficient space is a finite dimensional subspace $C(v)\subset\mathcal{A}$, and so we must have, for certain $k_1,\ldots,k_p$:

$$C(v)\subset C(u^{\otimes k_1}\oplus\ldots\oplus u^{\otimes k_p})$$

We deduce from this that we have an inclusion of corepresentations, as follows:

$$v\subset u^{\otimes k_1}\oplus\ldots\oplus u^{\otimes k_p}$$

Together with Theorem 3.9, this leads to the conclusion in the statement. $\square$

In order to further advance, with some finer results, we need to integrate over G. In the classical case the existence of such an integration is well-known, as follows:

Proposition 3.13. *Any commutative Woronowicz algebra, $A = C(G)$ with $G \subset U_N$, has a unique faithful positive unital linear form $\int_G : A \to \mathbb{C}$ satisfying*

$$\int_G f(xy)dx = \int_G f(yx)dx = \int_G f(x)dx$$

called Haar integration. This Haar integration functional can be constructed by starting with any faithful positive unital form $\varphi \in A^*$, and taking the Cesàro limit

$$\int_G = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi^{*k}$$

where the convolution operation for linear forms is given by $\phi * \psi = (\phi \otimes \psi)\Delta$.

Proof. This is the existence theorem for the Haar measure of G, in functional analytic formulation. Observe first that the invariance conditions in the statement read:

$$d(xy) = d(yx) = dx \quad , \quad \forall y \in G$$

Thus, we are looking indeed for the integration with respect to the Haar measure on G. Now recall that this Haar measure exists, is unique, and can be constructed by starting with any probability measure μ, and performing the following Cesàro limit:

$$dx = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d\mu^{*k}(x)$$

In functional analysis terms, this corresponds precisely to the second assertion. □

The above statement and proof are of course more of a reminder, with all the details missing. However, we will reprove all this later on, as a particular case of a general Haar integration existence result, in the general Woronowicz algebra setting.

In general now, let us start with a definition, as follows:

Definition 3.14. *Given an arbitrary Woronowicz algebra $A = C(G)$, any positive unital tracial state $\int_G : A \to \mathbb{C}$ subject to the invariance conditions*

$$\left(\int_G \otimes id\right)\Delta = \left(id \otimes \int_G\right)\Delta = \int_G (.)1$$

is called Haar integration over G.

As a first observation, in the commutative case, this notion agrees with the one in Proposition 3.13. To be more precise, Proposition 3.13 tells us that any commutative Woronowicz algebra has a Haar integration in the above sense, which is unique, and which can be constructed by performing the Cesàro limiting procedure there.

Let us discuss now the group dual case. We have here the following result:

Proposition 3.15. *Given a discrete group $\Gamma =< g_1,\ldots,g_N >$, the Woronowicz algebra $A=C^*(\Gamma)$ has a Haar functional, given on the standard generators $g\in\Gamma$ by:*

$$\int_{\widehat{\Gamma}}g=\delta_{g,1}$$

This functional is faithful on the image on $C^*(\Gamma)$ in the regular representation. Also, in the abelian case, we obtain in this way the counit of $C(\widehat{\Gamma})$.

Proof. Consider indeed the left regular representation $\pi:C^*(\Gamma)\to B(l^2(\Gamma))$, given by $\pi(g)(h)=gh$, that we have already met, in the proof of Proposition 1.18 above.

By composing this representation with the linear functional $T\to< T1,1 >$, the functional $\int_{\widehat{\Gamma}}$ that we obtain is given by the following formula:

$$\int_{\widehat{\Gamma}}g=< g1,1 >=\delta_{g,1}$$

But this gives all the assertions in the statement, namely the existence, traciality, left and right invariance properties, and faithfulness on the reduced algebra.

As for the last assertion, this is clear from the Pontrjagin duality isomorphism. □

With a bit of functional analysis knowledge, one can improve the above result, with a proof of the fact that the Haar integration is unique, and appears via a Cesàro limiting procedure, as in Proposition 3.13. We will do this directly, in the general case.

In order to discuss now the general case, let us define the convolution operation for linear forms by $\phi*\psi=(\phi\otimes\psi)\Delta$. We have then the following result, from [98]:

Proposition 3.16. *Given an arbitrary unital linear form $\varphi\in A^*$, the limit*

$$\int_\varphi a=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(a)$$

exists, and for a coefficient of a corepresentation $a=(\tau\otimes id)v$, we have

$$\int_\varphi a=\tau(P)$$

where P is the orthogonal projection onto the 1-eigenspace of $(id\otimes\varphi)v$.

Proof. By linearity and continuity, it is enough to prove the first assertion for elements of type $a=(\tau\otimes id)v$, where v is one of the Peter-Weyl corepresentations, and τ is a linear form. Thus we are led into the second assertion, and more precisely we can have the whole result proved if we can establish the following formula, with $a=(\tau\otimes id)v$:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(a)=\tau(P)$$

In order to prove this latter formula, observe that we have:

$$\varphi^{*k}(a) = (\tau \otimes \varphi^{*k})v = \tau((id \otimes \varphi^{*k})v)$$

Also, in terms of the matrix $M = (id \otimes \varphi)v$, we have the following formula:

$$((id \otimes \varphi^{*k})v)_{i_0 i_{k+1}} = \sum_{i_1 \ldots i_k} M_{i_0 i_1} \ldots M_{i_k i_{k+1}} = (M^k)_{i_0 i_{k+1}}$$

Thus $(id \otimes \varphi^{*k})v = M^k$ for any $k \in \mathbb{N}$, and our Cesàro limit is given by:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}(a) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \tau(M^k) = \tau\left(\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n M^k\right)$$

Now since v is unitary we have $||v|| = 1$, and so $||M|| \leq 1$, and the Cesàro limit on the right exists, and equals the orthogonal projection onto the 1-eigenspace of M:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n M^k = P$$

Thus our initial Cesàro limit converges as well, to $\tau(P)$, as desired. $\square$

When φ is chosen faithful, we have the following finer result, also from [98]:

Proposition 3.17. *Given a faithful unital linear form $\varphi \in A^*$, the limit*

$$\int_\varphi a = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}(a)$$

exists, and is independent of φ, given on coefficients of corepresentations by

$$\left(id \otimes \int_\varphi\right) v = P$$

where P is the orthogonal projection onto $Fix(v) = \{\xi \in \mathbb{C}^n | v\xi = \xi\}$.

Proof. In view of Proposition 3.16, it remains to prove that when φ is faithful, the 1-eigenspace of $M = (id \otimes \varphi)v$ equals the fixed point space $Fix(v) = \{\xi \in \mathbb{C}^n | v\xi = \xi\}$.

"$\supset$" This is clear, and for any φ, because $v\xi = \xi$ implies $M\xi = \xi$.

"$\subset$" Here we must prove that, when φ is faithful, $M\xi = \xi$ implies $v\xi = \xi$. For this purpose, assume that we have $M\xi = \xi$, and consider the following element:

$$a = \sum_i \left(\sum_j v_{ij}\xi_j - \xi_i\right)\left(\sum_k v_{ik}\xi_k - \xi_i\right)^*$$

We must prove that we have $a=0$. Since v is biunitary, we have:

$$\begin{aligned}
a &= \sum_i\left(\sum_j\left(v_{ij}\xi_j-\frac{1}{N}\xi_i\right)\right)\left(\sum_k\left(v_{ik}^*\bar{\xi}_k-\frac{1}{N}\bar{\xi}_i\right)\right)\\
&= \sum_{ijk} v_{ij}v_{ik}^*\xi_j\bar{\xi}_k-\frac{1}{N}v_{ij}\xi_j\bar{\xi}_i-\frac{1}{N}v_{ik}^*\xi_i\bar{\xi}_k+\frac{1}{N^2}\xi_i\bar{\xi}_i\\
&= \sum_j|\xi_j|^2-\sum_{ij}v_{ij}\xi_j\bar{\xi}_i-\sum_{ik}v_{ik}^*\xi_i\bar{\xi}_k+\sum_i|\xi_i|^2\\
&= ||\xi||^2-<v\xi,\xi>-\overline{<v\xi,\xi>}+||\xi||^2\\
&= 2(||\xi||^2-Re(<v\xi,\xi>))
\end{aligned}$$

By using now our assumption $M\xi=\xi$, we obtain from this:

$$\begin{aligned}
\varphi(a) &= 2\varphi(||\xi||^2-Re(<v\xi,\xi>))\\
&= 2(||\xi||^2-Re(<M\xi,\xi>))\\
&= 2(||\xi||^2-||\xi||^2)\\
&= 0
\end{aligned}$$

Now since φ is faithful, this gives $a=0$, and so $v\xi=\xi$, as claimed. □

We can now formulate a main result, due to Woronowicz [98], is as follows:

Theorem 3.18. *Any Woronowicz algebra has a unique Haar integration functional, which can be constructed by starting with any faithful positive unital state $\varphi\in A^*$, and setting*

$$\int_G=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}$$

where $\phi*\psi=(\phi\otimes\psi)\Delta$. Moreover, for any corepresentation v we have

$$\left(id\otimes\int_G\right)v=P$$

where P is the orthogonal projection onto $Fix(v)=\{\xi\in\mathbb{C}^n|v\xi=\xi\}$.

Proof. Let us first go back to the general context of Proposition 3.16 above. Since convolving one more time with φ will not change the Cesàro limit appearing there, the functional $\int_\varphi\in A^*$ constructed there has the following invariance property:

$$\int_\varphi*\varphi=\varphi*\int_\varphi=\int_\varphi$$

In the case where φ is assumed to be faithful, as in Proposition 3.17 above, our claim is that we have the following formula, valid this time for any $\psi \in A^*$:

$$\int_\varphi *\psi = \psi * \int_\varphi = \psi(1)\int_\varphi$$

It is enough to prove this formula on a coefficient of a corepresentation, $a = (\tau \otimes id)v$. In order to do so, observe that with $P = (id \otimes \int_\varphi)v$ and $Q = (id \otimes \psi)v$ we have:

$$\left(\int_\varphi *\psi\right) a = \left(\tau \otimes \int_\varphi \otimes \psi\right)(v_{12}v_{13}) = \tau(PQ)$$

Similarly, we have the following computation:

$$\left(\psi * \int_\varphi\right) a = \left(\tau \otimes \psi \otimes \int_\varphi\right)(v_{12}v_{13}) = \tau(QP)$$

Finally, regarding the term on the right, this is given by:

$$\psi(1)\int_\varphi a = \psi(1)\tau(P)$$

Thus, our claim is equivalent to the following equality:

$$PQ = QP = \psi(1)P$$

But this latter equality follows from the fact, coming from Proposition 3.17 above, that $P = (id \otimes \int_\varphi)v$ equals the orthogonal projection onto $Fix(v) = \{\xi \in \mathbb{C}^n | v\xi = \xi\}$.

Thus, we have proved our claim. Now observe that our formula can be written as:

$$\psi\left(\int_\varphi \otimes id\right)\Delta = \psi\left(id \otimes \int_\varphi\right)\Delta = \psi\int_\varphi(.)1$$

This formula being true for any $\psi \in A^*$, we can simply delete ψ, and we conclude that the invariance formula in Definition 3.14 holds indeed, with $\int_G = \int_\varphi$.

Finally, assuming that we have two invariant integrals $\int_G, \int_G'$, we have:

$$\left(\int_G \otimes \int_G'\right)\Delta = \left(\int_G' \otimes \int_G\right)\Delta = \int_G(.)1 = \int_G'(.)1$$

Thus we have $\int_G = \int_G'$, and this finishes the proof. See [98]. $\square$

As a first illustration, for the basic product operations, we have:

Proposition 3.19. *We have the following results:*

(1) *For a product $G \times H$, we have $\int_{G\times H} = \int_G \otimes \int_H$.*
(2) *For a dual free product $G \,\hat{*}\, H$, we have $\int_{G \hat{*} H} = \int_G * \int_H$.*
(3) *For a quotient $G \to H$, we have $\int_H = \left(\int_G\right)_{|C(H)}$.*
(4) *For a projective version $G \to PG$, we have $\int_{PG} = \left(\int_G\right)_{|C(PG)}$.*

Proof. These formulae all follow from the invariance property, as follows:

(1) Here the tensor product form $\int_G \otimes \int_H$ satisfies the left and right invariance properties of the Haar functional $\int_{G\times H}$, and so by uniqueness, it is equal to it.

(2) Here the situation is similar, with the free product of linear forms being defined with some inspiration from the discrete group case, where $\int_{\widehat{\Gamma}} g = \delta_{g,1}$.

(3) Here the restriction $\left(\int_G\right)_{|C(H)}$ satisfies by definition the required left and right invariance properties, so once again we can conclude by uniqueness.

(4) Here we simply have a particular case of (3) above. □

We will need the following result, which is of independent interest:

Proposition 3.20. *We have a Frobenius type isomorphism*

$$Hom(v,w) \simeq Fix(v \otimes \bar{w})$$

valid for any two corepresentations v,w.

Proof. According to the definitions, we have the following equivalences:

$$T \in Hom(v,w) \iff Tv = wT \iff \sum_j T_{aj} v_{ji} = \sum_b w_{ab} T_{bi}, \forall a,i$$

$$T \in Fix(v \otimes \bar{w}) \iff (v \otimes \bar{w})T = \xi \iff \sum_{jb} v_{ij} w_{ab}^* T_{bj} = T_{ai} \forall a,i$$

With these formulae in hand, both inclusions follow from the biunitarity of v,w. □

We can now formulate our third Peter-Weyl theorem, from [98], as follows:

Theorem 3.21. *The dense subalgebra $\mathcal{A} \subset A$ decomposes as a direct sum*

$$\mathcal{A} = \bigoplus_{v \in Irr(A)} M_{\dim(v)}(\mathbb{C})$$

with this being an isomorphism of $*$-coalgebras, and with the summands being pairwise orthogonal with respect to the scalar product given by

$$< a,b >= \int_G ab^*$$

where $\int_G$ is the Haar integration over G.

Proof. By combining the previous Peter-Weyl results, from Theorem 3.9 and Theorem 3.12 above, we deduce that we have a linear space decomposition as follows:

$$\mathcal{A} = \sum_{v \in Irr(A)} C(v) = \sum_{v \in Irr(A)} M_{\dim(v)}(\mathbb{C})$$

Thus, in order to conclude, it is enough to prove that for any two irreducible corepresentations $v,w\in Irr(A)$, the corresponding spaces of coefficients are orthogonal:

$$v\not\sim w\implies C(v)\perp C(w)$$

But this follows from Theorem 3.18, via Proposition 3.20. Let us set indeed:

$$P_{ia,jb}=\int_G v_{ij}w_{ab}^*$$

Then P is the orthogonal projection onto the following vector space:

$$Fix(v\otimes\bar{w})\simeq Hom(v,w)=\{0\}$$

Thus we have $P=0$, and this gives the result. □

We can obtain further results by using characters, which are defined as follows:

Proposition 3.22. *The characters of the corepresentations, given by*

$$\chi_v=\sum_i v_{ii}$$

behave as follows, in respect to the various operations:

$$\chi_{v+w}=\chi_v+\chi_w\quad,\quad\chi_{v\otimes w}=\chi_v\chi_w\quad,\quad\chi_{\bar{v}}=\chi_v^*$$

In addition, given two equivalent corepresentations, $v\sim w$, we have $\chi_v=\chi_w$.

Proof. The three formulae in the statement are all clear from definitions. Regarding now the last assertion, assuming that we have $v=T^{-1}wT$, we obtain:

$$\chi_v=Tr(v)=Tr(T^{-1}wT)=Tr(w)=\chi_w$$

We conclude that $v\sim w$ implies $\chi_v=\chi_w$, as claimed. □

We have the following result, also from [98], completing the Peter-Weyl theory:

Theorem 3.23. *The characters of irreducible corepresentations belong to the algebra*

$$\mathcal{A}_{central}=\left\{a\in\mathcal{A}\Big|\Sigma\Delta(a)=\Delta(a)\right\}$$

of "smooth central functions" on G, and form an orthonormal basis of it.

Proof. As a first remark, the linear space $\mathcal{A}_{central}$ defined above is indeed an algebra. In the classical case, we obtain the usual algebra of smooth central functions. Also, in the group dual case, where we have $\Sigma\Delta=\Delta$, we obtain the whole convolution algebra.

Observe also that $\mathcal{A}_{central}$ contains all the characters, because we have:

$$\Delta(\chi_v)=\Delta\left(\sum_i v_{ii}\right)=\sum_{ij}v_{ij}\otimes v_{ji}$$

Conversely, for an element $a \in \mathcal{A}$, written $a = \sum_{v \in Irr(A)} a_v$, the condition $a \in \mathcal{A}_{central}$ is equivalent to $a_v \in \mathcal{A}_{central}$ for any $v \in Irr(A)$. But $a_v \in \mathcal{A}_{central}$ means that a_v must be a scalar multiple of χ_v, and so the characters form a basis of $\mathcal{A}_{central}$, as stated.

Finally, the fact that we have an orthogonal basis follows from Theorem 3.21. As for the fact that the characters have norm 1, this follows from:

$$\int_G \chi_v \chi_v^* = \sum_{ij} \int_G v_{ii} v_{jj}^* = \sum_i \frac{1}{N} = 1$$

Here we have used the fact that the integrals $\int_G v_{ij} v_{kl}^*$ form the orthogonal projection onto the vector space $Fix(v \otimes \bar{v}) \simeq End(v) = \mathbb{C}1$, coming from Proposition 3.20. □

As a first application of the Peter-Weyl theory, we have:

Proposition 3.24. *Let $\Gamma =< g_1, \ldots, g_N >$ be a finitely generated discrete group.*

(1) *The 1-dimensional corepresentations of $C^*(\Gamma)$ are the group elements $g \in \Gamma$.*
(2) *The corepresentations of $C^*(\Gamma)$ are the direct sums of such group elements.*

Proof. This follows from the Peter-Weyl theory. Indeed, the tensor products between $u, \bar{u}$ are the matrices of type $u^{\otimes k} = diag(g_{i_1} \ldots g_{i_k})$, and so we are done. □

We can solve as well now a problem that we left open in section 2, namely:

Proposition 3.25. *The cocommutative Woronowicz algebras appear as the quotients*

$$C^*(\Gamma) \to A \to C^*_{red}(\Gamma)$$

given by $A = C^*_\pi(\Gamma)$ with $\pi \otimes \pi \subset \pi$, with Γ being a discrete group.

Proof. This follows as well from the Peter-Weyl theory. Observe that the assumption $\pi \otimes \pi \subset \pi$, which should be taken in a weak containment sense, is satisfied for the regular representation, as well as the universal representation. □

At the level of the examples coming from operations, we have:

Proposition 3.26. *We have the following results:*

(1) *The irreducible corepresentations of $C(G \times H)$ are the tensor products of the form $v \otimes w$, with v, w being irreducible corepresentations of $C(G), C(H)$.*
(2) *The irreducible corepresentations of $C(G \hat{*} H)$ appear as alternating tensor products of irreducible corepresentations of $C(G)$ and of $C(H)$.*
(3) *The irreducible corepresentations of $C(H) \subset C(G)$ are the irreducible corepresentations of $C(G)$ whose coefficients belong to $C(H)$.*
(4) *The irreducible corepresentations of $C(PG) \subset C(G)$ are the irreducible corepresentations of $C(G)$ which appear by decomposing the tensor powers of $u \otimes \bar{u}$.*

Proof. This is routine, the idea being as follows:

(1) Here we can integrate characters, by using Proposition 3.19 (1), and we conclude that if v, w are irreducible corepresentations of $C(G), C(H)$, then $v \otimes w$ is an irreducible corepresentation of $C(G \times H)$. Now since the coefficients of these latter corepresentations span $\mathcal{C}(G \times H)$, by Peter-Weyl these are all the irreducible corepresentations.

(2) Here we can use a similar method. By using Proposition 3.19 (2) we conclude that if $v_1, v_2, \ldots$ are irreducible corepresentations of $C(G)$ and $w_1, w_2, \ldots$ are irreducible corepresentations of $C(H)$, then $v_1 \otimes w_1 \otimes v_2 \otimes w_2 \otimes \ldots$ is an irreducible corepresentation of $C(G \,\hat{*}\, H)$, and then we can conclude by using the Peter-Weyl theory.

(3) This is clear from definitions, and from the Peter-Weyl theory.

(4) This is a particular case of the result (3) above. $\square$

Let us discuss now the notion of amenability. We have the following result:

Theorem 3.27. *Let A_{full} be the enveloping C^*-algebra of $\mathcal{A}$, and let A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:*

1. *The Haar functional of A_{full} is faithful.*
2. *The projection map $A_{full} \to A_{red}$ is an isomorphism.*
3. *The counit map $\varepsilon : A_{full} \to \mathbb{C}$ factorizes through A_{red}.*
4. *We have $N \in \sigma(Re(\chi_u))$, the spectrum being taken inside A_{red}.*

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

Proof. This is well-known in the group dual case, $A = C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) $\iff$ (2) This follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red}.

(2) $\iff$ (3) The implication $\implies$ is trivial. Conversely, assume that we have a counit map $\varepsilon : A_{red} \to \mathbb{C}$. We use the standard fact that the comultiplication of $\mathcal{A}$ can be extended, via a formula of type $\Phi(a) = W(a \otimes 1)W^*$, into a map as follows:

$$\Phi : A_{red} \to A_{red} \otimes A_{full}$$

The composition $(\varepsilon \otimes id)\Phi$ is then our desired isomorphism.

(3) $\iff$ (4) The implication $\implies$ is clear, because from $\varepsilon(u_{ii}) = 1$ for any i, we obtain the following formula:

$$\varepsilon(N - Re(\chi(u))) = 0$$

Thus the element $N - Re(\chi(u))$ is not invertible in A_{red}, as claimed.

Conversely, with $v = u \oplus \bar{u}$, our assumption reads $\dim v \in \sigma(\chi_v)$. By functional calculus the same holds for $w = v + 1$, and then again by functional calculus, the same holds for any tensor power $w_k = w^{\otimes k}$. Now choose for each $k \in \mathbb{N}$ a state $\varepsilon_k \in A_{red}^*$ having the property $\varepsilon_k(w_k) = \dim w_k$. By Peter-Weyl we must have $\varepsilon_k(v) = \dim v$, for any $v \leq w_k$, and since, again by Peter-Weyl, each irreducible corepresentation of A appears in this way, the sequence ε_k converges to a counit map $\varepsilon : A_{red} \to \mathbb{C}$, as desired. See [75]. $\square$

Here are some basic applications of the above result:

Proposition 3.28. *We have the following results:*

(1) *The compact Lie groups $G \subset U_N$ are all coamenable.*
(2) *A group dual $G = \widehat{\Gamma}$ is coamenable precisely when Γ is amenable.*
(3) *A product $G \times H$ of coamenable compact quantum groups is coamenable.*

Proof. This follows indeed from the results that we have:

(1) This is clear by using any of the criteria in Theorem 3.27 above, because for an algebra of type $A = C(G)$, we have $A_{full} = A_{red}$.

(2) Here the various criteria in Theorem 3.27 above correspond to the various equivalent definitions of the amenability of a discrete group.

(3) This follows from the description of the Haar functional of $C(G \times H)$, from Proposition 3.19 (1) above. Indeed, if $\int_G, \int_H$ are both faithful, then so is $\int_G \times \int_H$. □

Summarizing, we have a fully satisfactory generalized Peter-Weyl theory, which can be used for various purposes, including the study of amenability.

4. Tannakian duality

In order to have more insight into the structure of the compact quantum groups, and to effectively compute their representations, we can use algebraic geometry methods, and more precisely Tannakian duality. We will present here Woronowicz's Tannakian duality result from [99], in its "soft" form, worked out by Malacarne in [70].

Let us start with the following definition:

Definition 4.1. *The Tannakian category associated to a Woronowicz algebra (A,u) is the collection $C=(C(k,l))$ of vector spaces*

$$C(k,l)=Hom(u^{\otimes k},u^{\otimes l})$$

where $u^{\otimes k}$ with $k=\circ\bullet\bullet\circ\ldots$ colored integer are the Peter-Weyl corepresentations.

We know from Proposition 3.7 above that C is a tensor $*$-category. To be more precise, if we denote by $H\simeq\mathbb{C}^N$ the Hilbert space where $u\in M_N(A)$ coacts, then C is a tensor $*$-subcategory of the tensor $*$-category formed by the following linear spaces:

$$E(k,l)=\mathcal{L}(H^{\otimes k},H^{\otimes l})$$

Here the tensor powers $H^{\otimes k}$ with $k=\circ\bullet\bullet\circ\ldots$ colored integer are those where the corepresentations $u^{\otimes k}$ act, defined by the following formulae, and multiplicativity:

$$H^{\otimes\emptyset}=\mathbb{C}\quad,\quad H^{\otimes\circ}=H\quad,\quad H^{\otimes\bullet}=\bar{H}\simeq H$$

Our purpose in what follows will be that of reconstructing (A,u) in terms of the category $C=(C(k,l))$. As a first, elementary result on the subject, we have:

Proposition 4.2. *Given a morphism $\pi:(A,u)\to(B,v)$ we have inclusions*

$$Hom(u^{\otimes k},u^{\otimes l})\subset Hom(v^{\otimes k},v^{\otimes l})$$

for any k,l, and if these inclusions are all equalities, π is an isomorphism.

Proof. The fact that we have indeed inclusions as in the statement is clear from definitions. As for the last assertion, this follows from the Peter-Weyl theory. Indeed, if we assume that π is not an isomorphism, then one of the irreducible corepresentations of A must become reducible as a corepresentation of B. But the irreducible corepresentations being subrepresentations of the Peter-Weyl corepresentations $u^{\otimes k}$, one of the spaces $End(u^{\otimes k})$ must therefore increase strictly, and this gives the desired contradiction. □

The Tannakian duality result that we want to prove states, in a simplified form, that in what concerns the last conclusion in the above statement, the assumption that we have a morphism $\pi:(A,u)\to(B,v)$ is not needed. In other words, if we know that the Tannakian categories of A,B are different, then A,B themselves must be different.

In order to get started now, our first goal will be that of gaining some familiarity with the notion of Tannakian category. And, as a starting point here, we have to use the only general fact that we know about u, namely that this matrix is biunitary.

The biunitarity condition translates as follows:

Proposition 4.3. *An abstract matrix $u\in M_N(A)$ is biunitary if and only if*

$$R\in Hom(1,u\otimes\bar{u})\quad,\quad R\in Hom(1,\bar{u}\otimes u)$$

$$R^*\in Hom(u\otimes\bar{u},1)\quad,\quad R^*\in Hom(\bar{u}\otimes u,1)$$

where $R:\mathbb{C}\to\mathbb{C}^N\otimes\mathbb{C}^N$ is the linear operator given by $R(1)=\sum_ie_i\otimes e_i$.

Proof. With R being as in the statement, we have the following computation:

$$(u\otimes\bar{u})(R(1)\otimes 1)=\sum_{ijk}e_i\otimes e_k\otimes u_{ij}u_{kj}^*=\sum_{ik}e_i\otimes e_k\otimes(uu^*)_{ik}$$

We conclude from this that we have:

$$R\in Hom(1,u\otimes\bar{u})\iff uu^*=1$$

Consider now the adjoint operator $R^*:\mathbb{C}^N\otimes\mathbb{C}^N\to\mathbb{C}$, which is given by:

$$R^*(e_i\otimes e_j)=\delta_{ij}$$

We have then the following computation:

$$(R^*\otimes id)(u\otimes\bar{u})(e_j\otimes e_l\otimes 1)=\sum_iu_{ij}u_{il}^*=(u^t\bar{u})_{jl}$$

We conclude from this that we have:

$$R^*\in Hom(u\otimes\bar{u},1)\iff u^t\bar{u}=1$$

Similarly, or simply by replacing u in the above two conclusions with its conjugate $\bar{u}$, which is a corepresentation too, we have as well the following two equivalences:

$$R\in Hom(1,\bar{u}\otimes u)\iff \bar{u}u^t=1$$

$$R^*\in Hom(\bar{u}\otimes u,1)\iff u^*u=1$$

Thus, we are led to the biunitarity conditions, and we are done. □

As a consequence of this computation, we have the following result:

Proposition 4.4. *The Tannakian category $C=(C(k,l))$ associated to a Woronowicz algebra (A,u) must contain the operators*

$$R:1\to\sum_ie_i\otimes e_i\quad,\quad R^*(e_i\otimes e_j)=\delta_{ij}$$

in the sense that we must have:

$$R\in C(\emptyset,\circ\bullet)\quad,\quad R\in C(\emptyset,\bullet\circ)\quad,\quad R^*\in C(\circ\bullet,\emptyset)\quad,\quad R^*\in C(\bullet\circ,\emptyset)$$

In fact, C must contain the whole tensor category $<R,R^*>$ generated by R,R^*.

Proof. The first assertion is clear from the above result. As for the second assertion, this is clear from definitions, because $C=(C(k,l))$ is indeed a tensor category. □

Let us formulate now the following key definition:

Definition 4.5. *Let H be a finite dimensional Hilbert space. A tensor category over H is a collection $C=(C(k,l))$ of subspaces $C(k,l)\subset\mathcal{L}(H^{\otimes k},H^{\otimes l})$ satisfying:*

(1) *$S,T\in C$ implies $S\otimes T\in C$.*
(2) *If $S,T\in C$ are composable, then $ST\in C$.*
(3) *$T\in C$ implies $T^*\in C$.*
(4) *Each $C(k,k)$ contains the identity operator.*
(5) *$C(\emptyset,\circ\bullet)$ and $C(\emptyset,\bullet\circ)$ contain the operator $R:1\to\sum_i e_i\otimes e_i$.*

As a first observation, this formalism generalizes the Tannakian category formalism from Definition 4.1 above, because we have:

Proposition 4.6. *Let (A,u) be a Woronowicz algebra, with fundamental corepresentation $u\in M_N(A)$. The associated Tannakian category $C=(C(k,l))$, given by*

$$C(k,l)=Hom(u^{\otimes k},u^{\otimes l})$$

is then a tensor category over the Hilbert space $H=\mathbb{C}^N$.

Proof. The fact that the above axioms (1-4) are indeed satisfied is clear, and the validity of the axiom (5) follows from Proposition 4.4 above. □

Our main purpose in what follows will be that of proving that the converse of the above statement holds. In other words, we would like to prove that any tensor category in the sense of Definition 4.5 must appear as a Tannakian category.

As a first result on this subject, we have:

Proposition 4.7. *Given a tensor category $C=(C(k,l))$, the following algebra, with u being the fundamental corepresentation of $C(U_N^+)$, is a Woronowicz algebra:*

$$A_C=C(U_N^+)\Big/\left\langle T\in Hom(u^{\otimes k},u^{\otimes l})\Big|\forall k,l,\forall T\in C(k,l)\right\rangle$$

In the case where C comes from a Woronowicz algebra (A,v), we have a quotient map $A_C\to A$. Moreover, this map is an isomorphism in the discrete group algebra case.

Proof. Given colored integers k,l and an arbitrary linear operator $T\in\mathcal{L}(H^{\otimes k},H^{\otimes l})$, consider the following $*$-ideal of the algebra $C(U_N^+)$:

$$I=\left\langle T\in Hom(u^{\otimes k},u^{\otimes l})\right\rangle$$

Our claim is that I is a Hopf ideal. Indeed, with $U=\sum_k u_{ik}\otimes u_{kj}$, it is elementary to check that we have the following implication, which proves our claim:

$$T\in Hom(u^{\otimes k},u^{\otimes l})\implies T\in Hom(U^{\otimes k},U^{\otimes l})$$

With this claim in hand, A_C appears from $C(U_N^+)$ by dividing by a collection of Hopf ideals, and is therefore a Woronowicz algebra. It is also clear that we have a quotient map $A_C\to A$, simply because the relations defining A_C are satisfied in A.

Regarding now the last assertion, assume that we are in the case $A = C^*(\Gamma)$, with $\Gamma =< g_1, \ldots, g_N >$ being a finitely generated discrete group. If we write $\Gamma = F_N/\mathcal{R}$, with $\mathcal{R}$ being the complete collection of relations between the generators, then we have:

$$A_C = C^*\left(F_N\Big/\Big\langle\mathcal{R}\Big\rangle\right)$$

Thus the quotient map $A_C \to A$ is indeed an isomorphism, as claimed. □

With the above construction in hand, the Tannakian duality theorem that we want to prove states that the operations $A \to A_C$ and $C \to C_A$ are inverse to each other.

We have the following result, which simplifies our work:

Proposition 4.8. *Consider the following conditions:*

(1) $C = C_{A_C}$*, for any Tannakian category* C*.*

(2) $A = A_{C_A}$*, for any Woronowicz algebra* (A, u)*.*

We have then $(1) \implies (2)$*. Also,* $C \subset C_{A_C}$ *is automatic.*

Proof. Given a Woronowicz algebra (A, u), let us set $C = C_A$. By using (1) we have then $C_A = C_{A_{C_A}}$. On the other hand, by Proposition 4.7 above we have an arrow $A_{C_A} \to A$. Thus, we are in the general situation from Proposition 4.2 above, with a surjective arrow of Woronowicz algebras, which becomes an isomorphism at the level of the associated Tannakian categories. We conclude that Proposition 4.2 can be applied, and this gives the isomorphism of the associated Woronowicz algebras, $A_{C_A} = A$, as desired.

Finally, the fact that we have an inclusion $C \subset C_{A_C}$ is clear from definitions. □

Summarizing, we would like to prove that we have $C_{A_C} \subset C$, for any Tannakian category C. Let us begin with some abstract constructions. Following [70], let us formulate:

Proposition 4.9. *Given a tensor category* $C = C((k, l))$ *over a Hilbert space* H*,*

$$E_C^{(s)} = \bigoplus_{|k|,|l|\leq s} C(k, l) \subset \bigoplus_{|k|,|l|\leq s} B(H^{\otimes k}, H^{\otimes l}) = B\left(\bigoplus_{|k|\leq s} H^{\otimes k}\right)$$

is a finite dimensional C^**-subalgebra. Also,*

$$E_C = \bigoplus_{k,l} C(k, l) \subset \bigoplus_{k,l} B(H^{\otimes k}, H^{\otimes l}) \subset B\left(\bigoplus_{k} H^{\otimes k}\right)$$

is a closed $*$*-subalgebra.*

Proof. This is clear indeed from the categorical axioms from Definition 4.5. □

Now back to our reconstruction question, given a tensor category $C = (C(k, l))$, we want to prove that we have $C = C_{A_C}$, which is the same as proving that we have $E_C = E_{C_{A_C}}$. Equivalently, we want to prove that we have isomorphisms as follows, for any $s \in \mathbb{N}$:

$$E_C^{(s)} = E_{C_{A_C}}^{(s)}$$

We will use a standard commutant trick, as follows:

Proposition 4.10. *For any C^*-algebra $B \subset M_n(\mathbb{C})$ we have*

$$B = B''$$

where prime denotes the commutant, $X' = \{T \in M_n(\mathbb{C}) | Tx = xT, \forall x \in X\}$.

Proof. This is a particular case of von Neumann's bicommutant theorem [73], which follows as well from the explicit description of B given in Proposition 3.8 above. To be more precise, let us decompose B as there, as a direct sum of matrix algebras:

$$B = M_{r_1}(\mathbb{C}) \oplus \ldots \oplus M_{r_k}(\mathbb{C})$$

The center of each matrix algebra being reduced to the scalars, the commutant of this algebra is then as follows, with each copy of $\mathbb{C}$ corresponding to a matrix block:

$$B' = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$

By taking once again the commutant we obtain B itself, and we are done. □

We recall that we want to prove that we have $C = C_{A_C}$, for any Tannakian category C. By using the above notions, and the bicommutant theorem, we have:

Proposition 4.11. *Given a Tannakian category C, the following are equivalent:*

1. $C = C_{A_C}$.
2. $E_C = E_{C_{A_C}}$.
3. $E_C^{(s)} = E_{C_{A_C}}^{(s)}$, *for any $s \in \mathbb{N}$.*
4. $E_C^{(s)'} = E_{C_{A_C}}^{(s)'}$, *for any $s \in \mathbb{N}$.*

In addition, the inclusions $\subset$, $\subset$, $\subset$, $\supset$ are automatically satisfied.

Proof. Here (1) $\iff$ (2) is clear from definitions, (2) $\iff$ (3) is clear from definitions as well, and (3) $\iff$ (4) comes from the bicommutant theorem. As for the last assertion, we have indeed $C \subset C_{A_C}$ from Proposition 4.8, and this shows that we have as well $E_C \subset E_{C_{A_C}}$, and then $E_C^{(s)} \subset E_{C_{A_C}}^{(s)}$, and finally $E_C^{(s)} \supset E_{C_{A_C}}^{(s)}$, by taking the commutants. □

Summarizing, in order to finish, given a tensor category $C = (C(k,l))$, we would like to prove that we have inclusions $E_C^{(s)'} \subset E_{C_{A_C}}^{(s)'}$, for any $s \in \mathbb{N}$.

Let us first study the commutant on the right. As a first observation, we have:

Proposition 4.12. *Given a Woronowicz algebra (A, u), we have*

$$E_{C_A}^{(s)} = End\left(\bigoplus_{|k| \leq s} u^{\otimes k}\right)$$

as subalgebras of $B(\oplus_{|k| \leq s} H^{\otimes k})$.

Proof. The category C_A is given by $C_A(k,l)=Hom(u^{\otimes k},u^{\otimes l})$, so according to Proposition 4.9 above, the corresponding algebra $E_{C_A}^{(s)}$ appears as follows:

$$E_{C_A}^{(s)}=\bigoplus_{|k|,|l|\leq s}Hom(u^{\otimes k},u^{\otimes l})\subset\bigoplus_{|k|,|l|\leq s}B(H^{\otimes k},H^{\otimes l})=B\left(\bigoplus_{|k|\leq s}H^{\otimes k}\right)$$

On the other hand, the algebra of intertwiners of $\bigoplus_{|k|\leq s}u^{\otimes k}$ is given by:

$$End\left(\bigoplus_{|k|\leq s}u^{\otimes k}\right)=\bigoplus_{|k|,|l|\leq s}Hom(u^{\otimes k},u^{\otimes l})\subset\bigoplus_{|k|,|l|\leq s}B(H^{\otimes k},H^{\otimes l})=B\left(\bigoplus_{|k|\leq s}H^{\otimes k}\right)$$

Thus we have indeed the same algebra, and we are done. □

We have to compute the commutant of the above algebra. For this purpose, we can use the following general result, valid for any corepresentation:

Proposition 4.13. *Given a corepresentation $v\in M_n(A)$, we have a representation*

$$\pi_v:A^*\to M_n(\mathbb{C})\quad,\quad\varphi\to(\varphi(v_{ij}))_{ij}$$

whose image is given by $Im(\pi_v)=End(v)'$.

Proof. The first assertion is clear, with the multiplicativity claim coming from:

$$\begin{aligned}(\pi_v(\varphi*\psi))_{ij}&=(\varphi\otimes\psi)\Delta(v_{ij})\\&=\sum_k\varphi(v_{ik})\psi(v_{kj})\\&=\sum_k(\pi_v(\varphi))_{ik}(\pi_v(\psi))_{kj}\\&=(\pi_v(\varphi)\pi_v(\psi))_{ij}\end{aligned}$$

Let us first prove the inclusion $\subset$. Given $\varphi\in A^*$ and $T\in End(v)$, we have:

$$\begin{aligned}[\pi_v(\varphi),T]=0&\iff\sum_k\varphi(v_{ik})T_{kj}=\sum_kT_{ik}\varphi(v_{kj}),\forall i,j\\&\iff\varphi\left(\sum_kv_{ik}T_{kj}\right)=\varphi\left(\sum_kT_{ik}v_{kj}\right),\forall i,j\\&\iff\varphi((vT)_{ij})=\varphi((Tv)_{ij}),\forall i,j\end{aligned}$$

But this latter formula is true, because $T\in End(v)$ means that we have $vT=Tv$.

As for the converse inclusion $\supset$, the proof is quite similar. Indeed, by using the bicommutant theorem, this is the same as proving that we have $Im(\pi_v)'\subset End(v)$. But, by using the above equivalences, we have the following computation:

$$\begin{aligned} T \in Im(\pi_v)' &\iff [\pi_v(\varphi), T] = 0, \forall \varphi \\ &\iff \varphi((vT)_{ij}) = \varphi((Tv)_{ij}), \forall \varphi, i, j \\ &\iff vT = Tv \end{aligned}$$

Thus, we have obtained the desired inclusion, and we are done. □

By combining the above results, we obtain:

Proposition 4.14. *Given a Woronowicz algebra (A, u), we have*

$$E_{C_A}^{(s)'} = Im(\pi_v)$$

as subalgebras of $B(\oplus_{|k|\leq s} H^{\otimes k})$, where the corepresentation v is the sum

$$v = \bigoplus_{|k|\leq s} u^{\otimes k}$$

and where $\pi_v : A^* \to M_n(\mathbb{C})$ is given by $\varphi \to (\varphi(v_{ij}))_{ij}$.

Proof. This follows indeed from Proposition 4.12 and Proposition 4.13. □

We recall that we want to prove that we have $E_C^{(s)'} \subset E_{C_{A_C}}^{(s)'}$, for any $s \in \mathbb{N}$. For this purpose, we must first refine Proposition 4.14, in the case $A = A_C$.

Generally speaking, in order to prove anything about A_C, we are in need of an explicit model for this algebra. In order to construct such a model, let $< u_{ij} >$ be the free $*$-algebra over $\dim(H)^2$ variables, with comultiplication and counit as follows:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij}$$

Following [70], we can model this $*$-bialgebra, in the following way:

Proposition 4.15. *Consider the following pair of dual vector spaces,*

$$F = \bigoplus_k B\left(H^{\otimes k}\right) \quad , \quad F^* = \bigoplus_k B\left(H^{\otimes k}\right)^*$$

and let $f_{ij}, f_{ij}^* \in F^*$ be the standard generators of $B(H)^*, B(\bar{H})^*$.

1. *F^* is a $*$-algebra, with multiplication $\otimes$ and involution $f_{ij} \leftrightarrow f_{ij}^*$.*
2. *F^* is a $*$-bialgebra, with $\Delta(f_{ij}) = \sum_k f_{ik} \otimes f_{kj}$ and $\varepsilon(f_{ij}) = \delta_{ij}$.*
3. *We have a $*$-bialgebra isomorphism $< u_{ij} >\simeq F^*$, given by $u_{ij} \to f_{ij}$.*

Proof. Since F^* is spanned by the various tensor products between the variables f_{ij}, f_{ij}^*, we have a vector space isomorphism $< u_{ij} >\simeq F^*$ given by $u_{ij} \to f_{ij}, u_{ij}^* \to f_{ij}^*$, and the corresponding $*$-bialgebra structure induced on F^* is the one in the statement. □

Now back to our algebra A_C, we have the following modelling result for it:

Proposition 4.16. *The smooth part of the algebra A_C is given by*

$$\mathcal{A}_C \simeq F^*/J$$

where $J \subset F^*$ is the ideal coming from the following relations,

$$\sum_{p_1,\ldots,p_k} T_{i_1\ldots i_l,p_1\ldots p_k} f_{p_1j_1} \otimes \ldots \otimes f_{p_kj_k} = \sum_{q_1,\ldots,q_l} T_{q_1\ldots q_l,j_1\ldots j_k} f_{i_1q_1} \otimes \ldots \otimes f_{i_lq_l} \quad , \quad \forall i,j$$

one for each pair of colored integers k,l, and each $T \in C(k,l)$.

Proof. Our first claim is that A_C appears as enveloping C^*-algebra of the following universal $*$-algebra, where $u = (u_{ij})$ is regarded as a formal corepresentation:

$$\mathcal{A}_C = \left\langle (u_{ij})_{i,j=1,\ldots,N} \middle| T \in Hom(u^{\otimes k}, u^{\otimes l}), \forall k,l, \forall T \in C(k,l) \right\rangle$$

Indeed, this follows from Proposition 4.3 above, because according to the result there, the relations defining $C(U_N^+)$ are included into those that we impose.

With this claim in hand, the conclusion is that we have a formula as follows, where I is the ideal coming from the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C(k,l)$:

$$\mathcal{A}_C = < u_{ij} > /I$$

Now if we denote by $J \subset F^*$ the image of the ideal I via the $*$-algebra isomorphism $< u_{ij} > \simeq F^*$ from Proposition 4.15, we obtain an identification as follows:

$$\mathcal{A}_C \simeq F^*/J$$

In order to compute J, let us go back to I. With standard multi-index notations, and by assuming that $k,l \in \mathbb{N}$ are usual integers, for simplifying, a relation of type $T \in Hom(u^{\otimes k}, u^{\otimes l})$ inside $< u_{ij} >$ is equivalent to the following conditions:

$$\sum_{p_1,\ldots,p_k} T_{i_1\ldots i_l,p_1\ldots p_k} u_{p_1j_1} \ldots u_{p_kj_k} = \sum_{q_1,\ldots,q_l} T_{q_1\ldots q_l,j_1\ldots j_k} u_{i_1q_1} \ldots u_{i_lq_l} \quad , \quad \forall i,j$$

Now by recalling that the isomorphism of $*$-algebras $< u_{ij} > \to F^*$ is given by $u_{ij} \to f_{ij}$, and that the multiplication operation of F^* corresponds to the tensor product operation $\otimes$, we conclude that $J \subset F^*$ is the ideal from the statement. $\square$

With the above result in hand, let us go back to Proposition 4.14. We have:

Proposition 4.17. *The linear space $\mathcal{A}_C^*$ is given by the formula*

$$\mathcal{A}_C^* = \left\{ a \in F \middle| Ta_k = a_l T, \forall T \in C(k,l) \right\}$$

and $\pi_v : \mathcal{A}_C^* \to B(\oplus_{|k|\leq s} H^{\otimes k})$ appears diagonally, by truncating, $\pi_v : a \to (a_k)_{kk}$.

Proof. We know from Proposition 4.16 that we have $\mathcal{A}_C \simeq F^*/J$, and this gives a quotient map $F^* \to \mathcal{A}_C$, and so an inclusion $\mathcal{A}_C^* \subset F$. To be more precise, we have:

$$\mathcal{A}_C^* = \left\{ a \in \ F \middle| f(a) = 0, \forall f \in J \right\}$$

Now since $J =< f_T >$, where f_T are the relations in Proposition 4.16, we obtain:

$$\mathcal{A}_C^* = \left\{ a \in F \middle| f_T(a) = 0, \forall T \in C \right\}$$

Given $T \in C(k,l)$, for an arbitrary element $a = (a_k)$, we have:

$$\begin{aligned} & f_T(a) = 0 \\ \iff & \sum_{p_1,\ldots,p_k} T_{i_1\ldots i_l, p_1\ldots p_k}(a_k)_{p_1\ldots p_k, j_1\ldots j_k} = \sum_{q_1,\ldots,q_l} T_{q_1\ldots q_l, j_1\ldots j_k}(a_l)_{i_1\ldots i_l, q_1\ldots q_l}, \forall i,j \\ \iff & (Ta_k)_{i_1\ldots i_l, j_1\ldots j_k} = (a_l T)_{i_1\ldots i_l, j_1\ldots j_k}, \forall i,j \\ \iff & Ta_k = a_l T \end{aligned}$$

Thus, the dual space $\mathcal{A}_C^*$ is given by the formula in the statement.
It remains to compute the representation π_v, which appears as follows:

$$\pi_v : \mathcal{A}_C^* \to B\left(\bigoplus_{|k|\le s} H^{\otimes k}\right)$$

With $a = (a_k)$, we have the following computation:

$$\begin{aligned} \pi_v(a)_{i_1\ldots i_k, j_1\ldots j_k} &= a(v_{i_1\ldots i_k, j_1\ldots j_k}) \\ &= (f_{i_1 j_1} \otimes \ldots \otimes f_{i_k j_k})(a) \\ &= (a_k)_{i_1\ldots i_k, j_1\ldots j_k} \end{aligned}$$

Thus, our representation π_v appears diagonally, by truncating, as claimed. □

In order to further advance, consider the following vector spaces:

$$F_s = \bigoplus_{|k|\le s} B\left(H^{\otimes k}\right) \quad , \quad F_s^* = \bigoplus_{|k|\le s} B\left(H^{\otimes k}\right)^*$$

We denote by $a \to a_s$ the truncation operation $F \to F_s$. We have then:

Proposition 4.18. *The following hold:*

(1) $E_C^{(s)'} \subset F_s$.
(2) $E_C' \subset F$.
(3) $\mathcal{A}_C^* = E_C'$.
(4) $Im(\pi_v) = (E_C')_s$.

Proof. These results basically follow from what we have, as follows:

(1) Since $F_s \subset B\left(\bigoplus_{|k|\leq s} H^{\otimes k}\right)$ is the diagonal subalgebra, its commutant is:

$$F_s' = \left\{ b \in F_s \Big| b = (b_k), b_k \in \mathbb{C}, \forall k \right\}$$

On the other hand, we know from the identity axiom for C that this algebra is contained inside $E_C^{(s)}$. Thus, our result follows from the bicommutant theorem, as follows:

$$F_s' \subset E_C^{(s)} \implies F_s \supset E_C^{(s)'}$$

(2) This follows from (1), by taking inductive limits.

(3) With the present notations, the formula of $\mathcal{A}_C^*$ from Proposition 4.17 reads:

$$\mathcal{A}_C^* = F \cap E_C'$$

Now since by (2) we have $E_C' \subset F$, we obtain from this $\mathcal{A}_C^* = E_C'$.

(4) This follows from (3), and from the formula of π_v in Proposition 4.17. $\square$

Following [70], we can now state and prove our main result, as follows:

Theorem 4.19. *The Tannakian duality constructions*

$$C \to A_C \quad , \quad A \to C_A$$

are inverse to each other, modulo identifying full and reduced versions.

Proof. According to Proposition 4.8, Proposition 4.11, Proposition 4.14 and Proposition 4.18, we have to prove that, for any Tannakian category C, and any $s \in \mathbb{N}$:

$$E_C^{(s)'} \subset (E_C')_s$$

By taking duals, this is the same as proving that we have:

$$\left\{ f \in F_s^* \Big| f_{|(E_C')_s} = 0 \right\} \subset \left\{ f \in F_s^* \Big| f_{|E_C^{(s)'}} = 0 \right\}$$

In order to establish these inclusions, we use the formula $\mathcal{A}_C^* = E_C'$, from Proposition 4.18. Since we have $\mathcal{A}_C = F^*/J$, we conclude that the ideal J is given by:

$$J = \left\{ f \in F^* \Big| f_{|E_C'} = 0 \right\}$$

Our claim is that we have the following formula, for any $s \in \mathbb{N}$:

$$J \cap F_s^* = \left\{ f \in F_s^* \Big| f_{|E_C^{(s)'}} = 0 \right\}$$

Indeed, let us denote by X_s the spaces on the right. The categorical axioms for C show that these spaces are increasing, that their union $X = \cup_s X_s$ is an ideal, and that $X_s = X \cap F_s^*$. We must prove that we have $J = X$, and this can be done as follows:

"$\subset$" This follows from the following fact, for any $T \in C(k,l)$ with $|k|, |l| \leq s$:

$$(f_T)_{|\{T\}'} = 0 \implies (f_T)_{|E_C^{(s)'}} = 0 \implies f_T \in X_s$$

"$\supset$" This follows from our description of J, because from $E_C^{(s)} \subset E_C$ we obtain:

$$f_{|E_C^{(s)\prime}} = 0 \implies f_{|E'_C} = 0$$

Summarizing, we have proved our claim. On the other hand, we have:

$$\begin{aligned} J \cap F_s^* &= \left\{ f \in F^* \middle| f_{|E'_C} = 0 \right\} \cap F_s^* \\ &= \left\{ f \in F_s^* \middle| f_{|E'_C} = 0 \right\} \\ &= \left\{ f \in F_s^* \middle| f_{|(E'_C)_s} = 0 \right\} \end{aligned}$$

Thus, our claim is exactly the inclusion that we wanted to prove, and we are done. □

As a first application, let us record the following theoretical fact, from [14]:

Proposition 4.20. *Each closed subgroup $G \subset U_N^+$ appears as an algebraic manifold of the free complex sphere, $G \subset S^{N^2-1}_{\mathbb{C},+}$, the embedding being given by $x_{ij} = \frac{u_{ij}}{\sqrt{N}}$.*

Proof. This follows from Theorem 4.19, by using the inclusions $G \subset U_N^+ \subset S^{N^2-1}_{\mathbb{C},+}$. Indeed, both these inclusions are algebraic, and this gives the result. □

As a second application of the above results, let us study the quantum groups O_N^+, U_N^+. In order to get started, let us get back to the operators R, R^*. We have:

Proposition 4.21. *The tensor category $< R, R^* >$ generated by the operators*

$$R : 1 \to \sum_i e_i \otimes e_i \quad , \quad R^*(e_i \otimes e_j) = \delta_{ij}$$

produces via Tannakian duality the algebra $C(U_N^+)$.

Proof. By Proposition 4.4 the intertwining relations coming from R, R^*, and so from any element of the tensor category $< R, R^* >$, hold automatically, so the quotient operation in Proposition 4.7 is trivial, and we obtain the algebra $C(U_N^+)$ itself, as stated. □

Our goal now will be that of reaching to a better understanding of R, R^*. In order to do so, we use a diagrammatic formalism, as follows:

Definition 4.22. *Let k, l be two colored integers, having lengths $|k|, |l| \in \mathbb{N}$.*

1. *$P_2(k, l)$ is the set of pairings between an upper row of $|k|$ points, and a lower row of $|l|$ points, with these two rows of points colored by k, l.*
2. *$\mathcal{P}_2(k, l) \subset P_2(k, l)$ is the set of matching pairings, whose horizontal strings connect $\circ - \circ$ or $\bullet - \bullet$, and whose vertical strings connect $\circ - \bullet$.*
3. *$NC_2(k, l) \subset P_2(k, l)$ is the set of pairings which are noncrossing, in the sense that we can draw the pairing as for the strings to be noncrossing.*
4. *$\mathcal{NC}_2(k, l) \subset P_2(k, l)$ is the subset of noncrossing matching pairings, obtained as an intersection, $\mathcal{NC}_2(k, l) = NC_2(k, l) \cap \mathcal{P}_2(k, l)$.*

The relation with the Tannakian categories of linear maps comes from the fact that we can associate linear maps to the pairings, as in [35], as follows:

Definition 4.23. *Associated to any pairing $\pi\in P_2(k,l)$ and any $N\in\mathbb N$ is the linear map $T_\pi:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$ given by*

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

with the Kronecker type symbols $\delta_\pi\in\{0,1\}$ depending on whether the indices fit or not.

To be more precise here, in the definition of the Kronecker symbols, we agree to put the two multi-indices on the two rows of points of the pairing, in the obvious way. The Kronecker symbols are then defined by $\delta_\pi=1$ when all the strings of π join equal indices, and by $\delta_\pi=0$ otherwise. Observe that all this is independent of the coloring.

Here are a few basic examples of such linear maps:

Proposition 4.24. *The correspondence $\pi\to T_\pi$ has the following properties:*

(1) $T_\cap=R$.
(2) $T_\cup=R^*$.
(3) $T_{||\ldots||}=id$.
(4) $T_{\slash\!\!\!\backslash}=\Sigma$.

Proof. We can assume if we want that all the upper and lower legs of π are colored $\circ$. With this assumption made, the proof goes as follows:

(1) We have $\cap\in P_2(\emptyset,\circ\circ)$, and so the corresponding operator is a certain linear map $T_\cap:\mathbb C\to\mathbb C^N\otimes\mathbb C^N$. The formula of this map is as follows:

$$T_\cap(1)=\sum_{ij}\delta_\cap(i\ j)e_i\otimes e_j=\sum_{ij}\delta_{ij}e_i\otimes e_j=\sum_i e_i\otimes e_i$$

We recognize here the formula of $R(1)$, and so we have $T_\cap=R$, as claimed.

(2) Here we have $\cup\in P_2(\circ\circ,\emptyset)$, and so the corresponding operator is a certain linear form $T_\cap:\mathbb C^N\otimes\mathbb C^N\to\mathbb C$. The formula of this linear form is as follows:

$$T_\cap(e_i\otimes e_j)=\delta_\cap(i\ j)=\delta_{ij}$$

Since this is the same as $R^*(e_i\otimes e_j)$, we have $T_\cup=R^*$, as claimed.

(3) Consider indeed the "identity" pairing $||\ldots||\in P_2(k,k)$, with $k=\circ\circ\ldots\circ\circ$. The corresponding linear map is then the identity, because we have:

$$\begin{aligned}T_{||\ldots||}(e_{i_1}\otimes\ldots\otimes e_{i_k})&=\sum_{j_1\ldots j_k}\delta_{||\ldots||}\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_k\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_k}\\&=\sum_{j_1\ldots j_k}\delta_{i_1j_1}\ldots\delta_{i_kj_k}e_{j_1}\otimes\ldots\otimes e_{j_k}\\&=e_{i_1}\otimes\ldots\otimes e_{i_k}\end{aligned}$$

(4) In the case of the basic crossing $\slashed{\times} \in P_2(\circ\circ,\circ\circ)$, the corresponding linear map $T_{\slashed{\times}} : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N \otimes \mathbb{C}^N$ can be computed as follows:

$$T_{\slashed{\times}}(e_i \otimes e_j) = \sum_{kl} \delta_{\slashed{\times}} \begin{pmatrix} i & j \\ k & l \end{pmatrix} e_k \otimes e_l = \sum_{kl} \delta_{il}\delta_{jk} e_k \otimes e_l = e_j \otimes e_i$$

Thus we obtain the flip operator $\Sigma(a \otimes b) = b \otimes a$, as claimed. □

Summarizing, the correspondence $\pi \to T_\pi$ provides simple formulae for the operators R, R^* that we are interested in, and has as well some interesting categorical properties. Let us further explore these properties. We have the following result, from [35]:

Proposition 4.25. *The assignement $\pi \to T_\pi$ is categorical, in the sense that we have*

$$T_\pi \otimes T_\sigma = T_{[\pi\sigma]} \quad , \quad T_\pi T_\sigma = N^{c(\pi,\sigma)} T_{[^\sigma_\pi]} \quad , \quad T_\pi^* = T_{\pi^*}$$

where $c(\pi,\sigma)$ are certain integers, coming from the erased components in the middle.

Proof. The concatenation axiom follows from the following computation:

$$\begin{aligned}
& (T_\pi \otimes T_\sigma)(e_{i_1} \otimes \ldots \otimes e_{i_p} \otimes e_{k_1} \otimes \ldots \otimes e_{k_r}) \\
= & \sum_{j_1 \ldots j_q} \sum_{l_1 \ldots l_s} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_p \\ j_1 & \ldots & j_q \end{pmatrix} \delta_\sigma \begin{pmatrix} k_1 & \ldots & k_r \\ l_1 & \ldots & l_s \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_q} \otimes e_{l_1} \otimes \ldots \otimes e_{l_s} \\
= & \sum_{j_1 \ldots j_q} \sum_{l_1 \ldots l_s} \delta_{[\pi\sigma]} \begin{pmatrix} i_1 & \ldots & i_p & k_1 & \ldots & k_r \\ j_1 & \ldots & j_q & l_1 & \ldots & l_s \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_q} \otimes e_{l_1} \otimes \ldots \otimes e_{l_s} \\
= & T_{[\pi\sigma]}(e_{i_1} \otimes \ldots \otimes e_{i_p} \otimes e_{k_1} \otimes \ldots \otimes e_{k_r})
\end{aligned}$$

The composition axiom follows from the following computation:

$$\begin{aligned}
& T_\pi T_\sigma(e_{i_1} \otimes \ldots \otimes e_{i_p}) \\
= & \sum_{j_1 \ldots j_q} \delta_\sigma \begin{pmatrix} i_1 & \ldots & i_p \\ j_1 & \ldots & j_q \end{pmatrix} \sum_{k_1 \ldots k_r} \delta_\pi \begin{pmatrix} j_1 & \ldots & j_q \\ k_1 & \ldots & k_r \end{pmatrix} e_{k_1} \otimes \ldots \otimes e_{k_r} \\
= & \sum_{k_1 \ldots k_r} N^{c(\pi,\sigma)} \delta_{[^\sigma_\pi]} \begin{pmatrix} i_1 & \ldots & i_p \\ k_1 & \ldots & k_r \end{pmatrix} e_{k_1} \otimes \ldots \otimes e_{k_r} \\
= & N^{c(\pi,\sigma)} T_{[^\sigma_\pi]}(e_{i_1} \otimes \ldots \otimes e_{i_p})
\end{aligned}$$

Finally, the involution axiom follows from the following computation:

$$\begin{aligned} & T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q})\\ = & \sum_{i_1\ldots i_p} < T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q}), e_{i_1}\otimes\ldots\otimes e_{i_p} > e_{i_1}\otimes\ldots\otimes e_{i_p}\\ = & \sum_{i_1\ldots i_p}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix} e_{i_1}\otimes\ldots\otimes e_{i_p}\\ = & T_{\pi^*}(e_{j_1}\otimes\ldots\otimes e_{j_q})\end{aligned}$$

Summarizing, our correspondence is indeed categorical. □

We can now formulate a first result regarding O_N^+,U_N^+, as follows:

Theorem 4.26. *For the quantum groups O_N^+,U_N^+ we have*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

with $D=NC_2,\mathcal{NC}_2$ respectively, with $\pi\to T_\pi$ being constructed as above.

Proof. We know from Proposition 4.21 that U_N^+ corresponds via Tannakian duality to the category $<R,R^*>$. On the other hand, it follows from the above categorical considerations that this latter category is given by the following formula:

$$<R,R^*>=span\left(T_\pi\Big|\pi\in\mathcal{NC}_2\right)$$

As for the result from O_N^+, this follows by adding to the picture the self-adjointness condition $u=\bar{u}$, which corresponds, at the level of pairings, to removing the colors. □

As a first application, regarding O_N^+, we have the following result, from [1]:

Theorem 4.27. *The irreducible corepresentations of O_N^+ are labelled by positive integers, and their fusion rules are the Clebsch-Gordan ones, namely*

$$r_k\otimes r_l=r_{|k-l|}+r_{|k-l|+2}+\ldots+r_{k+l}$$

as for the group SU_2. The dimensions of these corepresentations are given by

$$\dim r_k=\frac{q^{k+1}-q^{-k-1}}{q-q^{-1}}$$

where q,q^{-1} are the solutions of $X^2-NX+1=0$.

Proof. Let $\{\chi_k\}_{k\in\mathbb{N}}$ be the characters of the irreducible representations of SU_2. These characters span a complex subalgebra $A\subset C(SU_2)$, which is isomorphic to $\mathbb{C}[X]$, via $X\to\chi_1$. We can find integers $c_{kl}\in\mathbb{N}$ such that $c_{kk}=1$ and:

$$\chi_1^k=\sum_{l=0}^k c_{kl}\chi_l$$

Also, we can define a morphism $\Psi : A \to C(O_N^+)$ by $\chi_1 \to f_1$, where f_1 is the character of the fundamental representation of O_N^+. The elements $f_k = \Psi(\chi_k)$ verify then:

$$f_k f_l = f_{|k-l|} + f_{|k-l|+2} + \ldots + f_{k+l}$$

We prove now by recurrence on k that each f_k is the character of an irreducible corepresentation r_k of $C(O_N^+)$, non-equivalent to $r_0, \ldots, r_{k-1}$. At $k = 0, 1$ this is clear.

Assume now that the result holds at $k-1$. We have $f_{k-2}f_1 = f_{k-3} + f_{k-1}$, and so we get $r_{k-2} \otimes r_1 = r_{k-3} + r_{k-1}$, which gives $r_{k-1} \subset r_{k-2} \otimes r_1$. Now since r_{k-2} is irreducible, by Frobenius reciprocity we have $r_{k-2} \subset r_{k-1} \otimes r_1$, so there exists a representation r_k such that $r_{k-1} \otimes r_1 = r_{k-2} + r_k$. Since $f_{k-1}f_1 = f_{k-2} + f_k$, the character of r_k is f_k.

It remains to prove that r_k is irreducible, and non-equivalent to $r_1, \ldots, r_{k-1}$. For this purpose, observe that we have inequalities as follows:

$$\sum_{l=0}^{k} c_{kl}^2 \leq \dim(End(u^{\otimes k})) \leq \# NC_2(k,k)$$

Indeed, the first inequality comes from the fact that we have $f_1^k = \sum_{l=0}^k c_{kl} f_l$, with the remark that the equality case holds precisely when r_k is irreducible, and non-equivalent to $r_1, \ldots, r_{k-1}$. As for the second inequality, this comes from Theorem 4.26.

Our claim now, which will end the proof, is that we have equalities everywhere. But in order to prove this claim, we can ignore of course the middle quantity, and we are left with a question regarding the numbers c_{kl}, and so with a question regarding SU_2.

In order to solve this latter question, let w be the fundamental representation of SU_2. We have then some well-known equalities, as follows:

$$\sum_{l=0}^{k} c_{kl}^2 = \dim(End(w^{\otimes k})) = \# NC_2(k,k)$$

Indeed, the first equality follows from the definition of the numbers c_{kl}. As for the second equality, this is something standard, coming for instance from the fact that we have an isomorphism $SU_2 \simeq S^3_{\mathbb{R}}$, as compact measured spaces, which makes the main character χ_w correspond to a semicircular variable, having the numbers $\# NC_2(k,k)$ as moments. All this is standard, and will be discussed in section 6 below, and everything can be deduced as well directly from the Clebsch-Gordan rules.

Summariring, we have equalities everywhere, and this proves our claim. Finally, since any irreducible representation of O_N^+ must appear in some tensor power of u, and we have a formula for decomposing each $u^{\otimes k}$ into sums of representations r_l, we conclude that these representations r_l are all the irreducible representations of O_N^+.

Finally, from the Clebsch-Gordan rules we have in particular:

$$r_k r_1 = r_{k-1} + r_{k+1}$$

But this gives the dimension formula in the statement, and we are done. □

Let us investigate now the quantum group U_N^+. We first have:

Theorem 4.28. *The canonical free complexification model*
$$C(U_N^+)\to C(\mathbb T)*C(O_N^+)$$
is faithful, at the level of the reduced version algebras.

Proof. We have embeddings as follows, with the first one coming by using the counit, and with the second one coming from the universality property of U_N^+:
$$O_N^+\subset\widetilde{O_N^+}\subset U_N^+$$

If we denote by v,zv,u the corresponding fundamental corepresentations, at the level of the associated Hom spaces we obtain reverse inclusions, as follows:
$$Hom(v^{\otimes k},v^{\otimes l})\supset Hom((zv)^{\otimes k},(zv)^{\otimes l})\supset Hom(u^{\otimes k},u^{\otimes l})$$

The spaces on the left and on the right are known from Theorem 4.26 above, the result there stating that these spaces are as follows:
$$span\left(T_\pi\Big|\pi\in NC_2(k,l)\right)\supset span\left(T_\pi\Big|\pi\in\mathcal{NC}_2(k,l)\right)$$

Regarding the spaces in the middle, these are obtained from those on the left by "coloring", so we obtain the same spaces as those on the right. Thus, by Tannakian duality, our embedding $\widetilde{O_N^+}\subset U_N^+$ is an isomorphism, modulo identifying full and reduced versions. $\square$

We can now compute the fusion rules for U_N^+, and we have:

Theorem 4.29. *The irreducible corepresentations of U_N^+ are labelled by $\mathbb N*\mathbb N$, and*
$$r_k\otimes r_l=\sum_{k=xy,l=\bar{y}z}r_{xz}$$
which appear as "free complexifications" of the Clebsch-Gordan ones.

Proof. This follows by combining Theorem 4.27 and Theorem 4.28 above. Indeed, the fusion rules for $\widetilde{O_N^+}$ can be explicitely computed, and appear as "free complexifications" of the fusion rules for O_N^+, which are the Clebsch-Gordan ones. Alternatively, we can use a recurrence argument as in the proof of Theorem 4.27 above, with $r_k\subset u^{\otimes k}$ being the "new components" which appear. All this is quite technical, see [1], [77] for details. $\square$

As a conclusion, the Tannakian duality methods allow us to have a very good insight into the structure of O_N^+,U_N^+, and the same methods can be in principle applied to any quantum group whose algebra of functions is given by a simple presentation formula.

We will systematically exploit this point of view, in what follows.

5. Easiness, examples

Our purpose now will be that of extending the main findings about O_N^+, U_N^+ from the previous section to O_N, U_N too, and to other compact quantum groups as well.

Let us begin with a general definition, from [35], [83], as follows:

Definition 5.1. *Let $P(k,l)$ be the set of partitions between an upper colored integer k, and a lower colored integer l. A set $D = \bigsqcup_{k,l} D(k,l)$ with $D(k,l) \subset P(k,l)$ is called a category of partitions when it has the following properties:*

1. *Stability under the horizontal concatenation, $(\pi, \sigma) \to [\pi\sigma]$.*
2. *Stability under vertical concatenation $(\pi, \sigma) \to [^\sigma_\pi]$, with matching middle symbols.*
3. *Stability under the upside-down turning $*$, with switching of colors, $\circ \leftrightarrow \bullet$.*
4. *Each set $P(k,k)$ contains the identity partition $|| \ldots ||$.*
5. *The sets $P(\emptyset, \circ\bullet)$ and $P(\emptyset, \bullet\circ)$ both contain the semicircle $\cap$.*

We have already met a number of such categories, in Definition 4.22 above. Indeed, the sets there are categories of pairings, with inclusions between them as follows:

$$\begin{array}{ccc}\mathcal{P}_2 & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow \\ P_2 & \longleftarrow & NC_2\end{array}$$

There are many other examples of such categories, as for instance P itself, or the category $NC \subset P$ of all noncrossing partitions. We will gradually explore these examples, in what follows. For the moment, we will rather focus on the categories of pairings.

The relation with the Tannakian categories comes from:

Proposition 5.2. *Each $\pi \in P(k,l)$ produces a linear map $T_\pi : (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l}$,*

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with the Kronecker type symbols $\delta_\pi \in \{0,1\}$ depending on whether the indices fit or not. The assignement $\pi \to T_\pi$ is categorical, in the sense that we have

$$T_\pi \otimes T_\sigma = T_{[\pi\sigma]} \quad , \quad T_\pi T_\sigma = N^{c(\pi,\sigma)} T_{[^\sigma_\pi]} \quad , \quad T_\pi^* = T_{\pi^*}$$

where $c(\pi, \sigma)$ are certain integers, coming from the erased components in the middle.

Proof. This is something that we already know for the pairings, from Proposition 4.25 above. In general, the proof is identical. To be more precise, the proof of Proposition 4.25 does not use the fact that the partitions there are actually pairings. $\square$

In relation with the quantum groups, we have the following result, from [35]:

Theorem 5.3. *Each category of partitions $D = (D(k,l))$ produces a family of compact quantum groups $G = (G_N)$, one for each $N \in \mathbb{N}$, via the formula*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

which produces a Tannakian category, and the Tannakian duality correspondence.

Proof. This follows indeed from Woronowicz's Tannakian duality, in its "soft" form from [70], as explained in section 4 above. Indeed, let us set:

$$C(k,l) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

By using the axioms in Definition 5.1, and the categorical properties of the operation $\pi \to T_\pi$, from Proposition 5.2 above, we deduce that $C = (C(k,l))$ is a Tannakian category. Thus the Tannakian duality applies, and gives the result. $\square$

We already know, from section 4 above, that the quantum groups O_N^+, U_N^+ appear in this way, with D being respectively $NC_2, \mathcal{NC}_2$. In general now, let us formulate:

Definition 5.4. *A closed subgroup $G \subset U_N^+$ is called easy when we have*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

for any colored integers k, l, for a certain category of partitions $D \subset P$.

In other words, a compact quantum group is called easy when its Tannakian category appears in the simplest possible way: from a category of partitions. The terminology is quite natural, because Tannakian duality is basically our only serious tool.

Observe that the category D is not unique, for instance because at $N = 1$ all the categories of partitions produce the same easy quantum group, namely $G = \{1\}$. We will be back to this issue on several occasions, with various results about it.

We will see in what follows, in this section and in the next few ones, that many interesting examples of compact quantum groups are easy. Moreover, most of the known series of "basic" compact quantum groups, $G = (G_N)$ with $N \in \mathbb{N}$, can be in principle made fit into some suitable extensions of the easy quantum group formalism.

In practice now, what we know so far is that O_N^+, U_N^+ are easy. Here is a simplified proof for this fact, using the main Tannakian result from section 4 as ingredient:

Proposition 5.5. *We have the following results:*

(1) *The quantum group U_N^+ is easy, coming from the category $\mathcal{NC}_2$.*
(2) *The quantum group O_N^+ is easy as well, coming from the category NC_2.*

Proof. We use the Tannakian duality result from section 4 above:

(1) U_N^+ is defined via the relations $u^* = u^{-1}, u^t = \bar{u}^{-1}$, which tell us that the operators T_π, with $\pi = {}^{\sqcap}_{\circ\bullet}$ and $\pi = {}^{\sqcap}_{\bullet\circ}$, must be in the associated Tannakian category C. We therefore obtain $C = span(T_\pi | \pi \in D)$, with $D = < {}^{\sqcap}_{\circ\bullet}\ , {}^{\sqcap}_{\bullet\circ} > = \mathcal{NC}_2$, as claimed.

(2) $O_N^+ \subset U_N^+$ is defined by imposing the relations $u_{ij} = \bar{u}_{ij}$, which tell us that the operators T_π, with $\pi =$ and $\pi =$, must be in the associated Tannakian category C. We therefore obtain $C = span(T_\pi | \pi \in D)$, with $D =< \mathcal{NC}_2, , >= NC_2$, as claimed. $\square$

Next in our lineup, we have the following result, due to Brauer [47]:

Proposition 5.6. *We have the following results:*

(1) *The unitary group U_N is easy, coming from the category $\mathcal{P}_2$.*
(2) *The orthogonal group O_N is easy as well, coming from the category P_2.*

Proof. As already mentioned, this result is due to Brauer [47]. The classical proof is via classical Tannakian duality, for the usual closed subgroups $G \subset U_N$.

In the present context, we can deduce this result from the one that we already have, for O_N^+, U_N^+. The idea is very simple, namely that of "adding crossings", as follows:

(1) $U_N \subset U_N^+$ is defined via the relations $[u_{ij}, u_{kl}] = 0$ and $[u_{ij}, \bar{u}_{kl}] = 0$, which tell us that the operators T_π, with $\pi =$ and $\pi =$, must be in the associated Tannakian category C. Thus $C = span(T_\pi | \pi \in D)$, with $D =< \mathcal{NC}_2, , >= \mathcal{P}_2$, as claimed.

(2) In order to deal now with O_N, we can simply use the formula $O_N = O_N^+ \cap U_N$. At the categorical level, this tells us that the associated Tannakian category is given by $C = span(T_\pi | \pi \in D)$, with $D =< NC_2, \mathcal{P}_2 >= P_2$, as claimed. $\square$

Regarding now the half-liberations, we have here:

Proposition 5.7. *We have the following results:*

(1) *U_N^* is easy, coming from the category $\mathcal{P}_2^* \subset \mathcal{P}_2$ of pairings having the property that, when the legs are relabelled clockwise $\circ \bullet \circ \bullet \ldots$, each string connects $\circ - \bullet$.*
(2) *O_N^* is easy too, coming from the category $P_2^* \subset P_2$ of pairings having the same property: when legs are labelled clockwise $\circ \bullet \circ \bullet \ldots$, each string connects $\circ - \bullet$.*

Proof. We can proceed here as in the proof of Proposition 5.6 above, by replacing the basic crossing by the half-commutation crossing, as follows:

(1) Regarding $U_N^* \subset U_N^+$, the corresponding Tannakian category is generated by the operators T_π, with $\pi =$, taken with all the possible $2^3 = 8$ matching colorings. Since these latter 8 partitions generate the category $\mathcal{P}_2^*$, we obtain the result.

(2) Finally, for O_N^* we can proceed similarly, by using the formula $O_N^* = O_N^+ \cap U_N^*$. At the categorical level, this tells us that the associated Tannakian category is given by $C = span(T_\pi | \pi \in D)$, with $D =< NC_2, \mathcal{P}_2^* >= P_2^*$, as claimed. $\square$

Let us collect now the results that we have so far in a single theorem, as follows:

Theorem 5.8. *The basic unitary quantum groups are all easy, as follows,*

$$\begin{array}{ccccc}U_N & \to & U_N^* & \to & U_N^+\\ \uparrow && \uparrow && \uparrow\\ O_N & \to & O_N^* & \to & O_N^+\end{array} \quad : \quad \begin{array}{ccccc}\mathcal{P}_2 & \leftarrow & \mathcal{P}_2^* & \leftarrow & \mathcal{NC}_2\\ \downarrow && \downarrow && \downarrow\\ P_2 & \leftarrow & P_2^* & \leftarrow & NC_2\end{array}$$

with the corresponding categories of partitions being those on the right.

Proof. This follows indeed from the above results. □

We have seen in section 4 above that the easiness property of O_N^+,U_N^+ leads to some interesting consequences. Regarding O_N^*,U_N^*, as a main consequence, we can now compute their projective versions, as part of the following general result:

Theorem 5.9. *The projective versions of the basic quantum groups are as follows,*

$$\begin{array}{ccccc}U_N & \to & U_N^* & \to & U_N^+\\ \uparrow && \uparrow && \uparrow\\ O_N & \to & O_N^* & \to & O_N^+\end{array} \quad \to \quad \begin{array}{ccccc}PU_N & \to & PU_N & \to & PU_N^+\\ \uparrow && \uparrow && \uparrow\\ PO_N & \to & PU_N & \to & PU_N^+\end{array}$$

when identifying, in the free case, full and reduced version algebras.

Proof. In the classical case, there is nothing to prove. Regarding the half-classical versions, consider the inclusions $O_N^*,U_N\subset U_N^*$. These induce inclusions $PO_N^*,PU_N\subset PU_N^*$, and our claim is that these latter inclusions are isomorphisms.

In order to prove this, let u,v,w be the fundamental corepresentations of O_N^*,U_N,U_N^*. We have then the following equalities, coming from Theorem 5.8 above:

$$Hom\left((u\otimes\bar{u})^k,(u\otimes\bar{u})^l\right)=span\left(T_\pi\Big|\pi\in P_2^*((\circ\bullet)^k,(\circ\bullet)^l)\right)$$

$$Hom\left((u\otimes\bar{u})^k,(u\otimes\bar{u})^l\right)=span\left(T_\pi\Big|\pi\in \mathcal{P}_2((\circ\bullet)^k,(\circ\bullet)^l)\right)$$

$$Hom\left((u\otimes\bar{u})^k,(u\otimes\bar{u})^l\right)=span\left(T_\pi\Big|\pi\in \mathcal{P}_2^*((\circ\bullet)^k,(\circ\bullet)^l)\right)$$

The sets on the right being equal, we conclude that the inclusions $O_N^*,U_N\subset U_N^*$ preserve the corresponding Tannakian categories, and so must be isomorphisms.

Finally, in the free case the result follows either from the free complexification result in Theorem 4.28, or from Theorem 5.8, by using the same method. □

The above result is quite interesting, philosophically, because it shows that, in the nocommutative setting, the distinction between $\mathbb{R}$ and $\mathbb{C}$ becomes "blurred". We will be back to this in section 11 below, with some noncommutative geometry considerations.

In order to enlarge now our list of examples, and develop some general theory as well, we have several directions to be explored. A first natural question is that of computing the quantum group associated to the category P itself, and we have here:

Theorem 5.10. *The symmetric group S_N, regarded as a compact quantum group,*

$$S_N \subset U_N \subset U_N^+$$

via the permutation matrices, is easy, coming from the category of all partitions P.

Proof. Consider indeed the symmetric group S_N, regarded as a group of unitary matrices, with each permutation $\sigma \in S_N$ corresponding to the associated permutation matrix, $\sigma(e_i) = e_{\sigma(i)}$. We have in this way an embedding $S_N \subset U_N \subset U_N^+$, as above.

Consider as well the easy quantum group $G \subset O_N^+$ coming from the category of all partitions P. Since P contains the basic crossing, we have $G \subset O_N$. Moreover, since P is generated by the one-block partition $\mu \in P(2,1)$, we have:

$$C(G) = C(O_N) \Big/ \Big\langle T_\mu \in Hom(u^{\otimes 2}, u) \Big\rangle$$

The linear map associated to μ being given by $T_\mu(e_i \otimes e_j) = \delta_{ij} e_i$, we have:

$$u = (u_{ij})_{ij} \quad , \quad u^{\otimes 2} = (u_{ij}u_{kl})_{ik,jl} \quad , \quad T_\mu = (\delta_{ijk})_{i,jk}$$

We therefore obtain the following formulae:

$$(T_\mu u^{\otimes 2})_{i,jk} = \sum_{lm} (T_\mu)_{i,lm} (u^{\otimes 2})_{lm,jk} = u_{ij} u_{ik}$$

$$(u T_\mu)_{i,jk} = \sum_l u_{il} (T_\mu)_{l,jk} = \delta_{jk} u_{ij}$$

Thus, the relation defining $G \subset O_N$ reformulates as follows:

$$T_\mu \in Hom(u^{\otimes 2}, u) \iff u_{ij} u_{ik} = \delta_{jk} u_{ij}, \forall i, j, k$$

In other words, the elements u_{ij} must be projections, and these projections must be pairwise orthogonal on the rows of $u = (u_{ij})$. We conclude that $G \subset O_N$ is the subgroup of matrices $U \in O_N$ having the property $U_{ij} \in \{0, 1\}$, and so $G = S_N$, as desired. $\square$

With the above result in hand, and after a quick look at Theorem 5.8, it is tempting to define the "quantum permutation group" as being the easy quantum group associated to the category of noncrossing partitions NC. We will discuss this in section 7 below.

For the moment, let us stay at the general level. We have:

Proposition 5.11. *Any easy quantum group $G \subset U_N^+$ appears as an intermediate subgroup $S_N \subset G \subset U_N^+$. Also, the intermediate subgroups $S_N \subset G \subset U_N^+$ are those satisfying*

$$Hom(u^{\otimes k}, u^{\otimes l}) \subset span\left(T_\pi \Big| \pi \in P(k,l)\right)$$

for any k, l, and the easy ones are those having the property that these inclusions come from an inclusion of categories of partitions $D \subset P$, via the $\pi \to T_\pi$ construction.

Proof. The first assertion follows from Theorem 5.10, by functoriality. The first part of the second assertion follows as well from Theorem 5.10, by functoriality. As for the second part of the second assertion, this is just a reformulation of Definition 5.4. $\square$

All this suggests looking, more generally, at the arbitrary intermediate closed subgroups $S_N \subset G \subset U_N^+$, by using combinatorial methods. We have here, as a basic result:

Proposition 5.12. *Given an intermediate quantum group $S_N \subset G \subset U_N^+$, with Tannakian category C_G, the following is a category of partitions,*

$$D' = \left\{\pi \in P \Big| T_\pi \in C_G\right\}$$

and the corresponding easy quantum group $G' \subset U_N^+$ is smallest easy quantum group containing G. Moreover, the same holds under the sole assumption $G \subset U_N^+$.

Proof. The fact that D' satisfies the axioms for the categories of partitions comes from the fact that C_G satisfies the axioms for the Tannakian categories. By functoriality we have an inclusion $G \subset G'$, and the fact that G' is the smallest easy quantum group having this property is clear as well, once again by functoriality considerations. $\square$

The quantum group G' constructed above is called "easy envelope" of G. The construction $G \to G'$ is something quite interesting, and we will be back to it, later on.

When G is easy, coming from a category of partitions D, we have of course $G = G'$. By functoriality we have $D \subset D'$, and the category D', called "saturation" of D, is the biggest category of partitions producing the same easy quantum group as D.

Observe however that the construction $D \to D'$ depends on $N \in \mathbb{N}$, so it is does not provide a good answer to the functoriality issues mentioned after Definition 5.4.

Let us discuss now composition operations. We will be interested in:

Proposition 5.13. *The closed subgroups of U_N^+ are subject to operations as follows:*

(1) *Intersection: $H \cap K$ is the biggest quantum subgroup of H, K.*
(2) *Generation: $< H, K >$ is the smallest quantum group containing H, K.*

Proof. We must prove that the universal quantum groups in the statement exist indeed. For this purpose, let us pick writings as follows, with I, J being Hopf ideals:

$$C(H) = C(U_N^+)/I \quad , \quad C(K) = C(U_N^+)/J$$

We can then construct our two universal quantum groups, as follows:

$$C(H\cap K)=C(U_N^+)/<I,J>$$

$$C(<H,K>)=C(U_N^+)/(I\cap J)$$

Thus, we obtain the result. □

In practice, the operation $\cap$ can be usually computed by using:

Proposition 5.14. *Assuming $H,K\subset G$, the intersection $H\cap K$ is given by*

$$C(H\cap K)=C(G)/\{\mathcal{R},\mathcal{P}\}$$

whenever $C(H)=C(G)/\mathcal{R}$ and $C(K)=C(G)/\mathcal{P}$, with $\mathcal{R},\mathcal{P}$ being certain sets of polynomial $*$-relations between the standard coordinates u_{ij}.

Proof. This follows from Proposition 5.13 above, or rather from its proof, and from the following trivial fact, regarding relations and ideals:

$$I=<\mathcal{R}>,J=<\mathcal{P}>\implies <I,J>=<\mathcal{R},\mathcal{P}>$$

Thus, we obtain the result. □

In order to discuss now the generation operation, let us call Hopf image of a representation $C(G)\to A$ the smallest Hopf algebra quotient $C(L)$ producing a factorization $C(G)\to C(L)\to A$. The fact that such a quotient exists indeed is routine, by dividing by a suitable ideal, and we will be back to this, with details, in section 12 below.

This notion can be generalized to families of representations, and we have:

Proposition 5.15. *Assuming $H,K\subset G$, the quantum group $<H,K>$ is such that*

$$C(G)\to C(H\cap K)\to C(H),C(K)$$

is the joint Hopf image of the quotient maps $C(G)\to C(H),C(K)$.

Proof. In the particular case from the statement, the joint Hopf image appears as the smallest Hopf algebra quotient $C(L)$ producing factorizations as follows:

$$C(G)\to C(L)\to C(H),C(K)$$

We conclude from this that we have $L=<H,K>$, as claimed. See [49]. □

In the Tannakian setting now, we have the following result:

Theorem 5.16. *The intersection and generation operations $\cap$ and $<,>$ can be constructed via the Tannakian correspondence $G\to C_G$, as follows:*

1. *Intersection: defined via $C_{G\cap H}=<C_G,C_H>$.*
2. *Generation: defined via $C_{<G,H>}=C_G\cap C_H$.*

Proof. This follows from Proposition 5.13, or rather from its proof, by taking I,J to be the ideals coming from Tannakian duality, in its soft form, from section 4 above. □

In relation now with our easiness questions, we first have the following result:

Proposition 5.17. *Assuming that H,K are easy, then so is $H\cap K$, and we have*
$$D_{H\cap K}=<D_H,D_K>$$
at the level of the corresponding categories of partitions.

Proof. We have indeed the following computation:
$$\begin{aligned}C_{H\cap K}&=&<C_H,C_K>\\&=&<span(D_H),span(D_K)>\\&=&span(<D_H,D_K>)\end{aligned}$$
Thus, by Tannakian duality we obtain the result. □

Regarding the generation operation, the situation is more complicated, as follows:

Proposition 5.18. *Assuming that H,K are easy, we have inclusions*
$$<H,K>\subset<H,K>'\subset\{H,K\}$$
coming from inclusions of Tannakian categories as follows,
$$C_H\cap C_K\supset span\left(T_\pi\Big|T_\pi\in C_H\cap C_K\right)\supset span(D_H\cap D_K)$$
where $\{H,K\}$ is the easy quantum group having as category of partitions $D_H\cap D_K$.

Proof. This follows from the definition and properties of the easy envelope operation, from Proposition 5.1 above, and from the following computation:
$$\begin{aligned}C_{<H,K>}&=&C_H\cap C_K\\&=&span(D_H)\cap span(D_K)\\&\supset&span(D_H\cap D_K)\end{aligned}$$
Indeed, by Tannakian duality we obtain from this all the assertions. □

It is not very clear if the various inclusions in Proposition 5.18 are isomorphisms or not, perhaps under a $N>>0$ assumption, and this is actually a topic of active research. Technically speaking, the problem comes from the fact that the operation $\pi\to T_\pi$ does not produce linearly independent maps. We will be back to this, later on.

Summarizing, we have a gap in our theory, and we must cheat, as follows:

Theorem 5.19. *The intersection and easy generation operations $\cap$ and $\{\,,\}$ can be constructed via the Tannakian correspondence $G\to D_G$, as follows:*

1. *Intersection: defined via $D_{G\cap H}=<D_G,D_H>$.*
2. *Easy generation: defined via $D_{\{G,H\}}=D_G\cap D_H$.*

Proof. Here (1) is an honest result, coming from Proposition 5.17, and (2) is an empty statement, related to the difficulties that we met in Proposition 5.18. □

Observe that we are actually cheating twice here, once for each of the 2 inclusions in Proposition 5.18. This is how mathematics goes. In what follows we will use the above operations $\cap$ and $\{\,,\}$ several times, in order to formulate certain results.

Let us go back now to more concrete things, and explore a number of further examples of easy quantum groups. With the convention that a matrix is called bistochastic when its entries sum up to 1, on each row and each column, we have:

Proposition 5.20. *We have the following groups and quantum groups:*
(1) $B_N \subset O_N$*, consisting of the orthogonal matrices which are bistochastic.*
(2) $C_N \subset U_N$*, consisting of the unitary matrices which are bistochastic.*
(3) $B_N^+ \subset O_N^+$*, coming via* $u\xi = \xi$*, where* ξ *is the all-one vector.*
(4) $C_N^+ \subset U_N^+$*, coming via* $u\xi = \xi$*, where* ξ *is the all-one vector.*

Also, we have inclusions $B_N \subset B_N^+$ *and* $C_N \subset C_N^+$*, which are both liberations.*

Proof. Here the fact that B_N, C_N are indeed groups is clear. As for B_N^+, C_N^+, these are quantum groups as well, because the relation $\xi \in Fix(u)$ is categorical.

Finally, observe that for $U \in U_N$ the condition $U\xi = \xi$ is equivalent to $U^*\xi = \xi$. By conjugating, these conditions are equivalent as well to $\bar{U}\xi = \xi$, and to $U^t\xi = \xi$. Thus $U \in U_N$ is bistochastic precisely when $U\xi = \xi$, and this gives the last assertion. □

The above quantum groups are all easy, and following [35], [83], we have:

Theorem 5.21. *The basic orthogonal and unitary quantum groups and their bistochastic versions are all easy, and they form a diagram as follows,*

$$\begin{tikzcd}
& C_N^+ \arrow[rr] & & U_N^+ \\
B_N^+ \arrow[ur] \arrow[rr] & & O_N^+ \arrow[ur] & \\
& C_N \arrow[uu] \arrow[rr] & & U_N \arrow[uu] \\
B_N \arrow[uu] \arrow[ur] \arrow[rr] & & O_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

which is an intersection and easy generation diagram, in the sense that any of its faces $P \subset Q, R \subset S$ *satisfies the condition* $P = Q \cap R, \{Q, R\} = S$*.*

Proof. The first assertion comes from the fact that the all-one vector ξ used in Proposition 5.20 above is the vector associated to the singleton partition, $\xi = T_|$. Indeed, we obtain from this that the quantum groups B_N, C_N, B_N^+, C_N^+ are indeed easy, appearing from the categories of partitions for O_N, U_N, O_N^+, U_N^+, by adding singletons.

In practice now, according to this observation, and to Theorem 5.8 above, the corresponding categories of partitions are as follows, where the symbol 12 stands for "singletons and pairings", in the same way as the symbol 2 stands for "pairings":

$$\begin{tikzcd}[row sep=small, column sep=small]
& \mathcal{NC}_{12} \arrow[dl] \arrow[dd] & & \mathcal{NC}_2 \arrow[ll] \arrow[dl] \arrow[dd] \\
NC_{12} \arrow[dd] & & NC_2 \arrow[ll] \arrow[dd] & \\
& \mathcal{P}_{12} \arrow[dl] & & \mathcal{P}_2 \arrow[ll] \arrow[dl] \\
P_{12} & & P_2 \arrow[ll] &
\end{tikzcd}$$

Now since both this diagram and the one the statement are intersection diagrams, the quantum groups form an intersection and easy generation diagram, as stated. □

The above result looks quite nice, theoretically speaking, but there are a few problems with it. First, we cannot really merge it with Theorem 5.8, as to obtain a nice cubic diagram, containing all the quantum groups considered so far, and this because the half-classical versions of the bistochastic quantum groups collapse, as follows:

Proposition 5.22. *The half-classical versions of B_N^+, C_N^+ are given by:*

$$B_N^+ \cap O_N^* = B_N \quad , \quad C_N^+ \cap U_N^* = C_N$$

In other words, the half-classical versions collapse to the classical versions.

Proof. This follows for instance from Tannakian duality, by using the fact that when capping the half-classical crossing with 2 singletons, we obtain the classical crossing. □

Yet another problem with the bistochastic groups and quantum groups comes from the fact that these objects are not really "new", because, following [77], we have:

Proposition 5.23. *We have isomorphisms as follows:*

(1) $B_N \simeq O_{N-1}$, $B_N^+ \simeq O_{N-1}^+$.
(2) $C_N \simeq U_{N-1}$, $C_N^+ \simeq U_{N-1}^+$.

Proof. Let us pick $F \in U_N$ satisfying $Fe_0 = \frac{1}{\sqrt{N}}\xi$, where ξ is the all-one vector, the basic example here being the Fourier matrix, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$. We have then:

$$\begin{aligned}
u\xi = \xi \quad &\iff \quad uFe_0 = Fe_0 \\
&\iff \quad F^*uFe_0 = e_0 \\
&\iff \quad F^*uF = diag(1, w)
\end{aligned}$$

Thus we have isomorphisms as in the statement, given by $w_{ij} \to (F^*uF)_{ij}$. □

Summarizing, we have some examples of easy quantum groups, but we are not ready yet for some serious structure and classification work. We will be back to all this later on, in section 9 below, after introducing and studying some more examples.

Back to generalities now, let us point out the fact that the easy quantum groups are not the only ones "coming from partitions", but are rather the simplest ones having this property. There are many other classes of such quantum groups, see [4], [33], [59].

In what follows we discuss one such construction, which is of particular interest, coming from the twisting philosophy of Drinfeld [58] and Jimbo [65]. Their idea was to deform the compact Lie groups with the help of a parameter $q \in \mathbb{C}$, the interesting case being $q \in \mathbb{T}$. However, as explained by Woronowicz in [97], in the extended compact quantum group setting the parameter needs to be real, $q \in \mathbb{R}$. We are therefore led to:

$$q \in \mathbb{T} \cap \mathbb{R} = \{\pm 1\}$$

In practice now, we can think for instance of the easy quantum groups as corresponding to the case $q = 1$, and we are led to the question of "twisting" them, at $q = -1$.

All this is quite tricky, and in order to start, best is to deform first the simplest objects that we have, namely the noncommutative spheres. Our starting point will be:

Proposition 5.24. *We have noncommutative spheres as follows, obtained via the twisted commutation relations $ab = \pm ba$, and twisted half-commutation relations $abc = \pm cba$,*

$$\begin{tikzcd}
\bar{S}^{N-1}_{\mathbb{C}} \arrow[r] & \bar{S}^{N-1}_{\mathbb{C},*} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\
\bar{S}^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & \bar{S}^{N-1}_{\mathbb{R},*} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{R},+} \arrow[u]
\end{tikzcd}$$

where the signs at left correspond to the anticommutation of distinct coordinates, and their adjoints, and the other signs come from functoriality.

Proof. For the spheres on the left, if we want to replace some of the commutation relations $z_i z_j = z_j z_i$ by anticommutation relations $z_i z_j = -z_j z_i$, the one and only natural choice is $z_i z_j = -z_j z_i$ for $i \neq j$. In other words, with the notation $\varepsilon_{ij} = 1 - \delta_{ij}$, we must have:

$$z_i z_j = (-1)^{\varepsilon_{ij}} z_j z_i$$

Regarding now the spheres in the middle, the situation is a priori a bit more tricky, because we have to take into account the various possible collapsings of $\{i, j, k\}$. However, if we want to have embeddings as above, there is only one choice, namely:

$$z_i z_j z_k = (-1)^{\varepsilon_{ij} + \varepsilon_{jk} + \varepsilon_{ik}} z_k z_j z_i$$

Thus, we have constructed our spheres, and embeddings, as needed. □

Let us discuss now the quantum group case. The situation here is considerably more complicated, because the coordinates u_{ij} depend on double indices, and finding for instance the correct signs for $u_{ij}u_{kl}u_{mn} = \pm u_{mn}u_{kl}u_{ij}$ looks nearly impossible.

However, we can solve this problem by taking some inspiration from the sphere case, which was already solved. A bit of thinking suggests that each row and column of $u = (u_{ij})$ must be subject to the relations found above for the spheres, and if we want for instance to have coactions as well, we end up with a complete picture, as follows:

Theorem 5.25. *We have quantum groups as follows, obtained via the twisted commutation relations $ab = \pm ba$, and twisted half-commutation relations $abc = \pm cba$,*

$$\begin{tikzcd}
\bar{U}_N \arrow[r] & \bar{U}_N^* \arrow[r] & U_N^+ \\
\bar{O}_N \arrow[r] \arrow[u] & \bar{O}_N^* \arrow[r] \arrow[u] & O_N^+ \arrow[u]
\end{tikzcd}$$

where the signs at left correspond to anticommutation for distinct entries on rows and columns, and commutation otherwise, and the other signs come from functoriality.

Proof. As explained above, there is only one reasonable way of arranging the signs, as for everything to work fine. So let us go ahead now, and present the solution.

Given abstract coordinates $a, b, c, \ldots \in \{u_{ij}\}$, let us set $span(a, b, c, \ldots) = (r, c)$, where $r, c \in \{1, 2, 3, \ldots\}$ are the numbers of rows and columns spanned by $a, b, c, \ldots$, inside the matrix $u = (u_{ij})$. Also, we make the conventions $\alpha = a, a^*$, $\beta = b, b^*$, and so on.

With these conventions, the relations for the quantum groups on the left, which are the only possible ones, as for having a good compatibility with the spheres, are:

$$\alpha\beta = \begin{cases} -\beta\alpha & \text{for } a, b \in \{u_{ij}\} \text{ with } span(a, b) = (1, 2) \text{ or } (2, 1) \\ \beta\alpha & \text{otherwise} \end{cases}$$

As for the relations for the quantum groups in the middle, once again these are uniquely determined by various functoriality considerations, and must be as follows:

$$\alpha\beta\gamma = \begin{cases} -\gamma\beta\alpha & \text{for } a, b, c \in \{u_{ij}\} \text{ with } span(a, b, c) = (\leq 2, 3) \text{ or } (3, \leq 2) \\ \gamma\beta\alpha & \text{otherwise} \end{cases}$$

Summarizing, we are done with the difficult part, namely guessing the signs. What is left is to prove that the above relations produce indeed quantum groups, with inclusions between them, as in the statement. But this is routine, by proceeding as in the non-twisted case, and adding signs where needed. For details on this, as well on the fact that we have indeed quantum isometry group results in this setting, we refer to [4]. □

All the above, while definitely not random, remains however a bit mysterious, and uncomplete. Our purpose now will be that of showing that the quantum groups constructed above can be in fact defined in a very conceptual way, as "Schur-Weyl twists".

Let $P_{even}(k,l) \subset P(k,l)$ be the set of partitions with blocks having even size, and $NC_{even}(k,l) \subset P_{even}(k,l)$ be the subset of noncrossing partitions. Also, we use the standard embedding $S_k \subset P_2(k,k)$, via the pairings having only up-to-down strings.

Given a partition $\tau \in P(k,l)$, we call "switch" the operation which consists in switching two neighbors, belonging to different blocks, in the upper row, or in the lower row.

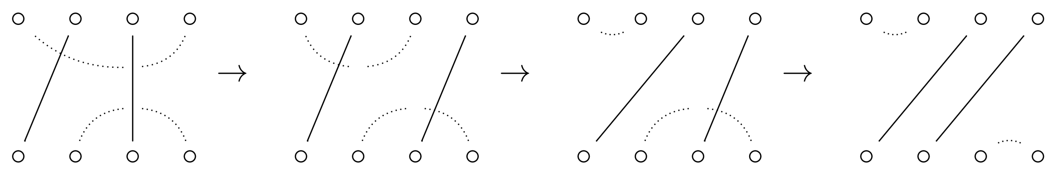
With these conventions, we have the following result:

Proposition 5.26. *There is a signature map $\varepsilon : P_{even} \to \{-1,1\}$, given by $\varepsilon(\tau) = (-1)^c$, where c is the number of switches needed to make τ noncrossing. In addition:*
1. *For $\tau \in S_k$, this is the usual signature.*
2. *For $\tau \in P_2$ we have $(-1)^c$, where c is the number of crossings.*
3. *For $\tau \leq \pi \in NC_{even}$, the signature is 1.*

Proof. In order to show that ε is well-defined, we must prove that the number c in the statement is well-defined modulo 2. It is enough to perform the verification for the noncrossing partitions. More precisely, given $\tau, \tau' \in NC_{even}$ having the same block structure, we must prove that the number of switches c required for the passage $\tau \to \tau'$ is even.

In order to do so, observe that any partition $\tau \in P(k,l)$ can be put in "standard form", by ordering its blocks according to the appearence of the first leg in each block, counting clockwise from top left, and then by performing the switches as for block 1 to be at left, then for block 2 to be at left, and so on. Here the required switches are also uniquely determined, by the order coming from counting clockwise from top left.

Here is an example of such an algorithmic switching operation, with block 1 being first put at left, by using two switches, then with block 2 left unchanged, and then with block 3 being put at left as well, but at right of blocks 1 and 2, with one switch:



The point now is that, under the assumption $\tau \in NC_{even}(k,l)$, each of the moves required for putting a leg at left, and hence for putting a whole block at left, requires an even number of switches. Thus, putting τ is standard form requires an even number of switches. Now given $\tau, \tau' \in NC_{even}$ having the same block structure, the standard form coincides, so the number of switches c required for the passage $\tau \to \tau'$ is indeed even.

Regarding now the remaining assertions, these are all elementary:

(1) For $\tau \in S_k$ the standard form is $\tau' = id$, and the passage $\tau \to id$ comes by composing with a number of transpositions, which gives the signature.

(2) For a general $\tau\in P_2$, the standard form is of type $\tau'=|\ldots|^{\cup\ldots\cup}_{\cap\ldots\cap}$, and the passage $\tau\to\tau'$ requires c mod 2 switches, where c is the number of crossings.

(3) Assuming that $\tau\in P_{even}$ comes from $\pi\in NC_{even}$ by merging a certain number of blocks, we can prove that the signature is 1 by proceeding by recurrence. $\square$

We define the kernel of a multi-index $\binom{i}{j}$ to be the partition obtained by joining the equal indices. Also, we write $\pi\leq\sigma$ if each block of π is contained in a block of σ.

With these conventions, and the above result in hand, we can now formulate:

Definition 5.27. *Associated to a partition $\pi\in P_{even}(k,l)$ is the linear map*

$$\bar{T}_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\bar{\delta}_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

where $\bar{\delta}_\pi\in\{-1,0,1\}$ is $\bar{\delta}_\pi=\varepsilon(\tau)$ if $\tau\geq\pi$, and $\bar{\delta}_\pi=0$ otherwise, with $\tau=\ker\binom{i}{j}$.

In other words, what we are doing here is to add signatures to the usual formula of T_π. Indeed, observe that the usual formula for T_π can be written as folllows:

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j:\ker\binom{i}{j}\geq\pi}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

Now by inserting signs, coming from the signature map $\varepsilon:P_{even}\to\{\pm1\}$, we are led to the following formula, which coincides with the one given above:

$$\bar{T}_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{\tau\geq\pi}\varepsilon(\tau)\sum_{j:\ker\binom{i}{j}=\tau}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

We will be back later to this analogy, with more details on what can be done with it. For the moment, we must first prove a key categorical result, as follows:

Proposition 5.28. *The assignement $\pi\to\bar{T}_\pi$ is categorical, in the sense that*

$$\bar{T}_\pi\otimes\bar{T}_\sigma=\bar{T}_{[\pi\sigma]},\qquad \bar{T}_\pi\bar{T}_\sigma=N^{c(\pi,\sigma)}\bar{T}_{[^\sigma_\pi]},\qquad \bar{T}_\pi^*=\bar{T}_{\pi^*}$$

where $c(\pi,\sigma)$ are certain positive integers.

Proof. We have to go back to the proof of Proposition 5.2, or rather to the proof of Proposition 4.25, and insert signs. We have to check three conditions, as follows:

1. Concatenation. In the untwisted case, this was based on the following formula:

$$\delta_\pi\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1\ldots k_r\\ l_1\ldots l_s\end{pmatrix}=\delta_{[\pi\sigma]}\begin{pmatrix}i_1\ldots i_p&k_1\ldots k_r\\ j_1\ldots j_q&l_1\ldots l_s\end{pmatrix}$$

In the twisted case, it is enough to check the following formula:

$$\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\right)\varepsilon\left(\ker\begin{pmatrix}k_1\ldots k_r\\ l_1\ldots l_s\end{pmatrix}\right)=\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p&k_1\ldots k_r\\ j_1\ldots j_q&l_1\ldots l_s\end{pmatrix}\right)$$

Let us denote by τ,ν the partitions on the left, so that the partition on the right is of the form $\rho\leq[\tau\nu]$. Now by switching to the noncrossing form, $\tau\to\tau'$ and $\nu\to\nu'$, the partition on the right transforms into $\rho\to\rho'\leq[\tau'\nu']$. Now since $[\tau'\nu']$ is noncrossing, we can use Proposition 5.26 (3), and we obtain the result.

2. Composition. In the untwisted case, this was based on the following formula:

$$\sum_{j_1\ldots j_q}\delta_\pi\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}j_1\ldots j_q\\ k_1\ldots k_r\end{pmatrix}=N^{c(\pi,\sigma)}\delta_{[^\pi_\sigma]}\begin{pmatrix}i_1\ldots i_p\\ k_1\ldots k_r\end{pmatrix}$$

In order to prove now the result in the twisted case, it is enough to check that the signs match. More precisely, we must establish the following formula:

$$\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\right)\varepsilon\left(\ker\begin{pmatrix}j_1\ldots j_q\\ k_1\ldots k_r\end{pmatrix}\right)=\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ k_1\ldots k_r\end{pmatrix}\right)$$

Let τ,ν be the partitions on the left, so that the partition on the right is of the form $\rho\leq[^\tau_\nu]$. Our claim is that we can jointly switch τ,ν to the noncrossing form. Indeed, we can first switch as for $\ker(j_1\ldots j_q)$ to become noncrossing, and then switch the upper legs of τ, and the lower legs of ν, as for both these partitions to become noncrossing.

Now observe that when switching in this way to the noncrossing form, $\tau\to\tau'$ and $\nu\to\nu'$, the partition on the right transforms into $\rho\to\rho'\leq[^{\tau'}_{\nu'}]$. Now since $[^{\tau'}_{\nu'}]$ is noncrossing, we can apply Proposition 5.26 (3), and we obtain the result.

3. Involution. Here we must prove the following formula:

$$\bar{\delta}_\pi\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}=\bar{\delta}_{\pi^*}\begin{pmatrix}j_1\ldots j_q\\ i_1\ldots i_p\end{pmatrix}$$

But this is clear from the definition of $\bar{\delta}_\pi$, and we are done. □

As a conclusion, our twisted construction $\pi\to\bar{T}_\pi$ has all the needed properties for producing quantum groups, via Tannakian duality. So, let us formulate:

Theorem 5.29. *Given a category of partitions $D\subset P_{even}$, the construction*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(\bar{T}_\pi\Big|\pi\in D(k,l)\right)$$

produces via Tannakian duality a quantum group $\bar{G}_N\subset U_N^+$, for any $N\in\mathbb{N}$.

Proof. This follows indeed from the Tannakian results from section 4 above, exactly as in the easy case, by using this time Proposition 5.28 as technical ingredient. To be more precise, Proposition 5.28 shows that the linear spaces on the right form a Tannakian category, and so the results in section 4 apply, and give the result. □

We can unify the easy quantum groups, or at least the examples coming from categories $D\subset P_{even}$, with the quantum groups constructed above, as follows:

Definition 5.30. *A closed subgroup $G\subset U_N^+$ is called q-easy, or quizzy, with deformation parameter $q=\pm1$, when its tensor category appears as follows,*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(\dot{T}_\pi\Big|\pi\in D(k,l)\right)$$

for a certain category of partitions $D\subset P_{even}$, where $\dot{T}=\bar{T},T$ for $q=-1,1$. The Schur-Weyl twist of G is the quizzy quantum group $\bar{G}\subset U_N^+$ obtained via $q\to-q$.

The terminology here is of course a bit awkward, but there is no way of fixing this, because the construction of the signature in Proposition 5.26 really needs the partitions to have even blocks. We will see later on, in section 8 below, that the easy quantum group associated to P_{even} itself is the hyperochahedral group H_N, and so that our assumption $D\subset P_{even}$, replacing $D\subset P$, corresponds to $H_N\subset G$, replacing $S_N\subset G$. We will also see, in section 9, that for classification purposes, this is a good assumption.

In relation now with the twists, we have the following result:

Theorem 5.31. *The quantum groups $\bar{O}_N,\bar{O}_N^*,O_N^+,\bar{U}_N,\bar{U}_N^*,U_N^+$ introduced before appear as Schur-Weyl twists of the quantum groups $O_N,O_N^*,O_N^+,U_N,U_N^*,U_N^+$.*

Proof. The basic crossing, $\ker\begin{pmatrix}ij\\ji\end{pmatrix}$ with $i\neq j$, comes from the transposition $\tau\in S_2$, so its signature is -1. As for its degenerated version $\ker\begin{pmatrix}ii\\ii\end{pmatrix}$, this is noncrossing, so here the signature is 1. We conclude that the linear map associated to the basic crossing is:

$$\bar{T}_{\slash\!\!\!\backslash}(e_i\otimes e_j)=\begin{cases}-e_j\otimes e_i&\text{for } i\neq j\\ e_j\otimes e_i&\text{otherwise}\end{cases}$$

For the half-classical crossing, here the signature is once again -1, and by examining the signatures of its various degenerations, we are led to the following formula:

$$\bar{T}_{\slash\!\!\!|\!\!\!\backslash}(e_i\otimes e_j\otimes e_k)=\begin{cases}-e_k\otimes e_j\otimes e_i&\text{for } i,j,k\text{ distinct}\\ e_k\otimes e_j\otimes e_i&\text{otherwise}\end{cases}$$

We can proceed now as in the untwisted case, and since the intertwining relations coming from $\bar{T}_{\slash\!\!\!\backslash},\bar{T}_{\slash\!\!\!|\!\!\!\backslash}$ correspond to the relations defining $\bar{U}_N,\bar{U}_N^*$, we obtain the result. □

As a conclusion, we have a quite interesting notion of easy quantum group, basically coming from the Brauer philosophy for O_N,U_N, and notably covering O_N^+,U_N^+, along with some theory and examples, and with a twisting extension as well.

6. Probabilistic aspects

We discuss here the computation of the various integrals over the compact quantum groups, with respect to the Haar measure. In order to formulate our results in a conceptual form, we use the modern measure theory language, namely probability theory.

In the noncommutative setting, the starting definition is as follows:

Definition 6.1. *Let A be a C^*-algebra, given with a trace tr.*

1. *The elements $a \in A$ are called random variables.*
2. *The moments of such a variable are the numbers $M_k(a) = tr(a^k)$.*
3. *The law of such a variable is the functional $\mu : P \to tr(P(a))$.*

Here $k = \circ \bullet \bullet \circ \ldots$ is as usual a colored integer, and the powers a^k are defined by the usual formulae, namely $a^{\emptyset} = 1, a^{\circ} = a, a^{\bullet} = a^*$ and multiplicativity. As for the polynomial P, this is a noncommuting $*$-polynomial in one variable, $P \in \mathbb{C} < X, X^* >$.

Observe that the law is uniquely determined by the moments, because:

$$P(X) = \sum_k \lambda_k X^k \implies \mu(P) = \sum_k \lambda_k M_k(a)$$

In the self-adjoint case, the law is a usual probability measure, supported by the spectrum of a. This follows indeed from the Gelfand theorem, and the Riesz theorem.

Now back to the quantum groups, let us start with:

Proposition 6.2. *Given a Woronowicz algebra (A, u), with $u \in M_N(A)$, the moments of the main character $\chi = \sum_i u_{ii}$ are given by:*

$$\int_G \chi^k = \dim \left(Fix(u^{\otimes k}) \right)$$

In the case $u \sim \bar{u}$ the law of χ is a usual probability measure, supported on $[-N, N]$.

Proof. The first assertion follows from the Peter-Weyl theory, which tells us that we have the following formula, valid for any corepresentation $v \in M_n(A)$:

$$\int_G \chi_v = \dim(Fix(v))$$

Indeed, with $v = u^{\otimes k}$ the character is $\chi_v = \chi^k$, and we obtain the result.

As for the second assertion, if we assume $u \sim \bar{u}$ then we have $\chi = \chi^*$, and so the general theory, explained above, tells us that $law(\chi)$ is in this case a real probability measure, supported by the spectrum of χ. But, since $u \in M_N(A)$ is unitary, we have:

$$uu^* = 1 \implies ||u_{ij}|| \leq 1, \forall i, j \implies ||\chi|| \leq N$$

Thus the spectrum of the character satisfies $\sigma(\chi) \subset [-N, N]$, and we are done. □

Summarizing, the law of the main character encodes some important representation theory data. Here is a second motivation for the study of such laws:

Proposition 6.3. *Consider two Woronowicz algebras (A,u) and (B,v), and assume that we have a morphism $f:A\to B$, mapping $u_{ij}\to v_{ij}$.*

1. *We have $M_k(\chi_u)\leq M_k(\chi_v)$, for any colored integer k.*
2. *f is an isomorphism when all these inequalities are equalities.*
3. *Thus, f is an isomorphism precisely when $law(\chi_u)=law(\chi_v)$.*

Proof. This is a probabilistic version of Proposition 4.2 above:
(1) This follows indeed from Proposition 4.2, via Proposition 6.2.
(2) This follows once again from Proposition 4.2, via Proposition 6.2.
(3) This follows from (2), the laws being uniquely determined by the moments. □

Here is a third motivation as well, analytic this time:

Proposition 6.4. *A Woronowicz algebra (A,u), with $u\in M_N(A)$, is amenable when*

$$N\in supp\Big(law(Re(\chi))\Big)$$

and the support on the right depends only on $law(\chi)$.

Proof. According to the Kesten amenability criterion, from Theorem 3.27 (4) above, the algebra A is amenable when the following condition is satisfied:

$$N\in\sigma(Re(\chi))$$

Now since $Re(\chi)$ is self-adjoint, the support of its spectral measure $law(Re(\chi))$ is precisely its spectrum $\sigma(Re(\chi))$, and this gives the first assertion.

Regarding now the second assertion, once again the variable $Re(\chi)$ being self-adjoint, its law depends only on the moments $\int_G Re(\chi)^p$, with $p\in\mathbb{N}$. But, we have:

$$\int_G Re(\chi)^p=\int_G\left(\frac{\chi+\chi^*}{2}\right)^p=\frac{1}{2^p}\sum_{|k|=p}\int_G\chi^k$$

Thus $law(Re(\chi))$ depends only on $law(\chi)$, and this gives the result. □

As yet another motivation for the study of $law(\chi)$, let us work out the group dual case. Here we obtain is a very interesting measure, called Kesten measure [67], as follows:

Proposition 6.5. *In the case $A=C^*(\Gamma)$ and $u=diag(g_1,\ldots,g_N)$, and with the normalization assumption $1\in u=\bar{u}$ made, the moments of the main character are*

$$\int_{\widehat{\Gamma}}\chi^p=\#\left\{i_1,\ldots,i_p\Big|g_{i_1}\ldots g_{i_p}=1\right\}$$

counting the loops based at 1, having lenght p, on the corresponding Cayley graph.

Proof. Consider indeed a discrete group $\Gamma=<g_1,\ldots,g_N>$. The main character of $A=C^*(\Gamma)$, with fundamental corepresentation $u=diag(g_1,\ldots,g_N)$, is then:

$$\chi=g_1+\ldots+g_N$$

Given a colored integer $k = e_1 \ldots e_p$, the corresponding moment is given by:

$$\int_{\widehat{\Gamma}} \chi^k = \int_{\widehat{\Gamma}} (g_1 + \ldots + g_N)^k = \#\left\{i_1, \ldots, i_p \Big| g_{i_1}^{e_1} \ldots g_{i_p}^{e_p} = 1\right\}$$

In the self-adjoint case, $u \sim \bar{u}$, we are only interested in the moments with respect to usual integers, $p \in \mathbb{N}$, and the above formula becomes:

$$\int_{\widehat{\Gamma}} \chi^p = \#\left\{i_1, \ldots, i_p \Big| g_{i_1} \ldots g_{i_p} = 1\right\}$$

Assume now that we have in addition $1 \in u$, so that the normalization condition $1 \in u = \bar{u}$ in the statement is satisfied. At the level of the generating set $S = \{g_1, \ldots, g_N\}$ this means $1 \in S = S^{-1}$, and so the corresponding Cayley graph is well-defined, with the elements of Γ as vertices, and with the edges $g - h$ appearing when $gh^{-1} \in S$.

A loop on this graph based at 1, having lenght p, is then a sequence as follows:

$$(1) - (g_{i_1}) - (g_{i_1} g_{i_2}) - \ldots - (g_{i_1} \ldots g_{i_{p-1}}) - (g_{i_1} \ldots g_{i_p} = 1)$$

Thus the moments of χ count indeed such loops, as claimed. □

In order to generalize the above result to arbitrary Woronowicz algebras, we can use the discrete quantum group philosophy. The fundamental result here is as follows:

Proposition 6.6. *Let (A, u) be a Woronowicz algebra, and assume, by enlarging if necessary u, that we have $1 \in u = \bar{u}$. The following formula*

$$d(v, w) = \min\left\{k \in \mathbb{N} \Big| 1 \subset \bar{v} \otimes w \otimes u^{\otimes k}\right\}$$

defines then a distance on $Irr(A)$, which coincides with the geodesic distance on the associated Cayley graph. In the group dual case we obtain the usual distance.

Proof. The fact that the lengths are finite follows from Woronowicz's analogue of Peter-Weyl theory. The symmetry axiom is clear as well, and the triangle inequality is elementary to establish as well. Finally, the last assertion is elementary as well.

In the group dual case, where $A = C^*(\Gamma)$ with $\Gamma =< S >$ being a finitely generated discrete group, our normalization condition $1 \in u = \bar{u}$ means that the generating set must satisfy $1 \in S = S^{-1}$. But this is precisely the normalization condition for the discrete groups, and the fact that we obtain the same metric space is clear. □

We can now formulate a generalization of Proposition 6.5, as follows:

Proposition 6.7. *Let (A, u) be a Woronowicz algebra, with the normalization assumption $1 \in u = \bar{u}$ made. The moments of the main character,*

$$\int_G \chi^p = \dim\left(Fix(u^{\otimes p})\right)$$

count then the loops based at 1, having lenght p, on the corresponding Cayley graph.

Proof. Here the formula of the moments, with $p\in\mathbb N$, is the one coming from Proposition 6.2 above, and the Cayley graph interpretation comes from definitions. $\square$

Summarizing, the computation of the law of the main character is the "main" problem regarding a Woronowicz algebra (A,u), from a massive variety of viewpoints.

In what follows we will be interested in computing such laws, for the main examples of quantum groups that we have. In the easy quantum group case, we have:

Proposition 6.8. *Let G be an easy quantum group, coming from a category of partitions $D=(D(k,l))$. The moments of the main character are then given by*

$$\int_G\chi^k=\dim\Big(span\left(\xi_\pi\big|\pi\in D(k)\right)\Big)$$

where $D(k)=D(\emptyset,k)$, and where for $\pi\in D(k)$ we use the notation $\xi_\pi=T_\pi$.

Proof. We recall from section 5 above that for an easy quantum group G, coming from a category of partitions $D=(D(k,l))$, we have equalities as follows:

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

By interchanging $k\leftrightarrow l$ in this formula, and then setting $l=\emptyset$, we obtain:

$$Fix(u^{\otimes k})=span\left(\xi_\pi\Big|\pi\in D(k)\right)$$

By using now Proposition 6.2 above, we obtain the result. $\square$

Thus, in the easy case, we are led into linear independence questions for the vectors ξ_π. In order to investigate these questions, we will use the Gram matrix of these vectors. Let us begin with some standard combinatorial definitions, as follows:

Definition 6.9. *Let $P(k)$ be the set of partitions of $\{1,\ldots,k\}$, and let $\pi,\sigma\in P(k)$.*

1. *We write $\pi\leq\sigma$ if each block of π is contained in a block of σ.*
2. *We let $\pi\vee\sigma\in P(k)$ be the partition obtained by superposing π,σ.*

As an illustration here, at $k=2$ we have $P(2)=\{||,\sqcap\}$, and we have $||\leq\sqcap$. Also, at $k=3$ we have $P(3)=\{|||,\sqcap|,\sqcap\!\!\!\!|\,,|\sqcap,\sqcap\!\sqcap\}$, and the order relation is as follows:

$$|||\leq\sqcap|,\sqcap\!\!\!\!|\,,|\sqcap\leq\sqcap\!\sqcap$$

Observe also that we have $\pi,\sigma\leq\pi\vee\sigma$, and that $\pi\vee\sigma$ is the smallest partition with this property. Due to this fact, $\pi\vee\sigma$ is called supremum of π,σ.

Now back to the easy quantum groups, we have:

Proposition 6.10. *The Gram matrix $G_{kN}(\pi,\sigma)=<\xi_\pi,\xi_\sigma>$ is given by*

$$G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$$

where $|.|$ is the number of blocks.

Proof. According to the formula of the vectors ξ_π, we have:

$$\begin{aligned} < \xi_\pi, \xi_\sigma > &= \sum_{i_1 \ldots i_k} \delta_\pi(i_1, \ldots, i_k)\delta_\sigma(i_1, \ldots, i_k) \\ &= \sum_{i_1 \ldots i_k} \delta_{\pi\vee\sigma}(i_1, \ldots, i_k) \\ &= N^{|\pi\vee\sigma|} \end{aligned}$$

Thus, we have obtained the formula in the statement. □

In order to study the Gram matrix, and more specifically to compute its determinant, we will use several standard facts about the partitions. We have:

Definition 6.11. *The Möbius function of any lattice, and so of P, is given by*

$$\mu(\pi, \sigma) = \begin{cases} 1 & \text{if } \pi = \sigma \\ -\sum_{\pi\leq\tau<\sigma} \mu(\pi, \tau) & \text{if } \pi < \sigma \\ 0 & \text{if } \pi \not\leq \sigma \end{cases}$$

with the construction being performed by recurrence.

As an illustration here, let us go back to the set of 2-point partitions, $P(2) = \{||, \sqcap\}$. We have by definition $\mu(||, ||) = \mu(\sqcap, \sqcap) = 1$. Also, we know that we have $|| < \sqcap$, with no intermediate partition in between, and so the above recurrence procedure gives:

$$\mu(||, \sqcap) = -\mu(||, ||) = -1$$

Finally, we have $\sqcap \not\leq ||$, and so $\mu(\sqcap, ||) = 0$. Thus, as a conclusion, the Möbius matrix $M_{\pi\sigma} = \mu(\pi, \sigma)$ of the lattice $P(2) = \{||, \sqcap\}$ is as follows:

$$M = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

The interest in the Möbius function comes from the Möbius inversion formula:

$$f(\sigma) = \sum_{\pi\leq\sigma} g(\pi) \implies g(\sigma) = \sum_{\pi\leq\sigma} \mu(\pi, \sigma) f(\pi)$$

In linear algebra terms, the statement and proof of this formula are as follows:

Proposition 6.12. *The inverse of the adjacency matrix of P, given by*

$$A_{\pi\sigma} = \begin{cases} 1 & \text{if } \pi \leq \sigma \\ 0 & \text{if } \pi \not\leq \sigma \end{cases}$$

is the Möbius matrix of P, given by $M_{\pi\sigma} = \mu(\pi, \sigma)$.

Proof. This is well-known, coming for instance from the fact that A is upper triangular. Indeed, when inverting, we are led into the recurrence from Definition 6.11. □

As a first illustration, for $P(2)$ the formula $M=A^{-1}$ appears as follows:

$$\begin{pmatrix}1&-1\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}^{-1}$$

Also, for $P(3)=\{|||,\sqcap|,\sqcap\!\!\!\!\mid\,,|\sqcap,\sqcap\!\sqcap\}$ the formula $M=A^{-1}$ reads:

$$\begin{pmatrix}1&-1&-1&-1&2\\0&1&0&0&-1\\0&0&1&0&-1\\0&0&0&1&-1\\0&0&0&0&1\end{pmatrix}=\begin{pmatrix}1&1&1&1&1\\0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}^{-1}$$

Now back to our Gram matrix considerations, we have the following result:

Proposition 6.13. *The Gram matrix is given by $G_{kN}=AL$, where*

$$L(\pi,\sigma)=\begin{cases}N(N-1)\ldots(N-|\pi|+1)&\text{if }\sigma\leq\pi\\0&\text{otherwise}\end{cases}$$

and where $A=M^{-1}$ is the adjacency matrix of $P(k)$.

Proof. We have indeed the following computation:

$$\begin{aligned}N^{|\pi\vee\sigma|}&=\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i\geq\pi\vee\sigma\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i=\tau\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}N(N-1)\ldots(N-|\tau|+1)\end{aligned}$$

According to Proposition 6.10 and to the definition of A,L, this formula reads:

$$(G_{kN})_{\pi\sigma}=\sum_{\tau\geq\pi}L_{\tau\sigma}=\sum_\tau A_{\pi\tau}L_{\tau\sigma}=(AL)_{\pi\sigma}$$

Thus, we obtain in this way the formula in the statement. $\square$

As an illustration for the above result, at $k=2$ we have $P(2)=\{||,\sqcap\}$, and the above formula $G_{kN}=AL$ appears as follows:

$$\begin{pmatrix}N^2&N\\N&N\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}N^2-N&0\\N&N\end{pmatrix}$$

At $k=3$ we have $P(3)=\{|||,\sqcap|,\sqcap\!\!\!\!\mid\,,|\sqcap,\sqcap\!\sqcap\}$, which leads to a similar formula.

With the above result in hand, we can now investigate the linear independence properties of the vectors ξ_π. To be more precise, we have the following result:

Theorem 6.14. *The determinant of the Gram matrix G_{kN} is given by*

$$\det(G_{kN}) = \prod_{\pi \in P(k)} \frac{N!}{(N - |\pi|)!}$$

and in particular, for $N \geq k$, the vectors $\{\xi_\pi | \pi \in P(k)\}$ are linearly independent.

Proof. According to the formula in Proposition 6.13 above, we have:

$$\det(G_{kN}) = \det(A) \det(L)$$

Now if we order $P(k)$ as above, with respect to the number of blocks, and then lexicographically, we see that A is upper triangular, and that L is lower triangular.

Thus $\det(A)$ can be computed simply by making the product on the diagonal, and we obtain 1. As for $\det(L)$, this can computed as well by making the product on the diagonal, and we obtain the number in the statement, with the technical remark that in the case $N < k$ the convention is that we obtain a vanishing determinant. See [24]. □

Now back to the laws of characters, we can formulate:

Theorem 6.15. *For an easy quantum group $G = (G_N)$, coming from a category of partitions $D = (D(k,l))$, the asymptotic moments of the main character are given by*

$$\lim_{N \to \infty} \int_{G_N} \chi^k = \# D(k)$$

where $D(k) = D(\emptyset, k)$, with the limiting sequence on the left consisting of certain integers, and being stationary at least starting from the k-th term.

Proof. This follows indeed from the general formula from Proposition 6.8, by using the linear independence result from Theorem 6.14 above. □

Our next purpose will be that of understanding what happens for the basic classes of easy quantum groups. In order to deal with the orthogonal case, we will need:

Proposition 6.16. *We have the following formulae:*

(1) $\# P_2(2k) = (2k)!!$, *where* $(2k)!! = 1.3.5 \ldots (2k-3)(2k-1)$.
(2) $\# NC_2(2k) = C_k$, *where C_k are the Catalan numbers,* $C_k = \frac{1}{k+1}\binom{2k}{k}$.

Proof. This is very standard combinatorics, the proof being as follows:

(1) We have to count the pairings of $\{1, \ldots, 2k\}$. But, in order to construct such a pairing, we have $2k-1$ choices for the pair of the number 1, then $2k-3$ choices for the pair of the next number left, and so on. Thus, we obtain $(2k)!!$, as claimed.

(2) We have to count the noncrossing pairings of $\{1, \ldots, 2k\}$. But such a pairing appears by pairing 1 to an odd number, $2a+1$, and then inserting a noncrossing pairing

of $\{2,\ldots,2a\}$, and a noncrossing pairing of $\{2a+2,\ldots,2l\}$. We conclude from this that we have the following recurrence for the Catalan numbers, $C_k = \#NC_2(2k)$:

$$C_k = \sum_{a+b=k-1} C_a C_b$$

In terms of the generating series $f(z) = \sum_{k\geq 0} C_k z^k$, this recurrence gives:

$$zf^2 = \sum_{a,b\geq 0} C_a C_b z^{a+b+1} = \sum_{k\geq 1}\sum_{a+b=k-1} C_a C_b z^k = \sum_{k\geq 1} C_k z^k = f-1$$

Thus f satisfies the degree 2 equation $zf^2 - f + 1 = 0$, and by solving this equation, and choosing the solution which is bounded at $z=0$, we obtain:

$$f(z) = \frac{1-\sqrt{1-4z}}{2z}$$

By using now the Taylor formula for $\sqrt{x}$, we obtain the following formula:

$$f(z) = \sum_{k\geq 0} \frac{1}{k+1}\binom{2k}{k} z^k$$

Thus, the Catalan numbers are given by the formula in the statement. □

With these preliminaries in hand, we can now state and prove:

Theorem 6.17. *In the $N\to\infty$ limit, the law of the main character χ_u is as follows:*

(1) *For O_N we obtain a Gaussian law, $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx$.*

(2) *For O_N^+ we obtain a Wigner semicircle law, $\frac{1}{2\pi}\sqrt{4-x^2}dx$.*

Proof. These results follow from the general formula from Theorem 6.15 above, by using the knowledge of the associated categories of partitions, from section 5, then the counting formulae from Proposition 6.16, and finally by doing some calculus:

(1) For O_N the associated category of partitions is P_2, so the asymptotic moments of the main character are as follows, with the convention $k!! = 0$ when k is odd:

$$M_k = \#P_2(k) = k!!$$

In order to recapture now the corresponding measure, there are some tools here, such as the Stieltjes inversion formula, but all this is quite advanced and technical, so perhaps best is to use our intuition. A bit of thinking at O_N, and at the associated sphere S^{N-1} as well, leads to the conclusion that our asymptotic law is probably Gaussian.

With this guess in mind, what we have to do is simply take the Gaussian law, and compute its moments. And the computation here, by partial integration, gives:

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2} x^k dx = (k-1)\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2} x^{k-2} dx$$

By recurrence, we obtain from this the following moment formula:

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2}x^k dx = k!!$$

Thus our guess was right, and we have proved our result.

(2) For O_N^+ the associated category of partitions is NC_2, so the asymptotic moments of the main character are as follows, with the convention $C_k = 0$ for $k \notin \mathbb{Z}$:

$$M_k = \# NC_2(k) = C_{k/2}$$

The problem now is that none of the "classical" probability measures has the Catalan numbers as moments. Thus, we are in trouble here.

In short, we have to ask a fellow physicist. And the physicist will tell us to try the Wigner semicircle law [96]. The moments of this law can be computed with the change of variable $x = 2\cos t$, and we are led to the following formula:

$$\frac{1}{2\pi}\int_{-2}^{2}\sqrt{4-x^2}x^k dx = C_k$$

Thus, our guess was right, with the remark however that, honestly speaking, this was not really our guess, and we obtain the conclusion in the statement. □

In order to fully understand all this, and to further advance, we definitely must gain some more familiarity with the Gaussian law, and its versions. The Gaussian law traditionally appears via the Central Limit Theorem (CLT), which is as follows:

Theorem 6.18 (CLT)**.** *Given real random variables $x_1, x_2, x_3, \ldots$, which are i.i.d., centered, and with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i \sim g_t$$

where g_t is the Gaussian law of parameter t, having as density $\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}dx$.

Proof. This is something standard, the proof being in two steps, as follows:

(1) Linearization of the convolution. It well-known that the log of the Fourier transform $F_x(\xi) = \mathbb{E}(e^{i\xi x})$ does the job, in the sense that if x, y are independent, then $F_{x+y} = F_x F_y$. Let us record as well the following useful formula for F_x, in terms of moments:

$$F_x(\xi) = \sum_{k=0}^{\infty}\frac{i^k M_k(x)}{k!}\xi^k$$

(2) Study of the limit. The Fourier transform of the variable in the statement is:

$$F(\xi) = \left[F_x\left(\frac{\xi}{\sqrt{n}}\right)\right]^n = \left[1 - \frac{t\xi^2}{2n} + o(\xi^2)\right]^n \simeq e^{-t\xi^2/2}$$

But this being the Fourier transform of g_t, we obtain the result. □

In view of this, and of Theorem 6.17 above, we can expect the Winger semicircle law to appear, conceptually speaking, via some kind of "free CLT", at $t = 1$.

This is indeed true, but quite tricky. We need to develop here a free analogue of probability theory, and our starting point will be the following definition:

Definition 6.19. *Given a pair (A, tr), we call two subalgebras $B, C \subset A$ free when the following condition is satisfied, for any $x_i \in B$ and $y_i \in C$:*

$$tr(x_i) = tr(y_i) = 0 \implies tr(x_1 y_1 x_2 y_2 \ldots) = 0$$

Also, two noncommutative random variables $b, c \in A$ are called free when the C^*-algebras $B =< b >, C =< c >$ that they generate inside A are free, in the above sense.

As a first observation, there is a similarity here with the classical notion of independence. Indeed, modulo some standard identifications, two subalgebras $B, C \subset L^\infty(X)$ are independent when the following condition is satisfied, for any $x \in B$ and $y \in C$:

$$tr(x) = tr(y) = 0 \implies tr(xy) = 0$$

Thus, freeness appears as a kind of "free analogue" of independence. As a basic result now regarding these notions, and providing us with examples, we have:

Proposition 6.20. *We have the following results, valid for group algebras:*

(1) *$C^*(\Gamma), C^*(\Lambda)$ are independent inside $C^*(\Gamma \times \Lambda)$.*
(2) *$C^*(\Gamma), C^*(\Lambda)$ are free inside $C^*(\Gamma * \Lambda)$.*

Proof. In order to prove these results, we can use the fact that each group algebra is spanned by the corresponding group elements. Thus, it is enough to check the independence and freeness formulae on group elements, and this is in turn trivial. □

There are many things that can be said about the analogy between independence and freeness. We have in particular the following result, due to Voiculescu [86]:

Theorem 6.21 (Free CLT)**.** *Given self-adjoint variables $x_1, x_2, x_3, \ldots$, which are f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \sim \gamma_t$$

where γ_t is the Wigner semicircle law of parameter t, having density $\frac{1}{2\pi t}\sqrt{4t^2 - x^2}dx$.

Proof. We follow the same idea as in the proof of Theorem 6.18 above:

(1) Linearization of the free convolution. In order to model the free convolution, the best is to use the monoid algebra $C^*(\mathbb{N} * \mathbb{N})$. Indeed, we have freeness here, a bit in the same way as for the above group algebras $C^*(\Gamma * \Lambda)$, and the point is that the variables of type $S^* + f(S)$, with $S \in C^*(\mathbb{N})$ being the shift, and with $f \in \mathbb{C}[X]$ being a polynomial, are easily seen to model in moments all the distributions $\mu : \mathbb{C}[X] \to \mathbb{C}$.

Now let $f,g\in\mathbb{C}[X]$ and consider the variables $S^*+f(S)$ and $T^*+g(T)$, where $S,T\in C^*(\mathbb{N}*\mathbb{N})$ are the shifts corresponding to the generators of $\mathbb{N}*\mathbb{N}$. These variables are free, and by using a 45° argument, their sum has the same law as $S^*+(f+g)(S)$.

Thus the operation $\mu\to f$ linearizes the free convolution. We are therefore left with a computation inside $C^*(\mathbb{N})$, which is elementary, and whose conclusion is that $R_\mu=f$ can be recaptured from μ via the Cauchy transform G_μ, in the following way:

$$G_\mu(\xi)=\int_\mathbb{R}\frac{d\mu(t)}{\xi-t}\implies G_\mu\left(R_\mu(\xi)+\frac{1}{\xi}\right)=\xi$$

(2) Study of the limit. At $t=1$, the R-transform of the variable in the statement can be computed by using the linearization property, and is given by:

$$R(\xi)=nR_x\left(\frac{\xi}{\sqrt{n}}\right)\simeq\xi$$

On the other hand, the computations from the proof of Theorem 6.17 (2) show that the Cauchy transform of the Wigner law γ_1 satisfies the following equation:

$$G_{\gamma_1}\left(\xi+\frac{1}{\xi}\right)=\xi$$

Thus we have $R_{\gamma_1}(\xi)=\xi$, which by the way follows as well from $\frac{S^*+S}{2}\sim\gamma_1$, and this gives the result. The passage to the general case, $t>0$, is routine, by dilation. □

Summarizing, we have now a more conceptual understanding of Theorem 6.17, and we can actually reprove now this theorem, without any help from a fellow physicist.

In order to discuss now the unitary case, concerning U_N,U_N^+, the most straightforward way, which would allow us to reach to the results without troubles, is by working out first the complex versions of the above results. In the classical case, we have:

Theorem 6.22 (Complex CLT)**.** *Given variables $x_1,x_2,x_3,\ldots$, whose real and imaginary parts are i.i.d., centered, and with variance $t>0$, we have, with $n\to\infty$,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i\sim G_t$$

where G_t is the complex Gaussian law of parameter t, appearing as the law of $\frac{1}{\sqrt{2}}(a+ib)$, where a,b are real and independent, each following the law g_t.

Proof. This is clear from Theorem 6.18 above, by taking real and imaginary parts. □

Similarly, in the free case, we have the following result:

Theorem 6.23 (Free complex CLT). *Given variables $x_1, x_2, x_3, \ldots$, whose real and imaginary parts are f.i.d., centered, and with variance $t > 0$, we have, with $n \to \infty$,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i \sim \Gamma_t$$

where Γ_t is the Voiculescu circular law of parameter t, appearing as the law of $\frac{1}{\sqrt{2}}(a+ib)$, where a, b are self-adjoint and free, each following the law γ_t.

Proof. This is clear from Theorem 6.21 above, by taking real and imaginary parts. $\square$

There are of course many other things that can be said about $g_t, \gamma_t, G_t, \Gamma_t$, but for the moment, this is all we need. We will be back later to these laws, with more details.

We know that for O_N, O_N^+, the asymptotic law of the main character follows the laws g_1, γ_1. This suggests that for U_N, U_N^+ we should obtain G_1, Γ_1, and indeed, it is so:

Theorem 6.24. *In the $N \to \infty$ limit, the law of the main character χ_u is as follows:*

(1) *For U_N we obtain the complex Gaussian law G_1.*
(2) *For U_N^+ we obtain the Voiculescu circular law Γ_1.*

Proof. This basically follows from Theorem 6.17 above and its proof, by performing some suitable complexification manipulations, as in [89]. To be more precise:

(1) This follows from some combinatorics. To be more precise, the asymptotic moments of the main character, with respect to the colored integers, are as follows:

$$M_k = \#\mathcal{P}_2(k)$$

But, by doing some combinatorics, the moments of the variable $\frac{1}{\sqrt{2}}(a+ib)$, where a, b are real and independent, each following the law g_1, are given by the same formula.

(2) This follows too from some combinatorics. To be more precise, the asymptotic moments of the main character, with respect to the colored integers, are as follows:

$$M_k = \#\mathcal{NC}_2(k)$$

But, by doing some combinatorics, the moments of the variable $\frac{1}{\sqrt{2}}(a+ib)$, where a, b are self-adjoint and free, each following the law γ_1, are given by the same formula. $\square$

Summarizing, we have seen so far that for O_N, O_N^+, U_N, U_N^+, the asymptotic laws of the main characters are the laws $g_1, \gamma_1, G_1, \Gamma_1$ coming from the various CLT.

This is certainly nice, but there is still one conceptual problem, coming from:

Proposition 6.25. *The above convergences $law(\chi_u) \to g_1, \gamma_1, G_1, \Gamma_1$ are as follows:*

(1) *They are non-stationary in the classical case.*
(2) *They are stationary in the free case, starting from $N = 2$.*

Proof. This is something quite subtle, which can be proved as follows:

(1) Here we can use an amenability argument, based on the Kesten criterion. Indeed, O_N, U_N being coamenable, the upper bound of the support of the law of $Re(\chi_u)$ is precisely N, and we obtain from this that the law of χ_u itself depends on $N \in \mathbb{N}$.

(2) Here the result follows from the computations in section 4 above, performed when working out the representation theory of O_N^+, U_N^+, which show that the linear maps T_π associated to the noncrossing pairings are linearly independent, at any $N \geq 2$. □

In short, we are not over with our study, which seems to open more questions than it solves. Fortunately the solution to this latest question is quite simple. The idea will be that of improving our $g_1, \gamma_1, G_1, \Gamma_1$ results with certain $g_t, \gamma_t, G_t, \Gamma_t$ results, which will require $N \to \infty$ in both the classical and free cases, in order to hold at any t.

In practice, the definition that we will need is as follows:

Definition 6.26. *Given a Woronowicz algebra (A, u), the variable*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

is called truncation of the main character, with parameter $t \in (0, 1]$.

Our purpose in what follows will be that of proving that for O_N, O_N^+, U_N, U_N^+, the asymptotic laws of the truncated characters χ_t with $t \in (0, 1]$ are the laws $g_t, \gamma_t, G_t, \Gamma_t$. This is something quite technical, motivated by the findings in Proposition 6.25 above, and also by a number of more advanced considerations, to become clear later on.

In order to start now, the basic result from Proposition 6.2 is not useful in the general $t \in (0, 1]$ setting, and we must use instead general integration methods [51], [94]:

Proposition 6.27. *Assuming that $A = C(G)$ has Tannakian category $C = (C(k, l))$, the Haar integration over G is given by the Weingarten type formula*

$$\int_G u_{i_1 j_1}^{e_1} \ldots u_{i_k j_k}^{e_k} = \sum_{\pi, \sigma \in D_k} \delta_\pi(i) \delta_\sigma(j) W_k(\pi, \sigma)$$

for any colored integer $k = e_1 \ldots e_k$ and any multi-indices i, j, where D_k is a linear basis of $C(\emptyset, k)$, $\delta_\pi(i) =< \pi, e_{i_1} \otimes \ldots \otimes e_{i_k} >$, and $W_k = G_k^{-1}$, with $G_k(\pi, \sigma) =< \pi, \sigma >$.

Proof. We know from section 3 above that the integrals in the statement form altogether the orthogonal projection P^k onto the space $Fix(u^{\otimes k}) = span(D_k)$. Consider now the following linear map, with $D_k = \{\xi_k\}$ being as in the statement:

$$E(x) = \sum_{\pi \in D_k} < x, \xi_\pi > \xi_\pi$$

By a standard linear algebra computation, it follows that we have $P = WE$, where W is the inverse on $span(T_\pi | \pi \in D_k)$ of the restriction of E. But this restriction is the linear map given by G_k, and so W is the linear map given by W_k, and this gives the result. □

In the easy quantum group case, the above formula simplifies, as follows:

Theorem 6.28. *For an easy quantum group $G \subset U_N^+$, coming from a category of partitions $D = (D(k,l))$, we have the Weingarten integration formula*

$$\int_G u_{i_1 j_1}^{e_1} \dots u_{i_k j_k}^{e_k} = \sum_{\pi,\sigma \in D(k)} \delta_\pi(i)\delta_\sigma(j) W_{kN}(\pi,\sigma)$$

for any $k = e_1 \dots e_k$ and any i, j, where $D(k) = D(\emptyset, k)$, δ are usual Kronecker symbols, and $W_{kN} = G_{kN}^{-1}$, with $G_{kN}(\pi,\sigma) = N^{|\pi \vee \sigma|}$, where $|.|$ is the number of blocks.

Proof. With notations from Proposition 6.27, the Kronecker symbols are given by:

$$\delta_{\xi_\pi}(i) = < \xi_\pi, e_{i_1} \otimes \dots \otimes e_{i_k} > = \delta_\pi(i_1, \dots, i_k)$$

The Gram matrix being as well the correct one, we obtain the result. See [20]. □

We can apply this formula to truncated characters, and we obtain:

Proposition 6.29. *The moments of truncated characters are given by the formula*

$$\int_G (u_{11} + \dots + u_{ss})^k = Tr(W_{kN} G_{ks})$$

and with $N \to \infty$ this quantity equals $(s/N)^k \# D(k)$.

Proof. The first assertion follows from the following computation:

$$\begin{aligned}
\int_G (u_{11} + \dots + u_{ss})^k &= \sum_{i_1=1}^s \dots \sum_{i_k=1}^s \int u_{i_1 i_1} \dots u_{i_k i_k} \\
&= \sum_{\pi,\sigma \in D(k)} W_{kN}(\pi,\sigma) \sum_{i_1=1}^s \dots \sum_{i_k=1}^s \delta_\pi(i)\delta_\sigma(i) \\
&= \sum_{\pi,\sigma \in D(k)} W_{kN}(\pi,\sigma) G_{ks}(\sigma,\pi) \\
&= Tr(W_{kN} G_{ks})
\end{aligned}$$

We have $G_{kN}(\pi,\sigma) = N^k$ for $\pi = \sigma$, and $G_{kN}(\pi,\sigma) \leq N^{k-1}$ for $\pi \neq \sigma$. Thus with $N \to \infty$ we have $G_{kN} \sim N^k 1$, which gives:

$$\begin{aligned}
\int_G (u_{11} + \dots + u_{ss})^k &= Tr(G_{kN}^{-1} G_{ks}) \\
&\sim Tr((N^k 1)^{-1} G_{ks}) \\
&= N^{-k} Tr(G_{ks}) \\
&= N^{-k} s^k \# D(k)
\end{aligned}$$

Thus, we have obtained the formula in the statement. See [20]. □

In order to process the above formula, we will need some more theory. Given a random variable a, we write $\log F_a(\xi) = \sum_n k_n(a)\xi^n$ and $R_a(\xi) = \sum_n \kappa_n(a)\xi^n$, and call the coefficients $k_n(a), \kappa_n(a)$ cumulants, respectively free cumulants of a. With this notion in hand, we can define then more general quantities $k_\pi(a), \kappa_\pi(a)$, depending on partitions $\pi \in P(k)$, by multiplicativity over the blocks. We have then the following result:

Theorem 6.30. *We have the classical and free moment-cumulant formulae*

$$M_k(a) = \sum_{\pi\in P(k)} k_\pi(a) \quad , \quad M_k(a) = \sum_{\pi\in NC(k)} \kappa_\pi(a)$$

where $k_\pi(a), \kappa_\pi(a)$ are the generalized cumulants and free cumulants of a.

Proof. This is standard, by using the formulae of F_a, R_a, or by doing some direct combinatorics, based on the Möbius inversion formula from Proposition 6.12. See [76]. □

We can now improve our results about characters, as follows:

Theorem 6.31. *With $N \to \infty$, the laws of truncated characters are as follows:*
1. *For O_N we obtain the Gaussian law g_t.*
2. *For O_N^+ we obtain the Wigner semicircle law γ_t.*
3. *For U_N we obtain the complex Gaussian law G_t.*
4. *For U_N^+ we obtain the Voiculescu circular law Γ_t.*

Proof. With $s = [tN]$ and $N \to \infty$, the formula in Proposition 6.29 above gives:

$$\lim_{N\to\infty} \int_{G_N} \chi_t^k = \sum_{\pi\in D(k)} t^{|\pi|}$$

By using now the formulae in Theorem 6.30, this gives the results. See [20]. □

As an interesting consequence, related to [38], let us formulate as well:

Theorem 6.32. *The asymptotic laws of truncated characters for the liberation operations $O_N \to O_N^+$ and $U_N \to U_N^+$ are in Bercovici-Pata bijection, in the sense that the classical cumulants in the classical case equal the free cumulants in the free case.*

Proof. This follows indeed from the computations in the proof of Theorem 6.31. □

This result will be of great use for the liberation of more complicated compact Lie groups, because it provides us with a criterion for checking if our guesses are right.

Let us discuss now the other easy quantum groups that we have. Regarding O_N^*, U_N^* the situation is a bit complicated, but we have the following result, at $t = 1$:

Proposition 6.33. *The asymptotic laws of the main characters are as follows:*
1. *For O_N^* we obtain a symmetrized Rayleigh variable.*
2. *For U_N^* we obtain a complexification of this variable.*

Proof. The idea is to use a projective version trick. Indeed, assuming that $G = (G_N)$ is easy, coming from a category of pairings D, we have:

$$\lim_{N\to\infty}\int_{PG_N}(\chi\chi^*)^k = \#D((\circ\bullet)^k)$$

In our case, where $G_N = O_N^*, U_N^*$, we can therefore use Theorem 5.9 above, and we are led to the conclusions in the statement. See [25], [26], [83]. □

The above result is of course something quite modest. We will be back to the quantum groups O_N^*, U_N^* in section 12 below, with some powerful modelling results for them.

Next in our lineup, we have the bistochastic quantum groups. We have here:

Proposition 6.34. *For the bistochastic quantum groups B_N, B_N^+, C_N, C_N^+, the asymptotic laws of truncated characters appear as modified versions of $g_t, \gamma_t, G_t, \Gamma_t$, and the operations $O_N \to O_N^+$ and $U_N \to U_N^+$ are compatible with the Bercovici-Pata bijection.*

Proof. This follows indeed by using the same methods as for O_N, O_N^+, U_N, U_N^+, with the verification of the Bercovici-Pata bijection being elementary, and with the computation of the corresponding laws being routine as well. See [35], [26], [83]. □

Regarding now the twists, we have here the following general result:

Proposition 6.35. *The integration over $\bar{G}_N$ is given by the Weingarten type formula*

$$\int_{\bar{G}_N} u_{i_1j_1}\ldots u_{i_kj_k} = \sum_{\pi,\sigma\in D(k)} \bar{\delta}_\pi(i)\bar{\delta}_\sigma(j)W_{kN}(\pi,\sigma)$$

where W_{kN} is the Weingarten matrix of G_N.

Proof. This follows from the general Weingarten formula from Proposition 6.27, with the corresponding Gram matrix being computed exactly as in the untwisted case. See [4]. □

As a consequence of the above result, we have another general result, as follows:

Theorem 6.36. *The Schur-Weyl twisting operation $G_N \leftrightarrow \bar{G}_N$ leaves invariant:*

1. *The law of the main character.*
2. *The coamenability property.*
3. *The asymptotic laws of truncated characters.*

Proof. This basically follows from Proposition 6.35, as follows:

(1) This is clear from the integration formula.

(2) This follows from (1), and from the Kesten criterion.

(3) This follows once again from the integration formula. □

To summarize, we have results for all the easy quantum groups introduced so far, except for S_N, and in each case we obtain Gaussian laws, and their versions.

7. Quantum permutations

The quantum groups that we considered so far, namely O_N, U_N and their liberations and twists, are of "continuous" nature. In order to have as well "discrete" examples, the idea will be that of looking at the corresponding quantum reflection groups.

In this section we discuss the quantum permutation groups. These are the simplest quantum reflection groups, but are interesting as well, as objects on their own.

Let us start with a functional analytic description of the usual symmetric group:

Proposition 7.1. *Consider the symmetric group S_N.*

1. *The standard coordinates $v_{ij} \in C(S_N)$, coming from the embedding $S_N \subset O_N$ given by the permutation matrices, are given by $v_{ij} = \chi(\sigma|\sigma(j) = i)$.*
2. *The matrix $v = (v_{ij})$ is magic, in the sense that its entries are orthogonal projections, summing up to 1 on each row and each column.*
3. *The algebra $C(S_N)$ is isomorphic to the universal commutative C^*-algebra generated by the entries of a $N \times N$ magic matrix.*

Proof. These results are all elementary, as follows:

(1) We recall that the canonical embedding $S_N \subset O_N$, coming from the standard permutation matrices, is given by $\sigma(e_j) = e_{\sigma(j)}$. Thus, we have:

$$\sigma = \sum_j e_{\sigma(j)j}$$

It follows that the standard coordinates on $S_N \subset O_N$ are given by $v_{ij}(\sigma) = \delta_{i,\sigma(j)}$. In other words, we must have $v_{ij} = \chi(\sigma|\sigma(j) = i)$, as claimed.

(2) Any characteristic function $\chi \in \{0, 1\}$ being a projection in the operator algebra sense ($\chi^2 = \chi^* = \chi$), we have indeed a matrix of projections. As for the sum 1 condition on rows and columns, this is clear from the formula of the elements v_{ij}.

(3) Consider the universal algebra in the statement, namely:

$$A = C^*_{comm}\left((w_{ij})_{i,j=1,\ldots,N}\Big|w = \text{magic}\right)$$

We have a quotient map $A \to C(S_N)$, given by $w_{ij} \to v_{ij}$. On the other hand, by using the Gelfand theorem we can write $A = C(X)$, with X being a compact space, and by using the coordinates w_{ij} we have $X \subset O_N$, and then $X \subset S_N$. Thus we have as well a quotient map $C(S_N) \to A$ given by $v_{ij} \to w_{ij}$, and this gives (3). See Wang [92]. □

With the above result in hand, we can now formulate, following [92]:

Proposition 7.2. *The following is a Woronowicz algebra,*

$$C(S_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u = \text{magic}\right)$$

and the compact quantum group S_N^+ is called quantum permutation group.

Proof. As a first remark, the algebra $C(S_N^+)$ is indeed well-defined, because the magic condition forces $||u_{ij}||\leq 1$, for any C^*-norm. Our claim now is that, by using the universal property of this algebra, we can define maps Δ,ε,S as in Definition 2.1.

Consider indeed the matrix $U_{ij}=\sum_k u_{ik}\otimes u_{kj}$. We have $U_{ij}=U_{ij}^*$, and in fact the entries U_{ij} are orthogonal projections, because we have as well:

$$U_{ij}^2=\sum_{kl}u_{ik}u_{il}\otimes u_{kj}u_{lj}=\sum_k u_{ik}\otimes u_{kj}=U_{ij}$$

In order to prove now that the matrix $U=(U_{ij})$ is magic, it remains to verify that the sums on the rows and columns are 1. But this can be checked as follows:

$$\sum_j U_{ij}=\sum_{jk}u_{ik}\otimes u_{kj}=\sum_k u_{ik}\otimes 1=1\otimes 1$$

$$\sum_i U_{ij}=\sum_{ik}u_{ik}\otimes u_{kj}=\sum_k 1\otimes u_{kj}=1\otimes 1$$

Thus the matrix $U=(U_{ij})$ is magic indeed, and so we can define a comultiplication map by setting $\Delta(u_{ij})=U_{ij}$. By using a similar reasoning, we can define as well a counit map by $\varepsilon(u_{ij})=\delta_{ij}$, and an antipode map by $S(u_{ij})=u_{ji}$. Thus the Woronowicz algebra axioms from Definition 2.1 are satisfied, and this finishes the proof. □

The terminology comes from the following result, also from [92]:

Proposition 7.3. *The quantum permutation group S_N^+ acts on the set $X=\{1,\ldots,N\}$, the corresponding coaction map $\Phi:C(X)\to C(S_N^+)\otimes C(X)$ being given by:*

$$\Phi(\delta_i)=\sum_j u_{ij}\otimes\delta_j$$

In fact, S_N^+ is the biggest compact quantum group acting on X, by leaving the counting measure invariant, in the sense that $(id\otimes tr)\Phi=tr(.)1$, where $tr(\delta_i)=\frac{1}{N},\forall i$.

Proof. Our claim is that given a compact matrix quantum group G, the formula $\Phi(\delta_i)=\sum_j u_{ij}\otimes\delta_j$ defines a morphism of algebras, which is a coaction map, leaving the trace invariant, precisely when the matrix $u=(u_{ij})$ is a magic corepresentation of $C(G)$.

Indeed, let us first determine when Φ is multiplicative. We have:

$$\Phi(\delta_i)\Phi(\delta_k)=\sum_{jl}u_{ij}u_{kl}\otimes\delta_j\delta_l=\sum_j u_{ij}u_{kj}\otimes\delta_j$$

$$\Phi(\delta_i\delta_k)=\delta_{ik}\Phi(\delta_i)=\delta_{ik}\sum_j u_{ij}\otimes\delta_j$$

We conclude that the multiplicativity of Φ is equivalent to the following conditions:

$$u_{ij}u_{kj}=\delta_{ik}u_{ij}\quad,\quad\forall i,j,k$$

Regarding now the unitality of Φ, we have the following formula:

$$\Phi(1)=\sum_i\Phi(\delta_i)=\sum_{ij}u_{ij}\otimes\delta_j=\sum_j\left(\sum_i u_{ij}\right)\otimes\delta_j$$

Thus Φ is unital when the following conditions are satisfied:

$$\sum_i u_{ij}=1\quad,\quad\forall j$$

Finally, the fact that Φ is a $*$-morphism translates into:

$$u_{ij}=u_{ij}^*\quad,\quad\forall i,j$$

Summing up, in order for $\Phi(\delta_i)=\sum_j u_{ij}\otimes\delta_j$ to be a morphism of C^*-algebras, the elements u_{ij} must be projections, summing up to 1 on each column of u. Regarding now the preservation of the trace condition, observe that we have:

$$(id\otimes tr)\Phi(\delta_i)=\frac{1}{n}\sum_j u_{ij}$$

Thus the trace is preserved precisely when the elements u_{ij} sum up to 1 on each of the rows of u. We conclude from this that $\Phi(\delta_i)=\sum_j u_{ij}\otimes\delta_j$ is a morphism of C^*-algebras preserving the trace precisely when u is magic, and since the coaction conditions on Φ are equivalent to the fact that u must be a corepresentation, this finishes the proof of our claim. But this claim proves all the assertions in the statement. □

As a perhaps quite surprising result now, also from [92], we have:

Proposition 7.4. *We have an embedding of compact quantum groups $S_N\subset S_N^+$, given at the algebra level by*

$$u_{ij}\to\chi\left(\sigma\middle|\sigma(j)=i\right)$$

and this embedding is an isomorphism at $N\leq 3$, but not at $N\geq 4$, where S_N^+ is non-classical, infinite compact quantum group.

Proof. The fact that we have indeed an embedding as above is clear from Proposition 7.1 and Proposition 7.2. Note that this follows as well from Proposition 7.3. Regarding now the second assertion, we can prove this in four steps, as follows:

Case $N=2$. The result here is trivial, the 2×2 magic matrices being by definition as follows, with p being a projection:

$$U=\begin{pmatrix}p&1-p\\1-p&p\end{pmatrix}$$

Indeed, this shows that the entries of a 2×2 magic matrix must pairwise commute, and so the algebra $C(S_2^+)$ follows to be commutative, which gives the result.

Case $N=3$. This is more tricky, and we present here a simple, recent proof, from [69]. By using the same abstract argument as in the $N=2$ case, and by permuting rows and columns, it is enough to check that u_{11},u_{22} commute. But this follows from:

$$\begin{aligned}
u_{11}u_{22} &= u_{11}u_{22}(u_{11}+u_{12}+u_{13})\\
&= u_{11}u_{22}u_{11}+u_{11}u_{22}u_{13}\\
&= u_{11}u_{22}u_{11}+u_{11}(1-u_{21}-u_{23})u_{13}\\
&= u_{11}u_{22}u_{11}
\end{aligned}$$

Indeed, by applying the involution to this formula, we obtain from this that we have $u_{22}u_{11}=u_{11}u_{22}u_{11}$ as well, and so we get $u_{11}u_{22}=u_{22}u_{11}$, as desired.

Case $N=4$. In order to prove our various claims about S_4^+, consider the following matrix, with p,q being projections, on some infinite dimensional Hilbert space:

$$U=\begin{pmatrix}p&1-p&0&0\\1-p&p&0&0\\0&0&q&1-q\\0&0&1-q&q\end{pmatrix}$$

This matrix is magic, and if we choose p,q as for the algebra $<p,q>$ to be not commutative, and infinite dimensional, we conclude that $C(S_4^+)$ is not commutative and infinite dimensional as well, and in particular is not isomorphic to $C(S_4)$.

Case $N\geq 5$. Here we can use the standard embedding $S_4^+\subset S_N^+$, obtained at the level of the corresponding magic matrices in the following way:

$$u\to\begin{pmatrix}u&0\\0&1_{N-4}\end{pmatrix}$$

Indeed, with this embedding in hand, the fact that S_4^+ is a non-classical, infinite compact quantum group implies that S_N^+ with $N\geq 5$ has these two properties as well. $\square$

At the representation theory level now, we have the following result, from [35]:

Theorem 7.5. *For the quantum permutation group S_N^+, the intertwining spaces for the tensor powers of the fundamental corepresentation $u=(u_{ij})$ are given by:*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in NC(k,l)\right)$$

In other words, S_N^+ is easy, coming from the category of noncrossing partitions NC.

Proof. We use the Tannakian duality results from sections 4 and 5 above. According to Proposition 7.2, the algebra $C(S_N^+)$ appears as follows:

$$C(S_N^+)=C(O_N^+)\Big/\Big\langle u=\text{magic}\Big\rangle$$

On the other hand, we know from the proof of Theorem 5.10 that if we denote by $\mu \in P(2,1)$ the one-block partition, then we have:

$$T_\mu \in Hom(u^{\otimes 2}, u) \iff u_{ij}u_{ik} = \delta_{jk}u_{ij}, \forall i, j, k$$

The condition on the right being equivalent to the magic condition, we obtain:

$$C(S_N^+) = C(O_N^+) \Big/ \Big\langle T_\mu \in Hom(u^{\otimes 2}, u) \Big\rangle$$

By using now the general theory from section 5, we conclude that the quantum group S_N^+ is indeed easy, with the corresponding category of partitions being:

$$D =< \mu >$$

But this latter category is NC, as one can see by "chopping" arbitrary noncrossing partitions into μ-shaped components, and so we obtain the result. □

The above result is in clear analogy with Theorem 5.10, stating that the usual permutation group S_N is easy, coming from the category of all partitions P.

As a technical comment, there might seem to be a bit of a clash between Theorem 5.10 and Theorem 7.5 at $N = 2, 3$, where $S_N = S_N^+$. However, there is no clash, because the implementation of the partitions is not faithful. We will be back to this.

In order to clarify all this, and to understand as well the representation theory of S_N^+ at $N \geq 4$, we will need some combinatorics. Let us start with:

Proposition 7.6. *We have a bijection $NC(k) \simeq NC_2(2k)$, constructed as follows:*

(1) *The application $NC(k) \to NC_2(2k)$ is the "fattening" one, obtained by doubling all the legs, and doubling all the strings as well.*
(2) *Its inverse $NC_2(2k) \to NC(k)$ is the "shrinking" application, obtained by collapsing pairs of consecutive neighbors.*

Proof. The fact that the two operations in the statement are indeed inverse to each other is clear, by computing the corresponding two compositions, with the remark that the construction of the fattening operation requires the partitions to be noncrossing. □

The above result suggests using the theory of O_N^+, from section 4 above, in order to obtain results about S_N^+. This will be indeed our idea, and we have:

Proposition 7.7. *The Gram matrices of $NC_2(2k), NC(k)$ are related as follows, where $\pi \to \pi'$ is the shrinking operation, and Δ_{kn} is the diagonal of G_{kn}:*

$$G_{2k,n}(\pi, \sigma) = n^k(\Delta_{kn}^{-1} G_{k,n^2} \Delta_{kn}^{-1})(\pi', \sigma')$$

In particular, we have $\det(G_{k,n^2}) \neq 0$ for any $n \geq 2$, and so the family of vectors $\{\xi_\pi | \pi \in NC(k)\} \subset (\mathbb{C}^N)^{\otimes k}$, with $N = n^2$, is linearly independent.

Proof. In the context of Proposition 7.6 above, it is elementary to see that we have:

$$|\pi \vee \sigma| = k + 2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|$$

We therefore have the following formula, valid for any $n \in \mathbb{N}$:

$$n^{|\pi \vee \sigma|} = n^{k+2|\pi' \vee \sigma'|-|\pi'|-|\sigma'|}$$

Thus, we obtain the formula in the statement. Now by applying the determinant, we obtain from this of formula of the following type, with $C > 0$ being a constant:

$$\det(G_{2k,n}) = C \cdot \det(G_{k,n^2})$$

Since we know from section 4 above, from our results regarding O_n^+, that we have $\det(G_{2k,n}) \neq 0$, we conclude that we have as well $\det(G_{k,n^2}) \neq 0$, as claimed. □

Summarizing, we have now some good knowledge of $NC(k)$, which includes a linear independence result for the associated vectors ξ_π, valid at any $N = n^2$ with $n \geq 2$.

This technology covers for instance the quantum group S_4^+, whose understanding would be our first objective here. However, in order to deal directly with the $N \geq 4$ case, we would need linear independence results at any $N \geq 4$. We have here:

Theorem 7.8. *Consider the Temperley-Lieb algebra of index $N \geq 4$, defined as*

$$TL_N(k) = span(NC_2(k,k))$$

with product given by the rule $\bigcirc = N$, when concatenating.

1. *We have a representation $i : TL_N(k) \to B((\mathbb{C}^N)^{\otimes k})$, given by $\pi \to T_\pi$.*
2. *$Tr(T_\pi) = N^{loops(<\pi>)}$, where $\pi \to <\pi>$ is the closing operation.*
3. *The linear form $\tau = Tr \circ i : TL_N(k) \to \mathbb{C}$ is a faithful positive trace.*
4. *The representation $i : TL_N(k) \to B((\mathbb{C}^N)^{\otimes k})$ is faithful.*

In particular, the vectors $\{\xi_\pi | \pi \in NC(k)\} \subset (\mathbb{C}^N)^{\otimes k}$ are linearly independent.

Proof. All this is quite standard, but advanced, the idea being as follows:

(1) This is clear from the categorical properties of $\pi \to T_\pi$.

(2) This follows indeed from the following computation:

$$\begin{aligned} Tr(T_\pi) &= \sum_{i_1 \ldots i_k} \delta_\pi \begin{pmatrix} i_1 \ldots i_k \\ i_1 \ldots i_k \end{pmatrix} \\ &= \#\left\{ i_1, \ldots, i_k \in \{1, \ldots, N\} \middle| \ker \begin{pmatrix} i_1 \ldots i_k \\ i_1 \ldots i_k \end{pmatrix} \geq \pi \right\} \\ &= N^{loops(<\pi>)} \end{aligned}$$

(3) The traciality of τ is clear, because Tr is tracial. Regarding now the faithfulness, this is best viewed via the formula $\tau(\pi) = N^{loops(<\pi>)}$, coming from (2).

The point indeed is that the Temperley-Lieb algebra $TL_N(k)$ appears in a massive number of mathematical contexts, with its standard trace $\tau(\pi) = N^{loops(<\pi>)}$ playing a key role in each of these situations, and known in particular to be faithful.

The original argument here, due to Jones [66], is very beautiful. According to the work of von Neumann and others [73], [90], of particular importance in mathematical physics are the C^*-algebras $A \subset B(H)$ which are closed under the weak topology, making the maps $a \to a\xi$ with $\xi \in H$ continuous, which are called von Neumann algebras.

The structure and classification work for such algebras, basically due to von Neumann and Connes, leads to the conclusion that the "building blocks" of the theory are the von Neumann algebras A which are infinite dimensional, $\dim A = \infty$, have trivial center, $Z(A) = \mathbb{C}$, and possess a faithful trace $tr : A \to \mathbb{C}$. These are called II_1 factors.

Now given an inclusion of II_1 factors $A_1 \subset A_2$, which is once again something natural in physics, we can consider the orthogonal projection $e_1 : A_2 \to A_1$, and set $A_3 =< A_2, e_1 >$. This procedure, discovered by Jones and called "basic construction", can be iterated, and we obtain in this way a whole tower of II_1 factors, with projections, as follows:

$$A_1 \subset_{e_1} A_2 \subset_{e_2} A_3 \subset_{e_3} A_4 \subset \ldots\ldots$$

The point now is that the sequence of projections $e_1, e_2, e_3, \ldots \in B(H)$ behaves exactly as the sequence of diagrams $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \in TL_N$ given by $\varepsilon_1 = {}^{\cup}_{\cap}$, $\varepsilon_2 = |{}^{\cup}_{\cap}$, $\varepsilon_3 = ||{}^{\cup}_{\cap}$, and so on, with the parameter being the index, $N = [A_2, A_1]$. Thus, we have a model for the Temperley-Lieb algebra, and the trace is the one that we are interested in:

$$TL_N = span(e_i) \quad , \quad tr(\pi) = N^{loops(<\pi>)}$$

As a conclusion, in this situation, the Temperley-Lieb trace is the II_1 factor trace, so it is faithful. Together with the standard fact that inclusions of II_1 factors $A_1 \subset A_2$ can be constructed for any index values $N \geq 4$, this gives the result. See Jones [66].

(4) This follows from (3) above, via a standard positivity argument.

As for the last assertion, this follows from (4), by fattening the partitions. $\square$

For our purposes, the final conclusion of Theorem 7.8 is exactly what we need. The problem, however, is that the proof of this fact remains quite heavy, based on [66]. We will be back to this a bit later, with the outline of a few alternative arguments.

We can now work out the representation theory of S_N^+, as follows:

Theorem 7.9. *The quantum groups S_N^+ with $N \geq 4$ have the following properties:*

1. *The moments of the main character are the Catalan numbers, $\int \chi^k = C_k$.*
2. *The fusion rules are $r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+1} + \ldots + r_{k+l}$, as for SO_3.*

Proof. We know from Theorem 7.8 above that the vectors $\{\xi_\pi | \pi \in NC(k)\} \subset (\mathbb{C}^N)^{\otimes k}$ are linearly independent, and by using this, the proof, from [2], goes as follows:

(1) We have indeed the following computation, with the various equalities coming from Proposition 6.2, Theorem 7.5, Proposition 7.6 and Proposition 6.16:

$$\int_{S_N^+}\chi^k=\dim(Fix(u^{\otimes k}))=\#NC(k)=\#NC_2(2k)=C_k$$

(2) This is standard, by using the moment formula in (1), and the known theory of SO_3. Let indeed $A=span(\chi_k|k\in\mathbb N)$ be the algebra of characters of SO_3. We can define a morphism $\Psi:A\to C(S_N^+)$ by $\chi_1\to f_1-1$, where f_1 is the character of the fundamental representation of S_N^+. The elements $f_k=\Psi(\chi_k)$ verify then:

$$f_kf_l=f_{|k-l|}+f_{|k-l|+1}+\ldots+f_{k+l}$$

We prove now by recurrence on k that each f_k is the character of an irreducible corepresentation r_k of $C(S_N^+)$, non-equivalent to $r_0,\ldots,r_{k-1}$. At $k=0,1$ this is clear.

Assume now that the result holds at $k-1$. By integrating characters we have then $r_{k-2},r_{k-1}\subset r_{k-1}\otimes r_1$, exactly as for SO_3, so there exists a corepresentation r_k such that $r_{k-1}\otimes r_1=r_{k-2}+r_{k-1}+r_k$. Once again by integrating characters, we conclude that r_k is irreducible, and non-equivalent to $r_1,\ldots,r_{k-1}$, as for SO_3. This proves our claim.

Finally, since any irreducible representation of S_N^+ must appear in some tensor power of u, and we have a formula for decomposing each $u^{\otimes k}$ into sums of representations r_l, we conclude that these representations r_l are all the irreducible representations of S_N^+. □

The above result is quite surprising, and raises a massive number of questions. We would like to better understand the relation with SO_3, and more generally see what happens at values $N=n^2$ with $n\geq 2$, and also compute the law of χ, and so on.

We will come up with answers to all these questions, but we will do this slowly.

One way of understanding the relation with SO_3 comes from noncommutative geometry considerations. We recall that, according to the general theory from section 1, each finite dimensional C^*-algebra A can be written as $A=C(X)$, with X being a "noncommutative finite space". We make the convention that each such space X is endowed with its counting measure, corresponding to the canonical trace $tr:A\subset B(A)\to\mathbb C$.

We have then the following general result, also from [2]:

Theorem 7.10. *Given a noncommutative finite space X, there exists a universal compact quantum group $G^+(X)$ acting on X, by leaving the counting measure invariant, and:*

1. *For $X=\{1,\ldots,N\}$ we have $G^+(X)=S_N^+$.*
2. *For $X=Spec(M_2(\mathbb C))$ we have $G^+(X)=SO_3$.*
3. *The fusion semiring of $G^+(X)$ is independent of X.*
4. *Thus, S_N^+ with $N\geq 4$ has the same fusion semiring as SO_3.*

Proof. In this statement the first assertion follows as in Proposition 7.3 above, with the remark that what we have here is a theoretical result, and so we are not in need of working out the precise algebraic formulae between generators. In fact, if we denote by m, u the multiplication and the unit of $A = C(X)$, the coaction conditions reformulate as:

$$m \in Hom(u^{\otimes 2}, u) \quad , \quad u \in Fix(u)$$

Thus, we can define our quantum group by imposing these relations, and all this is quite standard. Regarding now the other assertions, the proofs here are as follows:

(1) This is something that we already know, from Proposition 7.3 above.

(2) We use here some standard facts about SU_2, SO_3. We have an action by conjugation $SU_2 \curvearrowright M_2(\mathbb{C})$, and this action produces, via the canonical quotient map $SU_2 \to SO_3$, an action $SO_3 \curvearrowright M_2(\mathbb{C})$. On the other hand, it is quite routine to check, by using arguments like those in the proof of Proposition 7.4, that any action $G \curvearrowright M_2(\mathbb{C})$ must come from a classical group. Thus the action $SO_3 \curvearrowright M_2(\mathbb{C})$ is universal, as claimed.

(3) This follows a bit as in the proof of Theorem 7.9 above. To be more precise, this follows from the fact that the multiplication and unit of any complex algebra, and in particular of $\mathbb{C}^N$, can be modelled by the following two diagrams:

$$m = |\cup| \quad , \quad u = \cap$$

But these diagrams generate the Temperley-Lieb algebra TL, and by collapsing neighbors, as in the proof of Theorem 7.9, we are led in this way to the category NC.

(4) This follows indeed from (1,2,3) above. □

All this is certainly quite conceptual, but perhaps a bit too abstract. At $N = 4$ we can formulate a more concrete result on the subject, by using the following construction:

Definition 7.11. $C(SO_3^{-1})$ *is the universal* C^**-algebra generated by the entries of a* 3×3 *orthogonal matrix* $a = (a_{ij})$*, with the following relations:*

1. *Skew-commutation:* $a_{ij}a_{kl} = \pm a_{kl}a_{ij}$*, with sign* $+$ *if* $i \neq k, j \neq l$*, and* $-$ *otherwise.*
2. *Twisted determinant condition:* $\Sigma_{\sigma \in S_3} a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)} = 1$.

Observe the similarity with the twising constructions from section 5. However, SO_3 being not easy, we are not exactly in the Schur-Weyl twisting framework from there.

Our first task would be to prove that $C(SO_3^{-1})$ is a Woronowicz algebra. This is of course possible, by doing some computations, but we will not need to do these computations, because the result follows from the following theorem, from [11]:

Theorem 7.12. *We have an isomorphism of compact quantum groups*

$$S_4^+ = SO_3^{-1}$$

given by the Fourier transform over the Klein group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Consider indeed the matrix $a^+ = diag(1, a)$, corresponding to the action of SO_3^{-1} on $\mathbb{C}^4$, and apply to it the Fourier transform over the Klein group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$:

$$u = \frac{1}{4}\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&-1&1&-1\\1&1&-1&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&a_{11}&a_{12}&a_{13}\\0&a_{21}&a_{22}&a_{23}\\0&a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&-1&1&-1\\1&1&-1&-1\end{pmatrix}$$

It is routine to check that this matrix is magic, and vice versa, i.e. that the Fourier transform over K converts the relations in Definition 7.11 into the magic relations in Definition 7.1. Thus, we obtain the identification from the statement. □

It is possible to get beyond this, with an ADE classification of the closed subgroups of $S_4^+ = SO_3^{-1}$. All this is, however, quite technical, and we refer here to [11].

As an overall conclusion, the twisting formula $S_4^+ = SO_3^{-1}$ ultimately comes from something of type $X_4 \simeq M_2$, where $X_4 = \{1, 2, 3, 4\}$ and $M_2 = Spec(M_2(\mathbb{C}))$.

Yet another extension of Theorem 7.12, which is however quite technical, comes by looking at the general case $N = n^2$, with $n \geq 2$. It is possible indeed to complement Theorem 7.10 above with a general twisting result of type $G^+(\widehat{H}_\sigma) = G^+(\widehat{H})^\sigma$, valid for any finite group H and any 2-cocycle σ on it. In the case $H = \mathbb{Z}_n^2$ with Fourier cocycle, this leads to the conclusion that PO_n^+ appears as a cocycle twist of $S_{n^2}^+$. See [17].

Let us just record here an interesting probabilistic fact, from [17] as well:

Theorem 7.13. *The following families of variables have the same joint law,*

(1) $\{u_{ij}^2\} \in C(O_n^+)$,

(2) $\{X_{ij} = \frac{1}{n}\sum_{ab} p_{ia,jb}\} \in C(S_{n^2}^+)$,

where $u = (u_{ij})$ and $p = (p_{ia,jb})$ are the corresponding fundamental corepresentations.

Proof. As already mentioned, this result can be obtained via twisting methods. An alternative approach is by using the Weingarten formula for our two quantum groups, and the shrinking operation $\pi \to \pi'$. Indeed, we obtain the following moment formulae:

$$\int_{O_n^+} u_{ij}^{2k} = \sum_{\pi,\sigma\in NC_2(2k)} W_{2k,n}(\pi,\sigma)$$

$$\int_{S_{n^2}^+} X_{ij}^k = \sum_{\pi,\sigma\in NC_2(2k)} n^{|\pi'|+|\sigma'|-k} W_{k,n^2}(\pi',\sigma')$$

According to Proposition 7.7 the summands coincide, and so the moments are equal, as desired. The proof in general, dealing with joint moments, is similar. □

The above result is quite interesting, because it makes a connection between free hyperspherical and free hypergeometric laws. We refer here to [17], [23].

Before getting into further probabilistic aspects, let us however go back to the linear independence result in Theorem 7.8, and discuss some alternative arguments.

We know from Proposition 7.7 that the result at $N = n^2$ with $n \in \mathbb{N}$ can be obtained from the results for O_n^+ from section 4, via the connecting formula for the Gram matrices. It is possible to use this kind of argument in general as well, by using the extended compact quantum group framework, as in Woronowicz's papers [97], [98], [99], [100], with $S^2 \neq id$. Indeed, in this framework the quantum groups O_n^+ can be deformed, as to probabilistically have $n \in [2, \infty)$, and we can obtain the faithfulness result via Proposition 7.7.

Yet another approach, which is heavy too, but only computationally, not using any extra general theory, is by explicitely computing the determinants of the Gram matrices in Proposition 7.7. The formulae here, due to Di Francesco [57], are as follows:

Theorem 7.14. *The Gram determinants for O_n^+, S_n^+ are given by the formulae*

$$\det(G_{kn}) = \prod_{r=1}^{[k/2]} P_r(n)^{d_{k/2,r}} \quad , \quad \det(G_{kn}) = (\sqrt{n})^{a_k} \prod_{r=1}^{k} P_r(\sqrt{n})^{d_{kr}}$$

where P_r are the Chebycheff polynomials, $P_0 = 1, P_1 = n$ and $P_{r+1} = nP_r - P_{r-1}$,

$$f_{kr} = \binom{2k}{k-r} - \binom{2k}{k-r-1}$$

depending on $k, r \in \mathbb{Z}$, with $f_{kr} = 0$ for $k \notin \mathbb{Z}$, and $d_{kr} = f_{kr} - f_{k+1,r}$.

Proof. Let $\Gamma = \mathbb{N}$, regarded as Cayley graph of O_n^+, and consider its orthogonal polynomial eigenvector $\mu(r) = P_r(n)$, with eigenvalue n. Let also L_k be the set of length k loops $l = l_1 \ldots l_k$ based at 0, and $H_k = span(L_k)$. For $\pi \in NC_2(k)$ we define $f_\pi \in H_k$ by:

$$f_\pi = \sum_{l \in L_k} \left(\prod_{i \sim_\pi j} \delta(l_i, l_j^o) \gamma(l_i) \right) l$$

Here $e \to e^o$ is the edge reversing, and the "spin factor" is $\gamma = \sqrt{\mu(t)/\mu(s)}$, where s, t are the source and target of the edges. The point is that we have:

$$G_{kn}(\pi, \sigma) = < f_\pi, f_\sigma >$$

We have a bijection $NC_2(k) \to L_k$, constructed as follows. For $\pi \in NC_2(k)$ and $0 \leq i \leq k$ we define $h_\pi(i)$ to be the number of $1 \leq j \leq i$ which are joined by π to a number strictly larger than i. We then define a loop $l(\pi) = l(\pi)_1 \ldots l(\pi)_k$, where $l(\pi)_i$ is the edge from $h_\pi(i-1)$ to $h_\pi(i)$. Consider now the following matrix:

$$T_{kn}(\pi, \sigma) = \prod_{i \sim_\pi j} \delta(l(\sigma)_i, l(\sigma)_j^o) \gamma(l(\sigma)_i)$$

We have then $f_\pi = \sum_\sigma T_{kn}(\pi, \sigma) \cdot l(\sigma)$, and we obtain from this:

$$G_{kn} = T_{kn} T_{kn}^t$$

If we consider the partial order on $NC_2(k)$ given by $\pi \leq \sigma$ if $h_\pi(i) \leq h_\sigma(i)$ for $i = 1,\ldots,k$, then $\sigma \not\leq \pi$ implies $T_{kn}(\pi,\sigma) = 0$, and so T_{kn} is lower triangular. Thus:

$$\det(T_{kn}) = \prod_\pi T_{kn}(\pi,\pi) = \prod_\pi \prod_{i\sim_\pi j} \sqrt{\frac{P_{h_{\pi(i)}}}{P_{h_{\pi(i)-1}}}} = \prod_{r=1}^{k/2} P_r^{e_{kr}/2}$$

Here the exponents appearing on the right are by definition as follows:

$$e_{kr} = \sum_\pi \sum_{i\sim_\pi j} \delta_{h_\pi(i),r} - \delta_{h_\pi(i),r+1}$$

On the other hand, by doing some combinatorics, for $1 \leq r \leq k/2$ we have:

$$\sum_\pi \sum_{i\sim_\pi j} \delta_{h_\pi(i)r} = f_{k/2,r}$$

Thus, we obtain the formula for O_n^+ in the statement. As for the formula for S_n^+, this follows from this, and from the formula in Proposition 7.7. See [24], [57]. □

Summarizing, there is a lot of interesting mathematics in connnection with S_N^+. Passed the problem of understanding all this, at a first glance, this is a good thing.

Let us go back now to our main result so far, namely Theorem 7.9, and further build on that. The dimensions of the representations appearing there are as follows:

Proposition 7.15. *The dimensions of the irreducible corepresentations of S_N^+ are*

$$\dim(r_k) = \frac{q^{k+1} - q^{-k}}{q-1}$$

where q, q^{-1} are the roots of $X^2 - (N-2)X + 1 = 0$.

Proof. From the Clebsch-Gordan rules we have, in particular:

$$r_k r_1 = r_{k-1} + r_k + r_{k+1}$$

We are therefore led to a recurrence, and the initial data being $\dim(r_0) = 1$ and $\dim(r_1) = N - 1 = q + 1 + q^{-1}$, we are led to the following formula:

$$\dim(r_k) = q^k + q^{k-1} + \ldots + q^{1-k} + q^{-k}$$

In more compact form, this gives the formula in the statement. □

Let us work out now some probabilistic consequences. Following [10], we have:

Theorem 7.16. *The spectral measure of the main character of S_N^+ with $N \geq 4$ is the Marchenko-Pastur law of parameter 1, having the following density:*

$$\pi_1 = \frac{1}{2\pi}\sqrt{4x^{-1} - 1}dx$$

Also, S_4^+ is coamenable, and S_N^+ with $N \geq 5$ is not coamenable.

Proof. Here the first assertion follows from the following formula, which can be established by doing some calculus, and more specifically by setting $x = 4\sin^2 t$:

$$\frac{1}{2\pi}\int_0^4 \sqrt{1-4x^{-1}}\,x^k dx = C_k$$

Of course, it is a bit unclear where the Marchenko-Pastur law [72] comes from. This comes of course from our physicist friend, as it was the case with the Wigner law [96].

As for the second assertion, this follows from this, and from the Kesten criterion. □

Summarizing, we have some results, but all this is still dependent on our physicist friend, and his excellent knowledge of calculus, SU_2/SO_3, and other things, like [88].

So, our next purpose will be that of understanding, probabilistically speaking, and in a conceptual way, out of nothing, the liberation operation $S_N \to S_N^+$.

Let us begin our study with the classical case computation, for the symmetric group S_N. Here the result, which is truly remarkable, and well-known, is as follows:

Theorem 7.17. *Consider the symmetric group S_N, regarded as a compact group of matrices, $S_N \subset O_N$, via the standard permutation matrices.*

1. *The main character $\chi \in C(S_N)$, defined as usual as $\chi = \sum_i u_{ii}$, counts the number of fixed points, $\chi(\sigma) = \#\{i|\sigma(i) = i\}$.*
2. *The probability for a permutation $\sigma \in S_N$ to be a derangement, meaning to have no fixed points at all, becomes, with $N \to \infty$, equal to $1/e$.*
3. *The law of the main character $\chi \in C(S_N)$ becomes, with $N \to \infty$, a Poisson law of parameter 1, with respect to the counting measure.*

Proof. This is something very classical, and beautiful, as follows:

(1) We have indeed the following computation:

$$\chi(\sigma) = \sum_i u_{ii}(\sigma) = \sum_i \delta_{\sigma(i)i} = \#\left\{i\big|\sigma(i) = i\right\}$$

(2) This is best viewed by using the inclusion-exclusion principle. Let us set:

$$S_N^{i_1\ldots i_k} = \left\{\sigma \in S_N\Big|\sigma(i_1) = i_1, \ldots, \sigma(i_k) = i_k\right\}$$

By using the inclusion-exclusion principle, we have:

$$\mathbb{P}(\chi = 0) = \frac{1}{N!}\left(|S_N| - \sum_i |S_N^i| + \sum_{i<j}|S_N^{ij}| - \ldots + (-1)^N \sum_{i_1<\ldots<i_N} |S_N^{i_1\ldots i_N}|\right)$$

Now since $|S_N^{i_1\ldots i_k}| = (N-k)!$ for any $i_1 < \ldots < i_k$, we obtain from this:

$$\mathbb{P}(\chi = 0) = 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^{N-1}\frac{1}{(N-1)!} + (-1)^N\frac{1}{N!}$$

Since on the right we have the expansion of $\frac{1}{e}$, we conclude that we have:

$$\lim_{N\to\infty}\mathbb{P}(\chi=0)=\frac{1}{e}$$

(3) This follows by generalizing the computation in (2). To be more precise, a similar application of the inclusion-exclusion principle gives the following formula:

$$\lim_{N\to\infty}\mathbb{P}(\chi=k)=\frac{1}{k!e}$$

Thus, we obtain in the limit a Poisson law, as stated. □

In order to talk about free analogues of this, we will need some theory. Let us denote by $*$ the usual convolution of measures, and by $\boxplus$ its free version. We have then:

Theorem 7.18. *The following Poisson type limits converge, for any $t>0$,*

$$p_t=\lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)\delta_0+\frac{1}{n}\delta_t\right)^{*n}\quad,\quad \pi_t=\lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)\delta_0+\frac{1}{n}\delta_t\right)^{\boxplus n}$$

the limiting measures being the Poisson law p_t, and the Marchenko-Pastur law π_t,

$$p_t=\frac{1}{e^t}\sum_{k=0}^\infty\frac{t^k\delta_k}{k!}\quad,\quad \pi_t=\max(1-t,0)\delta_0+\frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

whose moments are given by the following formulae:

$$M_k(p_t)=\sum_{\pi\in P(k)}t^{|\pi|}\quad,\quad M_k(\pi_t)=\sum_{\pi\in NC(k)}t^{|\pi|}$$

The Marchenko-Pastur measure π_t is also called free Poisson law.

Proof. This is something standard, which follows by using either $\log F,R$ and calculus, or classical and free cumulants, and combinatorics. In combinatorial terms, the point is that the limiting measures must be those having classical and free cumulants $t,t,t,\ldots$ But this gives all the assertions, the density computations being standard. See [76]. □

We can now formulate a conceptual result about $S_N\to S_N^+$, as follows:

Theorem 7.19. *The law of the main character χ_u is as follows:*

(1) *For S_N with $N\to\infty$ we obtain a Poisson law p_1.*
(2) *For S_N^+ with $N\geq 4$ we obtain a free Poisson law π_1.*

In addition, these laws are related by the Bercovici-Pata correspondence.

Proof. This follows indeed from the computations that we have, from Theorem 7.16 and Theorem 7.17, by using the various theoretical results from Theorem 7.18. □

As in the continuous case, our purpose now will be that of extending this result to the truncated characters. In order to discuss the classical case, we first have:

Proposition 7.20. *Consider the symmetric group S_N, together with its standard matrix coordinates $u_{ij} = \chi(\sigma \in S_N | \sigma(j) = i)$. We have the formula*

$$\int_{S_N} u_{i_1 j_1} \ldots u_{i_k j_k} = \begin{cases} \frac{(N - |\ker i|)!}{N!} & \text{if } \ker i = \ker j \\ 0 & \text{otherwise} \end{cases}$$

where $\ker i$ *denotes as usual the partition of* $\{1, \ldots, k\}$ *whose blocks collect the equal indices of i, and where $|.|$ denotes the number of blocks.*

Proof. According to the definition of u_{ij}, the integrals in the statement are given by:

$$\int_{S_N} u_{i_1 j_1} \ldots u_{i_k j_k} = \frac{1}{N!} \# \left\{ \sigma \in S_N \middle| \sigma(j_1) = i_1, \ldots, \sigma(j_k) = i_k \right\}$$

Since the existence of $\sigma \in S_N$ as above requires $i_m = i_n \iff j_m = j_n$, this integral vanishes when $\ker i \neq \ker j$. As for the case $\ker i = \ker j$, if we denote by $b \in \{1, \ldots, k\}$ the number of blocks of this partition, we have $N - b$ points to be sent bijectively to $N - b$ points, and so $(N - b)!$ solutions, and the integral is $\frac{(N-b)!}{N!}$, as claimed. □

We can now compute the laws of truncated characters, and we obtain:

Proposition 7.21. *For the symmetric group $S_N \subset O_N$, regarded as a compact group of matrices, $S_N \subset O_N$, via the standard permutation matrices, the truncated character*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

counts the number of fixed points among $\{1, \ldots, [tN]\}$, and its law with respect to the counting measure becomes, with $N \to \infty$, a Poisson law of parameter t.

Proof. The first assertion comes from the formula $u_{ij} = \chi(\sigma | \sigma(j) = i)$. Regarding now the second assertion, a first proof can be obtained via inclusion-exclusion, along the lines of the proof of Theorem 7.17. However, simplest here is to use the integration formula in Proposition 7.20. With $S_{k,b}$ being the Stirling numbers, we have:

$$\begin{aligned} \int_{S_N} \chi_t^k &= \sum_{i_1 \ldots i_k = 1}^{[tN]} \int_{S_N} u_{i_1 i_1} \ldots u_{i_k i_k} \\ &= \sum_{\pi \in P_k} \frac{[tN]!}{([tN] - |\pi|!)} \cdot \frac{(N - |\pi|!)}{N!} \\ &= \sum_{b=1}^{[tN]} \frac{[tN]!}{([tN] - b)!} \cdot \frac{(N - b)!}{N!} \cdot S_{k,b} \end{aligned}$$

In particular with $N\to\infty$ we obtain the following formula:

$$\lim_{N\to\infty}\int_{S_N}\chi_t^k=\sum_{b=1}^kS_{k,b}t^b$$

But this is a Poisson(t) moment, and so we are done. □

In the free case now, the integration formula, from [20], is as follows:

Proposition 7.22. *We have the Weingarten formula*

$$\int_{S_N^+}u_{i_1j_1}\ldots u_{i_kj_k}=\sum_{\pi,\sigma\in NC(k)}\delta_\pi(i)\delta_\sigma(j)W_{kN}(\pi,\sigma)$$

where the Kronecker symbols and Weingarten matrix are defined as usual. In particular, at $k\leq 3$ we obtain in this way the same integrals as those over S_N.

Proof. The formula in the statement is the usual one. Regarding the second assertion, we can write a Weingarten formula for the usual symmetric group S_N as well, as follows:

$$\int_{S_N}u_{i_1j_1}\ldots u_{i_kj_k}=\sum_{\pi,\sigma\in P(k)}\delta_\pi(i)\delta_\sigma(j)W'_{kN}(\pi,\sigma)$$

Now since at $k\leq 3$ all the partitions of $\{1,\ldots,k\}$ are noncrossing, we have $P(k)=NC(k)$, the Weingarten functions for S_N,S_N^+ coincide, and we obtain the result. □

We can now finish our computations, and generalize Theorem 7.19, as follows:

Theorem 7.23. *The laws of truncated characters $\chi_t=\sum_{i=1}^{[tN]}u_{ii}$ are as follows:*

(1) *For S_N with $N\to\infty$ we obtain a Poisson law p_t.*
(2) *For S_N^+ with $N\to\infty$ we obtain a free Poisson law π_t.*

In addition, these laws are related by the Bercovici-Pata correspondence.

Proof. This follows from the above results:

(1) This is something that we already know, from Proposition 7.21.

(2) This is something that we know so far only at $t=1$, from Theorem 7.19. In order to deal with the general $t\in(0,1]$ case, we can use the same method as for the orthogonal and unitary quantum groups, from section 6, and we obtain the following moments:

$$M_k=\sum_{\pi\in NC(k)}t^{|\pi|}$$

But these being the moments of the free Poisson law of parameter t, as explained in Theorem 7.18 above, or in section 8 below, we obtain the result. See [21]. □

Summarizing, the liberation operation $S_N\to S_N^+$ has many common features with the liberation operations $O_N\to O_N^+$ and $U_N\to U_N^+$, studied in section 6 above.

8. Quantum reflections

We have seen that the quantum permutation groups S_N^+ are understood quite well. In this section we explore, with similar methods, some of the subgroups $G\subset S_N^+$.

Many interesting examples of quantum permutation groups appear as particular cases of the following general construction from [3], involving finite graphs:

Proposition 8.1. *Given a finite graph X, with adjacency matrix $d\in M_N(0,1)$, the following construction produces a quantum permutation group,*

$$C(G^+(X))=C(S_N^+)\Big/\Big\langle du=ud\Big\rangle$$

whose classical version $G(X)$ is the usual automorphism group of X.

Proof. The fact that we have a quantum group comes from the fact that $du=ud$ reformulates as $d\in End(u)$, which makes it clear that we are dividing by a Hopf ideal.

Regarding the second assertion, we must establish here the following equality:

$$C(G(X))=C(S_N)\Big/\Big\langle d\in End(u)\Big\rangle$$

For this purpose, observe that with $u_{ij}=\chi(\sigma|\sigma(j)=i)$, as in Proposition 7.1 above, which is the same as saying that $u_{ij}(\sigma)=\delta_{\sigma(j)i}$, we have:

$$(du)_{ij}(\sigma)=\sum_k d_{ik}u_{kj}(\sigma)=\sum_k d_{ik}\delta_{\sigma(j)k}=d_{i\sigma(j)}$$

$$(ud)_{ij}(\sigma)=\sum_k u_{ik}(\sigma)d_{kj}=\sum_k\delta_{\sigma(k)i}d_{kj}=d_{\sigma^{-1}(i)j}$$

Thus $du=ud$ reformulates as $d_{ij}=d_{\sigma(i)\sigma(j)}$, and we are led to the usual notion of an action of a permutation group on X, which leaves invariant the edges, as claimed. $\square$

Let us work out some basic examples. We have the following result:

Proposition 8.2. *The construction $X\to G^+(X)$ has the following properties:*

1. *For the N-point graph, having no edges at all, we obtain S_N^+.*
2. *For the N-simplex, having edges everywhere, we obtain as well S_N^+.*
3. *We have $G^+(X)=G^+(X^c)$, where X^c is the complementary graph.*
4. *For a disconnected union, we have $G^+(X)\,\hat{*}\,G^+(Y)\subset G^+(X\sqcup Y)$.*
5. *For the square, we obtain a non-classical, proper subgroup of S_4^+.*

Proof. All these results are elementary, the proofs being as follows:

(1) This follows from definitions, because here we have $d=0$.

(2) Here $d=\mathbb{I}$ is the all-one matrix, and since the magic condition gives $u\mathbb{I}=\mathbb{I}u=N\mathbb{I}$, we conclude that $du=ud$ is automatic in this case, and so $G^+(X)=S_N^+$.

(3) The adjacency matrices of X, X^c being related by the formula $d_X + d_{X^c} = \mathbb{I}$, we can use here the above formula $u\mathbb{I} = \mathbb{I}u = N\mathbb{I}$, and we conclude that $d_X u = u d_X$ is equivalent to $d_{X^c} u = u d_{X^c}$. Thus, we obtain $G^+(X) = G^+(X^c)$, as claimed.

(4) The adjacency matrix of a disconnected union is given by $d_{X \sqcup Y} = diag(d_X, d_Y)$. Now let $w = diag(u, v)$ be the fundamental corepresentation of $G^+(X) \,\hat{*}\, G^+(Y)$. Since $d_X u = u d_X$ and $d_Y v = v d_Y$ imply $d_{X \sqcup Y} w = w d_{X \sqcup Y}$, this gives the result.

(5) We know from (3) that we have $G^+(\square) = G^+(|\ |)$, and we know as well from (4) that we have $\mathbb{Z}_2 \,\hat{*}\, \mathbb{Z}_2 \subset G^+(|\ |)$. It follows that $G^+(\square)$ is non-classical. Finally, the inclusion $G^+(\square) \subset S_4^+$ is indeed proper, because $S_4 \subset S_4^+$ does not act on the square. $\square$

In order to further advance, we can use the spectral decomposition of d:

Proposition 8.3. *A closed subgroup $G \subset S_N^+$ acts on a graph X precisely when*

$$P_\lambda u = u P_\lambda \quad , \quad \forall \lambda \in \mathbb{R}$$

where $d = \sum_\lambda \lambda \cdot P_\lambda$ is the spectral decomposition of the adjacency matrix of X.

Proof. Since $d \in M_N(0,1)$ is a symmetric matrix, we can consider indeed its spectral decomposition, $d = \sum_\lambda \lambda \cdot P_\lambda$. We have then the following formula:

$$< d >= span\left\{ P_\lambda \Big| \lambda \in \mathbb{R} \right\}$$

But this shows that we have the following equivalence:

$$d \in End(u) \iff P_\lambda \in End(u), \forall \lambda \in \mathbb{R}$$

Thus, we are led to the conclusion in the statement. $\square$

In order to exploit this, we will often combine it with the following standard fact:

Proposition 8.4. *Consider a closed subgroup $G \subset S_N^+$, with associated coaction map $\Phi : \mathbb{C}^N \to C(G) \otimes \mathbb{C}^N$. For a linear subspace $V \subset \mathbb{C}^N$, the following are equivalent:*

1. *The magic matrix u commutes with P_V.*
2. *We have $\Phi(V) \subset C(G) \otimes V$.*

Proof. Let $P = P_V$. For any $i \in \{1, \ldots, N\}$ we have the following formula:

$$\Phi(P(\delta_i)) = \Phi\left(\sum_j P_{ij}\delta_j\right) = \sum_{jk} u_{jk} \otimes P_{ij}\delta_k = \sum_k (Pu)_{ik} \otimes \delta_k$$

On the other hand the linear map $(id \otimes P)\Phi$ is given by a similar formula:

$$(id \otimes P)(\Phi(\delta_i)) = \sum_j u_{ij} \otimes P(\delta_j) = \sum_{jk} u_{ij} \otimes P_{jk}\delta_k = \sum_k (uP)_{ik} \otimes \delta_k$$

Thus $\Phi P = (P \otimes id)\Phi$ is equivalent to $Pu = uP$, and the conclusion follows. $\square$

We have as well the following useful complementary result, from [3]:

Proposition 8.5. *Let $p \in M_N(\mathbb{C})$ be a matrix, and consider its "color" decomposition, obtained by setting $(p_c)_{ij} = 1$ if $p_{ij} = c$ and $(p_c)_{ij} = 0$ otherwise:*

$$p = \sum_{c \in \mathbb{C}} c \cdot p_c$$

Then $u = (u_{ij})$ commutes with p if and only if it commutes with all matrices p_c.

Proof. Since the multiplication $M : \delta_i \otimes \delta_j \to \delta_i \delta_j$ and the counit $C : \delta_i \to \delta_i \otimes \delta_i$ intertwine $u, u^{\otimes 2}$, their iterations $M^{(k)}, C^{(k)}$ intertwine $u, u^{\otimes k}$, and so we have:

$$p^{(k)} = M^{(k)} p^{\otimes k} C^{(k)} = \sum_{c \in \mathbb{C}} c^k p_c \in End(u)$$

Let $S = \{c \in \mathbb{C} | p_c \neq 0\}$, and $f(c) = c$. By Stone-Weierstrass we have $S =< f >$, and so for any $e \in S$ the Dirac mass at e is a linear combination of powers of f:

$$\delta_e = \sum_k \lambda_k f^k = \sum_k \lambda_k \left(\sum_{c \in S} c^k \delta_c \right) = \sum_{c \in S} \left(\sum_k \lambda_k c^k \right) \delta_c$$

The corresponding linear combination of matrices $p^{(k)}$ is given by:

$$\sum_k \lambda_k p^{(k)} = \sum_k \lambda_k \left(\sum_{c \in S} c^k p_c \right) = \sum_{c \in S} \left(\sum_k \lambda_k c^k \right) p_c$$

The Dirac masses being linearly independent, in the first formula all coefficients in the right term are 0, except for the coefficient of δ_e, which is 1. Thus the right term in the second formula is p_e, and it follows that we have $p_e \in End(u)$, as claimed. $\square$

The above results can be combined, and we are led into a "color-spectral" decomposition method for d, which can lead to a number of nontrivial results. See [3].

As a basic application of this, we can further study $G^+(\square)$, as follows:

Proposition 8.6. *The quantum automorphism group of the N-cycle is as follows:*

(1) *At $N \neq 4$ we have $G^+(X) = D_N$.*
(2) *At $N = 4$ we have $D_4 \subset G^+(X) \subset S_4^+$, with proper inclusions.*

Proof. We already know that the results hold at $N \leq 4$, so let us assume $N \geq 5$.

Given a N-th root of unity, $w^N = 1$, the vector $\xi = (w^i)$ is an eigenvector of d, with eigenvalue $w + w^{N-1}$. Now by taking $w = e^{2\pi i/N}$, it follows that $1, f, f^2, \ldots, f^{N-1}$ are eigenvectors of d. More precisely, the invariant subspaces of d are as follows, with the last subspace having dimension 1 or 2 depending on the parity of N:

$$\mathbb{C}1,\ \mathbb{C}f \oplus \mathbb{C}f^{N-1},\ \mathbb{C}f^2 \oplus \mathbb{C}f^{N-2}, \ldots$$

Consider now the associated coaction $\Phi : \mathbb{C}^N \to C(G) \otimes \mathbb{C}^N$, and write:

$$\Phi(f) = a \otimes f + b \otimes f^{N-1}$$

By taking the square of this equality we obtain:

$$\Phi(f^2) = a^2 \otimes f^2 + b^2 \otimes f^{N-2} + (ab + ba) \otimes 1$$

It follows that $ab = -ba$, and that $\Phi(f^2)$ is given by the following formula:

$$\Phi(f^2) = a^2 \otimes f^2 + b^2 \otimes f^{N-2}$$

By multiplying this with $\Phi(f)$ we obtain:

$$\Phi(f^3) = a^3 \otimes f^3 + b^3 \otimes f^{N-3} + ab^2 \otimes f^{N-1} + ba^2 \otimes f$$

Now since $N \geq 5$ implies that $1, N-1$ are different from $3, N-3$, we must have $ab^2 = ba^2 = 0$. By using this and $ab = -ba$, we obtain by recurrence on k that:

$$\Phi(f^k) = a^k \otimes f^k + b^k \otimes f^{N-k}$$

In particular at $k = N-1$ we obtain:

$$\Phi(f^{N-1}) = a^{N-1} \otimes f^{N-1} + b^{N-1} \otimes f$$

On the other hand we have $f^* = f^{N-1}$, so by applying $*$ to $\Phi(f)$ we get:

$$\Phi(f^{N-1}) = a^* \otimes f^{N-1} + b^* \otimes f$$

Thus $a^* = a^{N-1}$ and $b^* = b^{N-1}$. Together with $ab^2 = 0$ this gives:

$$(ab)(ab)^* = abb^*a^* = ab^N a^{N-1} = (ab^2)b^{N-2}a^{N-1} = 0$$

From positivity we get from this $ab = 0$, and together with $ab = -ba$, this shows that a, b commute. On the other hand $C(G)$ is generated by the coefficients of Φ, which are powers of a, b, and so $C(G)$ must be commutative, and we obtain the result. $\square$

Summarizing, this was a bad attempt in understanding $G^+(\square)$, which appears to be "exceptional" among the quantum symmetry groups of the N-cycles.

An alternative approach to $G^+(\square)$ comes by regarding the square as the $N = 2$ particular case of the N-hypercube $\square_N$. Indeed, the usual symmetry group of $\square_N$ is the hyperoctahedral group H_N, so we should have a formula of type $G(\square) = H_2^+$.

In order to clarify this, let us start with the following simple fact:

Proposition 8.7. *We have an embedding as follows, g_i being the generators of $\mathbb{Z}_2^N$,*

$$\widehat{\mathbb{Z}_2^N} \subset S^{N-1}_{\mathbb{R},+} \quad , \quad x_i = \frac{g_i}{\sqrt{N}}$$

whose image is the geometric hypercube $\square_N = \{x \in \mathbb{R}^N | x_i = \pm 1/\sqrt{N}, \forall i\}$.

Proof. This is something that we already know, from Theorem 1.21 above, and which comes from the fact that the standard generators $g_i \in C^*(\mathbb{Z}_2^N) = C(\widehat{\mathbb{Z}_2^N})$ satisfy:

$$g_i = g_i^* \quad , \quad g_i^2 = 1$$

Indeed, when rescaling by $1/\sqrt{N}$, we obtain the relations defining $\square_N$. $\square$

We can now study the quantum symmetry groups $G^+(\square_N)$, and we are led to the quite surprising conclusion, from [15], that these are the twisted orthogonal groups $\bar{O}_N$:

Theorem 8.8. *With $\mathbb{Z}_2^N =< g_1,\ldots,g_N >$ we have a coaction map*

$$\Phi : C^*(\mathbb{Z}_2^N) \to C(\bar{O}_N) \otimes C^*(\mathbb{Z}_2^N) \quad , \quad g_i \to \sum_j u_{ij} \otimes g_j$$

which makes $\bar{O}_N$ the quantum isometry group of the hypercube $\square_N = \widehat{\mathbb{Z}_2^N}$, as follows:

(1) *With $\square_N$ viewed as an algebraic manifold, $\square_N \subset S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},+}$.*
(2) *With $\square_N$ viewed as a graph, with 2^N vertices and $2^{N-1}N$ edges.*
(3) *With $\square_N$ viewed as a metric space, with metric coming from $\mathbb{R}^N$.*

Proof. Observe first that $\square_N$ is indeed an algebraic manifold, so (1) as formulated above makes sense, in the general framework of Proposition 2.23. The cube $\square_N$ is also a graph, as indicated, and so (2) makes sense as well, in the framework of Proposition 8.1. Finally, (3) makes sense as well, because we can define the quantum isometry group of a finite metric space exactly as for graphs, but with d being this time the distance matrix.

(1) In order for $G \subset O_N^+$ to act affinely on $\square_N$, the variables $G_i = \sum_j u_{ij} \otimes g_j$ must satisfy the same relations as the generators $g_i \in \mathbb{Z}_2^N$. The self-adjointness being automatic, the relations to be checked are therefore $G_i^2 = 1, G_iG_j = G_jG_i$. We have:

$$\begin{aligned} G_i^2 &= \sum_{kl} u_{ik}u_{il} \otimes g_kg_l = 1 + \sum_{k<l}(u_{ik}u_{il} + u_{il}u_{ik}) \otimes g_kg_l \\ [G_i, G_j] &= \sum_{k<l}(u_{ik}u_{jl} - u_{jk}u_{il} + u_{il}u_{jk} - u_{jl}u_{ik}) \otimes g_kg_l \end{aligned}$$

From the first relation we obtain $ab = 0$ for $a \neq b$ on the same row of u, and by using the antipode, the same happens for the columns. From the second relation we obtain $[u_{ik}, u_{jl}] = [u_{jk}, u_{il}]$ for $k \neq l$. Now by applying the antipode we obtain $[u_{lj}, u_{ki}] = [u_{li}, u_{kj}]$, and by relabelling, this gives $[u_{ik}, u_{jl}] = [u_{il}, u_{jk}]$ for $j \neq i$. Thus for $i \neq j, k \neq l$ we must have $[u_{ik}, u_{jl}] = [u_{jk}, u_{il}] = 0$, and we are therefore led to $G \subset \bar{O}_N$, as claimed.

(2) We can use here the fact that the cube $\square_N$, when regarded as a graph, is the Cayley graph of the group $\mathbb{Z}_2^N$. The eigenvectors and eigenvalues of $\square_N$ are as follows:

$$\begin{aligned} v_{i_1\ldots i_N} &= \sum_{j_1\ldots j_N} (-1)^{i_1j_1+\ldots+i_Nj_N} g_1^{j_1}\ldots g_N^{j_N} \\ \lambda_{i_1\ldots i_N} &= (-1)^{i_1} + \ldots + (-1)^{i_N} \end{aligned}$$

With this picture in hand, and by using Proposition 8.3 and Proposition 8.4 above, the result follows from the same computations as in the proof of (1). See [15].

(3) Our claim here is that we obtain the same symmetry group as in (2). Indeed, observe that distance matrix of the cube has a color decomposition as follows:

$$d = d_1 + \sqrt{2}d_2 + \sqrt{3}d_3 + \ldots + \sqrt{N}d_N$$

Since the powers of d_1 can be computed by counting loops on the cube, we have formulae as follows, with $x_{ij}\in\mathbb N$ being certain positive integers:

$$\begin{eqnarray*}
d_1^2&=&x_{21}1_N+x_{22}d_2\\
d_1^3&=&x_{31}1_N+x_{32}d_2+x_{33}d_3\\
&\ldots&\\
d_1^N&=&x_{N1}1_N+x_{N2}d_2+x_{N3}d_3+\ldots+x_{NN}d_N
\end{eqnarray*}$$

But this shows that we have $<d>=<d_1>$. Now since d_1 is the adjacency matrix of $\square_N$, viewed as graph, this proves our claim, and we obtain the result via (2). $\square$

Now back to our questions regarding the square, we have $G^+(\square)=\bar{O}_2$, and this formula appears as the $N=2$ particular case of a general formula, namely $G^+(\square_N)=\bar{O}_N$.

This is quite conceptual, but still not ok. The problem is that we have $G(\square_N)=H_N$, and so for our theory to be complete, we would need a formula of type $H_N^+=\bar{O}_N$.

And this latter formula is obviously wrong, because for $\bar{O}_N$ the character computations lead to Gaussian laws, who cannot appear as liberations of the character laws for H_N, that we have not computed yet, but which can only be something Poisson-related.

Summarizing, the problem of conceptually understanding $G(\square)$ remains open. In order to present now the correct, final solution, the idea will be that to look at the quantum group $G^+(|\ |)$ instead, which is equal to it, according to Proposition 8.2 (3).

We first have the following result, extending Proposition 8.2 (4) above:

Proposition 8.9. *For a disconnected union of graphs we have*

$$G^+(X_1)\,\hat{*}\,\ldots\,\hat{*}\,G^+(X_k)\subset G^+(X_1\sqcup\ldots\sqcup X_k)$$

and this inclusion is in general not an isomorphism.

Proof. The proof of the first assertion is nearly identical to the proof of Proposition 8.2 (4) above. Indeed, the adjacency matrix of the disconnected union is given by:

$$d_{X_1\sqcup\ldots\sqcup X_k}=diag(d_{X_1},\ldots,d_{X_k})$$

$$w=diag(u_1,\ldots,u_k)$$

We have then $d_{X_i}u_i=u_id_{X_i}$, and this implies $dw=wd$, which gives the result. As for the last assertion, this is something that we already know, from Proposition 8.6 (2). $\square$

In the case where the graphs $X_1,\ldots,X_k$ are identical, which is the one that we are truly interested in, we can further build on this. Let us first recall that we have:

Proposition 8.10. *Given closed subgroups $G\subset U_N^+$, $H\subset S_k^+$, with fundamental corepresentations u,v, the following construction produces a closed subgroup of U_{Nk}^+:*

$$C(G\wr_*H)=(C(G)^{*k}*C(H))/<[u_{ij}^{(a)},v_{ab}]=0>$$

In the case where G,H are classical, the classical version of $G\wr_*H$ is the usual wreath product $G\wr H$. Also, when G is a quantum permutation group, so is $G\wr_*H$.

Proof. Consider indeed the matrix $w_{ia,jb} = u_{ij}^{(a)} v_{ab}$, over the quotient algebra in the statement. It is routine to check that w is unitary, and in the case $G \subset S_N^+$, our claim is that this matrix is magic. Indeed, the entries are projections, because they appear as products of commuting projections, and the row and column sums are as follows:

$$\sum_{ia} w_{ia,jb} = \sum_{ia} u_{ij}^{(a)} v_{ab} = \sum_a v_{ab} \sum_i u_{ij}^{(a)} = 1$$

$$\sum_{jb} w_{ia,jb} = \sum_{jb} u_{ij}^{(a)} v_{ab} = \sum_b v_{ab} \sum_j u_{ij}^{(a)} = 1$$

With these observations in hand, it is routine to check that $G \wr_* H$ is indeed a quantum group, with fundamental corepresentation w, by constructing maps Δ, ε, S as in Definition 2.1, and in the case $G \subset S_N^+$, we obtain in this way a closed subgroup of S_{Nk}^+. Finally, the assertion regarding the classical version is standard as well. See [41]. □

We refer to [10], [41], [82] for more details regarding the above construction.

With this notion in hand, we can now formulate a non-trivial result, as follows:

Theorem 8.11. *Given a connected graph X, and $k \in \mathbb{N}$, we have the formulae*

$$G(kX) = G(X) \wr S_k \quad , \quad G^+(kX) = G^+(X) \wr_* S_k^+$$

where $kX = X \sqcup \ldots \sqcup X$ is the k-fold disjoint union of X with itself.

Proof. The first formula is something well-known, which follows as well from the second formula, by taking the classical version. Regarding now the second formula, it is quite elementary to check that we have an inclusion as follows, for any finite graph X:

$$G^+(X) \wr_* S_k^+ \subset G^+(kX)$$

Indeed, we want to construct an action $G^+(X) \wr_* S_k^+ \curvearrowright kX$, and this amounts in proving that we have $[w, d] = 0$. But, the matrices w, d are given by:

$$w_{ia,jb} = u_{ij}^{(a)} v_{ab} \quad , \quad d_{ia,jb} = \delta_{ij} d_{ab}$$

With these formulae in hand, we have the following computations:

$$(dw)_{ia,jb} = \sum_k d_{ik} w_{ka,jb} = \sum_k d_{ik} u_{kj}^{(a)} v_{ab} = (du^{(a)})_{ij} v_{ab}$$

$$(wd)_{ia,jb} = \sum_k w_{ia,kb} d_{kj} = \sum_k u_{ik}^{(a)} v_{ab} d_{kj} = (u^{(a)} d)_{ij} v_{ab}$$

Thus we have $[w, d] = 0$, and from this we obtain:

$$G^+(X) \wr_* S_k^+ \subset G^+(kX)$$

Regarding now the reverse inclusion, which requires X to be connected, this follows by doing some matrix analysis, by using the commutation with u. To be more precise, let us denote by w the fundamental corepresentation of $G^+(kX)$, and set:

$$u_{ij}^{(a)} = \sum_b w_{ia,jb} \quad , \quad v_{ab} = \sum_i v_{ab}$$

It is then routine to check, by using the fact that X is indeed connected, that we have here magic unitaries, as in the definition of the free wreath products. Thus we obtain the reverse inclusion $G^+(kX) \subset G^+(X) \wr_* S_k^+$, and this gives the result.

To be more precise, the key ingredient is the fact that when X is connected, the $*$-algebra generated by d_X contains a matrix having nonzero entries. See [10]. $\square$

We are led in this way to the following result:

Theorem 8.12. *Consider the graph consisting of N segments.*

(1) *Its symmetry group is the hyperoctahedral group $H_N = \mathbb{Z}_2 \wr S_N$.*
(2) *Its quantum symmetry group is the quantum group $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$.*

Proof. Here the first assertion is clear from definitions, with the remark that the relation with the formula $H_N = G(\square_N)$ comes by viewing the N segments as being the $[-1,1]$ segments on each of the N coordinate axes of $\mathbb{R}^N$. Indeed, a symmetry of the N-cube is the same as a symmetry of the N segments, and so $G(\square_N) = \mathbb{Z}_2 \wr S_N$, as desired.

As for the second assertion, this follows from Theorem 8.11 above, applied to the segment graph. Observe also that (2) implies (1), by taking the classical version. $\square$

Now back to the square, we have $G^+(\square) = H_2^+$, and our claim is that this is the "good" and final formula. In order to prove this, we must work out the easiness theory for H_N, H_N^+, and find a compatibility there. We first have the following result:

Proposition 8.13. *The algebra $C(H_N^+)$ can be presented in two ways, as follows:*

(1) *As the universal algebra generated by the entries of a $2N \times 2N$ magic unitary having the "sudoku" pattern $w = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, with a, b being square matrices.*
(2) *As the universal algebra generated by the entries of a $N \times N$ orthogonal matrix which is "cubic", in the sense that $u_{ij}u_{ik} = u_{ji}u_{ki} = 0$, for any $j \neq k$.*

As for $C(H_N)$, this has similar presentations, among the commutative algebras.

Proof. Here the first assertion follows from Theorem 8.12, via Proposition 8.10, and the last assertion is clear as well, because $C(H_N)$ is the abelianization of $C(H_N^+)$. Thus, we are left with proving that the algebras A_s, A_c coming from (1,2) coincide.

We construct first the arrow $A_c \to A_s$. The elements a_{ij}, b_{ij} being self-adjoint, their differences are self-adjoint as well. Thus $a - b$ is a matrix of self-adjoint elements. We have the following formula for the products on the columns of $a-b$:

$$\begin{aligned}(a-b)_{ik}(a-b)_{jk} &= a_{ik}a_{jk}-a_{ik}b_{jk}-b_{ik}a_{jk}+b_{ik}b_{jk}\\ &= \begin{cases}0 & \text{for } i\neq j\\ a_{ik}+b_{ik} & \text{for } i=j\end{cases}\end{aligned}$$

In the $i=j$ case the elements $a_{ik}+b_{ik}$ sum up to 1, so the columns of $a-b$ are orthogonal. A similar computation works for rows, so $a-b$ is orthogonal.

Now by using the $i\neq j$ computation, along with its row analogue, we conclude that $a-b$ is cubic. Thus we can define a morphism $A_c\to A_s$ by the following formula:

$$\varphi(u_{ij})=a_{ij}-b_{ij}$$

We construct now the inverse morphism. Consider the following elements:

$$\alpha_{ij}=\frac{u_{ij}^2+u_{ij}}{2}\quad,\quad \beta_{ij}=\frac{u_{ij}^2-u_{ij}}{2}$$

These are projections, and the following matrix is a sudoku unitary:

$$M=\begin{pmatrix}(\alpha_{ij})&(\beta_{ij})\\(\beta_{ij})&(\alpha_{ij})\end{pmatrix}$$

Thus we can define a morphism $A_s\to A_c$ by the following formula:

$$\psi(a_{ij})=\frac{u_{ij}^2+u_{ij}}{2}\quad,\quad \psi(b_{ij})=\frac{u_{ij}^2-u_{ij}}{2}$$

We check now the fact that ψ,φ are indeed inverse morphisms:

$$\psi\varphi(u_{ij})=\psi(a_{ij}-b_{ij})=\frac{u_{ij}^2+u_{ij}}{2}-\frac{u_{ij}^2-u_{ij}}{2}=u_{ij}$$

As for the other composition, we have the following computation:

$$\varphi\psi(a_{ij})=\varphi\left(\frac{u_{ij}^2+u_{ij}}{2}\right)=\frac{(a_{ij}-b_{ij})^2+(a_{ij}-b_{ij})}{2}=a_{ij}$$

A similar computation gives $\varphi\psi(b_{ij})=b_{ij}$, which completes the proof. □

We can now work out the easiness property of H_N,H_N^+, with respect to the cubic representations, and we are led to the following result, which is fully satisfactory:

Theorem 8.14. *The quantum groups H_N,H_N^+ are both easy, as follows:*

(1) *H_N corresponds to the category P_{even}.*
(2) *H_N^+ corresponds to the category NC_{even}.*

Proof. These assertions follow indeed from the fact that the cubic relations are implemented by the one-block partition in $P(2,2)$, which generates NC_{even}. □

There is a similarity here with the easiness results for permutations and quantum permutations, obtained in sections 5 and 7 above. In fact, the basic algebraic results regarding S_N, S_N^+ and H_N, H_N^+ appear as the $s = 1, 2$ particular cases of:

Theorem 8.15. *The complex reflection groups $H_N^s = \mathbb{Z}_s \wr S_N$ and their free analogues $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$, defined for any $s \in \mathbb{N}$, have the following properties:*

1. *They have N-dimensional coordinates $u = (u_{ij})$, which are subject to the relations $u_{ij}u_{ij}^* = u_{ij}^*u_{ij}$, $p_{ij} = u_{ij}u_{ij}^* = magic$, and $u_{ij}^s = p_{ij}$.*
2. *They are easy, the corresponding categories $P^s \subset P, NC^s \subset NC$ being given by the fact that we have $\# \circ - \# \bullet = 0(s)$, as a weighted sum, in each block.*

Proof. We already know that the results hold at $s = 1, 2$, and the proof in general is similar. With respect to the above proof at $s = 2$, the situation is as follows:

(1) Observe first that the result holds at $s = 1$, where we obtain the magic condition, and at $s = 2$ as well, where we obtain something equivalent to the cubic condition. In general, this follows from a $\mathbb{Z}_s$-analogue of Proposition 8.13. See [37].

(2) Once again, the result holds at $s = 1$, trivially, and at $s = 2$ as well, where our condition is equivalent to $\# \circ + \# \bullet = 0(2)$, in each block. In general, this follows as in the proof of Theorem 8.14, by using the one-block partition in $P(s, s)$. See [8]. □

The above proof is of course quite brief, but we will not be really interested here in the case $s \geq 3$, which is quite technical. In fact, the above result, dealing with the general case $s \in \mathbb{N}$, is here for providing an introduction to the case $s = \infty$, where we have:

Theorem 8.16. *The pure complex reflection groups $K_N = \mathbb{T} \wr S_N$ and their free analogues $K_N^+ = \mathbb{T} \wr_* S_N^+$ have the following properties:*

1. *They have N-dimensional coordinates $u = (u_{ij})$, which are subject to the relations $u_{ij}u_{ij}^* = u_{ij}^*u_{ij}$ and $p_{ij} = u_{ij}u_{ij}^* = magic$.*
2. *They are easy, the corresponding categories $\mathcal{P}_{even} \subset P, \mathcal{NC}_{even} \subset NC$ being given by the fact that we have $\#\circ = \#\bullet$, as a weighted equality, in each block.*

Proof. The assertions here appear as an $s = \infty$ extension of (1,2) in Theorem 8.15 above, and their proof can be obtained along the same lines, as follows:

(1) This follows indeed by working out a $\mathbb{T}$-analogue of the computations in the proof of Proposition 8.13 above. We refer here to [8].

(2) Once again, this appears as a $s = \infty$ extension of the results that we already have, and for details here, we refer once again to [8]. □

The above results at $s = 2, \infty$ are quite interesting for us, because we can now focus on the quantum reflection groups H_N, H_N^+, K_N, K_N^+, with the idea in mind of completing the orthogonal and unitary quantum group picture from section 5 above.

Before doing this, we have two more quantum groups to be introduced and study, namely the half-liberations H_N^*, K_N^*. We have here the following result:

Theorem 8.17. *We have quantum groups H_N^*, K_N^*, which are both easy, as follows,*

(1) $H_N^* = H_N^+ \cap O_N^*$, *corresponding to the category P_{even}^*,*
(2) $K_N^* = K_N^+ \cap U_N^*$, *corresponding to the category $\mathcal{P}_{even}^*$,*

with the symbol $*$ standing for the fact that the corresponding partitions, when relabelled clockwise $\circ\bullet\circ\bullet\ldots$, must contain the same number of $\circ,\bullet$, in each block.

Proof. This is standard, from the results that we already have, regarding the various quantum groups involved, because the interesection operations at the quantum group level correspond to generation operations, at the category of partitions level. □

We can now complete the "continuous" picture from section 5 above, as follows:

Theorem 8.18. *The basic orthogonal and unitary quantum groups are related to the basic real and complex quantum reflection groups as follows,*

$$\begin{tikzcd} U_N \arrow[r] & U_N^* \arrow[r] & U_N^+ \\ O_N \arrow[u] \arrow[r] & O_N^* \arrow[u] \arrow[r] & O_N^+ \arrow[u] \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} K_N \arrow[r] & K_N^* \arrow[r] & K_N^+ \\ H_N \arrow[u] \arrow[r] & H_N^* \arrow[u] \arrow[r] & H_N^+ \arrow[u] \end{tikzcd}$$

the connecting operations $U \leftrightarrow K$ being given by $K = U \cap K_N^+$ and $U = \{K, O_N\}$.

Proof. According to the general results in section 5 above, in terms of categories of partitions, the operations introduced in the statement reformulate as follows:

$$D_K = < D_U, \mathcal{NC}_{even} > \quad , \quad D_U = D_K \cap P_2$$

On the other hand, by putting together the various easiness results that we have, the categories of partitions for the quantum groups in the statement are as follows:

$$\begin{tikzcd} \mathcal{P}_2 \arrow[d] & \mathcal{P}_2^* \arrow[l] \arrow[d] & \mathcal{NC}_2 \arrow[l] \arrow[d] \\ P_2 & P_2^* \arrow[l] & NC_2 \arrow[l] \end{tikzcd} \quad : \quad \begin{tikzcd} \mathcal{P}_{even} \arrow[d] & \mathcal{P}_{even}^* \arrow[l] \arrow[d] & \mathcal{NC}_{even} \arrow[l] \arrow[d] \\ P_{even} & P_{even}^* \arrow[l] & NC_{even} \arrow[l] \end{tikzcd}$$

It is elementary to check that these categories are related by the above intersection and generation operations, and we conclude that the correspondence holds indeed. □

All this looks quite conceptual, but as a word of warning here, for more complicated intermediate liberations, such as those found in [71], [78], the problem of establishing correspondences is quite complicated. We will comment on this in section 9 below.

Our purpose now will be that of showing that a twisted analogue of the above result holds, with the quantum unitary groups being those in section 5 above, and with the quantum reflection groups being equal to their own Schur-Weyl twists.

It is convenient to include in our discussion two more important quantum groups, coming from [25], [78] and denoted $H_N^{[\infty]},K_N^{[\infty]}$, which are constructed as follows:

Theorem 8.19. *We have intermediate liberations $H_N^{[\infty]},K_N^{[\infty]}$ as follows, constructed by using the relations $\alpha\beta\gamma=0$, for any $a\neq c$ on the same row or column of u,*

$$\begin{array}{ccccccc}
K_N & \longrightarrow & K_N^* & \longrightarrow & K_N^{[\infty]} & \longrightarrow & K_N^+\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
H_N & \longrightarrow & H_N^* & \longrightarrow & H_N^{[\infty]} & \longrightarrow & H_N^+
\end{array}$$

with the convention $\alpha=a,a^*$, and so on. These quantum groups are easy, the corresponding categories $P_{even}^{[\infty]}\subset P_{even}$ and $\mathcal{P}_{even}^{[\infty]}\subset\mathcal{P}_{even}$ being generated by $\eta=\ker\binom{iij}{jii}$.

Proof. This is routine, by using the fact that the relations $\alpha\beta\gamma=0$ in the statement are equivalent to the condition $\eta\in End(u^{\otimes k})$, with $|k|=3$. We refer here to [25], [78]. $\square$

In order to discuss the twisting, we will need the following technical result:

Proposition 8.20. *We have the following equalities,*

$$\begin{aligned}
P_{even}^* &= \left\{\pi\in P_{even}\Big|\varepsilon(\tau)=1,\forall\tau\leq\pi,|\tau|=2\right\}\\
P_{even}^{[\infty]} &= \left\{\pi\in P_{even}\Big|\sigma\in P_{even}^*,\forall\sigma\subset\pi\right\}\\
P_{even}^{[\infty]} &= \left\{\pi\in P_{even}\Big|\varepsilon(\tau)=1,\forall\tau\leq\pi\right\}
\end{aligned}$$

where $\varepsilon:P_{even}\to\{\pm1\}$ is the signature of even permutations.

Proof. This is routine combinatorics, from [5], [78], the idea being as follows:

(1) Given $\pi\in P_{even}$, we have $\tau\leq\pi,|\tau|=2$ precisely when $\tau=\pi^\beta$ is the partition obtained from π by merging all the legs of a certain subpartition $\beta\subset\pi$, and by merging as well all the other blocks. Now observe that π^β does not depend on π, but only on β, and that the number of switches required for making π^β noncrossing is $c=N_\bullet-N_\circ$ modulo 2, where $N_\bullet/N_\circ$ is the number of black/white legs of β, when labelling the legs of π counterclockwise $\circ\bullet\circ\bullet\ldots$ Thus $\varepsilon(\pi^\beta)=1$ holds precisely when $\beta\in\pi$ has the same number of black and white legs, and this gives the result.

(2) This simply follows from the equality $P_{even}^{[\infty]}=<\eta>$ coming from Theorem 8.19, by computing $<\eta>$, and for the complete proof here we refer to [78].

(3) We use here the fact, also from [78], that the relations $g_ig_ig_j = g_jg_ig_i$ are trivially satisfied for real reflections. This leads to the following conclusion:

$$P_{even}^{[\infty]}(k,l) = \left\{\ker\begin{pmatrix}i_1 & \dots & i_k\\ j_1 & \dots & j_l\end{pmatrix}\Big| g_{i_1}\dots g_{i_k} = g_{j_1}\dots g_{j_l} \text{ inside } \mathbb{Z}_2^{*N}\right\}$$

In other words, the partitions in $P_{even}^{[\infty]}$ are those describing the relations between free variables, subject to the conditions $g_i^2 = 1$. We conclude that $P_{even}^{[\infty]}$ appears from NC_{even} by "inflating blocks", in the sense that each $\pi \in P_{even}^{[\infty]}$ can be transformed into a partition $\pi' \in NC_{even}$ by deleting pairs of consecutive legs, belonging to the same block.

Now since this inflation operation leaves invariant modulo 2 the number $c \in \mathbb{N}$ of switches in the definition of the signature, it leaves invariant the signature $\varepsilon = (-1)^c$ itself, and we obtain in this way the inclusion "$\subset$" in the statement.

Conversely, given $\pi \in P_{even}$ satisfying $\varepsilon(\tau) = 1$, $\forall \tau \leq \pi$, our claim is that:

$$\rho \leq \sigma \subset \pi, |\rho| = 2 \implies \varepsilon(\rho) = 1$$

Indeed, let us denote by α, β the two blocks of ρ, and by γ the remaining blocks of π, merged altogether. We know that the partitions $\tau_1 = (\alpha \wedge \gamma, \beta)$, $\tau_2 = (\beta \wedge \gamma, \alpha)$, $\tau_3 = (\alpha, \beta, \gamma)$ are all even. On the other hand, putting these partitions in noncrossing form requires respectively $s + t, s' + t, s + s' + t$ switches, where t is the number of switches needed for putting $\rho = (\alpha, \beta)$ in noncrossing form. Thus t is even, and we are done.

With the above claim in hand, we conclude, by using the second equality in the statement, that we have $\sigma \in P_{even}^*$. Thus we have $\pi \in P_{even}^{[\infty]}$, which ends the proof of "$\supset$". □

With the above result in hand, we can now prove:

Theorem 8.21. *We have the following results:*

1. *The quantum groups from Theorem 8.19 are equal to their own twists.*
2. *With input coming from this, a twisted version of Theorem 8.18 holds.*

Proof. This result, established in [5], basically comes from the results that we have.

(1) In the real case, the verifications are as follows:

– H_N^+. We know from Proposition 5.26 above that for $\pi \in NC_{even}$ we have $\bar{T}_\pi = T_\pi$, and since we are in the situation $D \subset NC_{even}$, the definitions of $G, \bar{G}$ coincide.

– $H_N^{[\infty]}$. Here we can use the same argument as in (1), based this time on the description of $P_{even}^{[\infty]}$ involving the signature found in Proposition 8.20.

– H_N^*. We have $H_N^* = H_N^{[\infty]} \cap O_N^*$, so $\bar{H}_N^* \subset H_N^{[\infty]}$ is the subgroup obtained via the defining relations for $\bar{O}_N^*$. But all the $abc = -cba$ relations defining $\bar{H}_N^*$ are automatic, of type $0 = 0$, and it follows that $\bar{H}_N^* \subset H_N^{[\infty]}$ is the subgroup obtained via the relations $abc = cba$, for any $a, b, c \in \{u_{ij}\}$. Thus we have $\bar{H}_N^* = H_N^{[\infty]} \cap O_N^* = H_N^*$, as claimed.

– H_N. We have $H_N = H_N^* \cap O_N$, and by functoriality, $\bar{H}_N = \bar{H}_N^* \cap \bar{O}_N = H_N^* \cap \bar{O}_N$. But this latter intersection is easily seen to be equal to H_N, as claimed.

In the complex case the proof is similar, and we refer here to [5].

(2) This can be proved by proceeding as in the proof of Theorem 8.18 above, with of course some care when formulating the result. Once again, we refer here to [5]. □

Let us go back now to the free examples H_N^+, K_N^+, or rather to the whole series H_N^{s+}, with $s\in\{1,2,\ldots,\infty\}$ and work out the fusion rules, and probabilistic aspects.

Regarding the fusion rules, one can prove that the irreducible representations of H_N^{s+} can be labeled r_x, with x being a word over $\mathbb Z_s$, such that the fusion rules are:

$$r_x\otimes r_y=\sum_{x=vz,y=\bar{z}w}r_{vw}+r_{v\cdot w}$$

Observe that at $s=1$ we have here, modulo some indentifications, the Clebsch-Gordan rules for S_N^+. In general, all this is quite technical, and we refer here to [36].

Regarding the probabilistic aspects, we will need some general theory. We have the following definition, extending the Poisson limit theory from section 6 above:

Definition 8.22. *Associated to any compactly supported positive measure ρ on $\mathbb R$ are the probability measures*

$$p_\rho=\lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0+\frac{1}{n}\rho\right)^{*n}\quad,\quad \pi_\rho=\lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0+\frac{1}{n}\rho\right)^{\boxplus n}$$

where $c=mass(\rho)$, called compound Poisson and compound free Poisson laws.

In what follows we will be interested in the case where ρ is discrete, as is for instance the case for $\rho=\delta_t$ with $t>0$, which produces the Poisson and free Poisson laws.

The following result allows one to detect compound Poisson/free Poisson laws:

Proposition 8.23. *For $\rho=\sum_{i=1}^s c_i\delta_{z_i}$ with $c_i>0$ and $z_i\in\mathbb R$, we have*

$$F_{p_\rho}(y)=\exp\left(\sum_{i=1}^s c_i(e^{-iyz_i}-1)\right)\quad,\quad R_{\pi_\rho}(y)=\sum_{i=1}^s\frac{c_iz_i}{1-yz_i}$$

where F,R denote respectively the Fourier transform, and Voiculescu's R-transform.

Proof. Let μ_n be the measure appearing in Definition 8.22, under the convolution signs. In the classical case, we have the following computation, with $F_{\delta_z}(y)=e^{-iyz}$:

$$\begin{aligned}F_{\mu_n}(y)=\left(1-\frac{c}{n}\right)+\frac{1}{n}\sum_{i=1}^s c_ie^{-iyz_i}\quad&\Longrightarrow\quad F_{\mu_n^{*n}}(y)=\left(\left(1-\frac{c}{n}\right)+\frac{1}{n}\sum_{i=1}^s c_ie^{-iyz_i}\right)^n\\&\Longrightarrow\quad F_{p_\rho}(y)=\exp\left(\sum_{i=1}^s c_i(e^{-iyz_i}-1)\right)\end{aligned}$$

In the free case now, we use a similar method. First, we have:

$$f_{\mu_n}(y) = \left(1 - \frac{c}{n}\right) + \frac{1}{n}\sum_{i=1}^s \frac{c_i}{1 - z_i y} \implies G_{\mu_n}(\xi) = \left(1 - \frac{c}{n}\right)\frac{1}{\xi} + \frac{1}{n}\sum_{i=1}^s \frac{c_i}{\xi - z_i}$$
$$\implies y = \left(1 - \frac{c}{n}\right)\frac{1}{K_{\mu_n}(y)} + \frac{1}{n}\sum_{i=1}^s \frac{c_i}{K_{\mu_n}(y) - z_i}$$

Now since $K_{\mu_n}(y) = y^{-1} + R_{\mu_n}(y) = y^{-1} + R/n$, where $R = R_{\mu_n^{\boxplus n}}(y)$, we get:

$$y = \left(1 - \frac{c}{n}\right)\frac{1}{y^{-1} + R/n} + \frac{1}{n}\sum_{i=1}^s \frac{c_i}{y^{-1} + R/n - z_i}$$
$$\implies 1 = \left(1 - \frac{c}{n}\right)\frac{1}{1 + yR/n} + \frac{1}{n}\sum_{i=1}^s \frac{c_i}{1 + yR/n - yz_i}$$

Now multiplying by n, rearranging the terms, and letting $n \to \infty$, we get:

$$\frac{c + yR}{1 + yR/n} = \sum_{i=1}^s \frac{c_i}{1 + yR/n - yz_i} \implies c + yR_{\pi_\rho}(y) = \sum_{i=1}^s \frac{c_i}{1 - yz_i}$$
$$\implies R_{\pi_\rho}(y) = \sum_{i=1}^s \frac{c_i z_i}{1 - yz_i}$$

This finishes the proof in the free case, and we are done. □

We have as well the following result, providing an alternative to Definition 8.22:

Theorem 8.24. *For $\rho = \sum_{i=1}^s c_i \delta_{z_i}$ with $c_i > 0$ and $z_i \in \mathbb{R}$, we have*

$$p_\rho/\pi_\rho = \mathrm{law}\left(\sum_{i=1}^s z_i \alpha_i\right)$$

where the variables α_i are Poisson/free Poisson(c_i), independent/free.

Proof. Let α be the sum of Poisson/free Poisson variables in the statement. We will show that the Fourier/R-transform of α is given by the formulae in Proposition 8.23.

Indeed, by using some well-known Fourier transform formulae, we have:

$$F_{\alpha_i}(y) = \exp(c_i(e^{-iy} - 1)) \implies F_{z_i\alpha_i}(y) = \exp(c_i(e^{-iyz_i} - 1))$$
$$\implies F_\alpha(y) = \exp\left(\sum_{i=1}^s c_i(e^{-iyz_i} - 1)\right)$$

Also, by using some well-known R-transform formulae, we have:

$$R_{\alpha_i}(y)=\frac{c_i}{1-y}\implies R_{z_i\alpha_i}(y)=\frac{c_iz_i}{1-yz_i}$$
$$\implies R_\alpha(y)=\sum_{i=1}^s\frac{c_iz_i}{1-yz_i}$$

Thus we have indeed the same formulae as those in Proposition 8.23. □

We can go back now to quantum reflection groups, and we have:

Theorem 8.25. *The asymptotic laws of truncated characters are as follows, where ε_s with $s\in\{1,2,\ldots,\infty\}$ is the uniform measure on the s-th roots of unity:*

(1) *For H_N^s we obtain the compound Poisson law $b_t^s=p_{t\varepsilon_s}$.*
(2) *For H_N^{s+} we obtain the compound free Poisson law $\beta_t^s=\pi_{t\varepsilon_s}$.*

These measures are in Bercovici-Pata bijection.

Proof. This follows from easiness, and from the Weingarten formula. To be more precise, at $t=1$ this follows by counting the partitions, and at $t\in(0,1]$ general, this follows in the usual way, for instance by using cumulants. For details here, we refer to [8]. □

The above measures are called Bessel and free Bessel laws. This is because at $s=2$ we have $b_t^2=e^{-t}\sum_{k=-\infty}^\infty f_k(t/2)\delta_k$, with f_k being the Bessel function of the first kind:

$$f_k(t)=\sum_{p=0}^\infty\frac{t^{|k|+2p}}{(|k|+p)!p!}$$

The Bessel and free Bessel laws have particularly interesting properties at the parameter values $s=2,\infty$. So, let us record the precise statement here:

Theorem 8.26. *The asymptotic laws of truncated characters are as follows:*

(1) *For H_N we obtain the real Bessel law $b_t^2=p_{t\varepsilon_2}$.*
(2) *For K_N we obtain the complex Bessel law $b_t^\infty=p_{t\varepsilon_\infty}$.*
(3) *For H_N^+ we obtain the free real Bessel law $\beta_t^2=\pi_{t\varepsilon_2}$.*
(4) *For K_N^+ we obtain the free complex Bessel law $\beta_t^\infty=\pi_{t\varepsilon_\infty}$.*

Proof. This follows indeed from Theorem 8.25 above, at $s=2,\infty$. □

In addition to what has been said above, there are as well some interesting results about the Bessel and free Bessel laws involving the multiplicative convolution $\times$, and the multiplicative free convolution $\boxtimes$ from [87]. For details, we refer here to [8].

9. Classification results

We discuss here classification questions for the closed subgroups $G_N\subset U_N^+$, in the easy case, and beyond. There has been a lot of work on the subject, and our objective will be that of presenting a few basic results, with proofs, along with some discussion.

We have already met a number of easy quantum groups, as follows:

Proposition 9.1. *We have the following basic examples of easy quantum groups:*

1. *Unitary quantum groups: $O_N,O_N^*,O_N^+,U_N,U_N^*,U_N^+$.*
2. *Bistochastic versions: B_N,B_N^+,C_N,C_N^+.*
3. *Quantum permutation groups: S_N,S_N^+.*
4. *Quantum reflections: $H_N,H_N^*,H_N^+,K_N,K_N^*,K_N^+$.*

Proof. This is something that we already know, the partitions being as follows:
(1) $P_2,P_2^*,NC_2,\mathcal{P}_2,\mathcal{P}_2^*,\mathcal{NC}_2$.
(2) $P_{12},NC_{12},\mathcal{P}_{12},\mathcal{NC}_{12}$.
(3) P,NC.
(4) $P_{even},P_{even}^*,NC_{even},\mathcal{P}_{even},\mathcal{P}_{even}^*,\mathcal{NC}_{even}$. □

In addition to the above list, we have the quantum groups H_N^s,H_N^{s+} with $3\leq s<\infty$, as well as the related series $H_N^{s*}=H_N^{s+}\cap U_N^*$. Further examples can be constructed via free complexification, or via operations of type $G_N\to\mathbb{Z}_r\times G_N$, or $G_N\to\mathbb{Z}_rG_N$, with $r\in\{2,3,\ldots,\infty\}$. There are as well many "exotic" intermediate liberation procedures, involving relations far more complicated than the half-commutation ones $abc=cba$.

All this makes the classification question particularly difficult. So, our first task in what follows will be that of cutting a bit from complexity, by adding some extra axioms, chosen as "natural" as possible. A first such axiom, very natural, is as follows:

Proposition 9.2. *For an easy quantum group $G=(G_N)$, coming from a category of partitions $D\subset P$, the following conditions are equivalent:*

1. *$G_{N-1}=G_N\cap U_{N-1}^+$, via the embedding $U_{N-1}^+\subset U_N^+$ given by $u\to diag(u,1)$.*
2. *$G_{N-1}=G_N\cap U_{N-1}^+$, via the N possible diagonal embeddings $U_{N-1}^+\subset U_N^+$.*
3. *D is stable under the operation which consists in removing blocks.*

If these conditions are satisfied, we say that $G=(G_N)$ is "uniform".

Proof. We use here the general theory from section 5 above.

(1) $\iff$ (2) This is something standard, coming from the inclusion $S_N\subset G_N$, which makes everything S_N-invariant. The result follows as well from the proof of (1) $\iff$ (3) below, which can be converted into a proof of (2) $\iff$ (3), in the obvious way.

(1) $\iff$ (3) Given a subgroup $K\subset U_{N-1}^+$, with fundamental corepresentation u, consider the $N\times N$ matrix $v=diag(u,1)$. Our claim is that for any $\pi\in P(k)$ we have:

$$\xi_\pi\in Fix(v^{\otimes k})\iff \xi_{\pi'}\in Fix(v^{\otimes k'}),\ \forall\pi'\in P(k'),\pi'\subset\pi$$

In order to prove this, we must study the condition on the left. We have:

$$\begin{aligned}\xi_\pi \in Fix(v^{\otimes k}) &\iff (v^{\otimes k}\xi_\pi)_{i_1\ldots i_k} = (\xi_\pi)_{i_1\ldots i_k}, \forall i\\ &\iff \sum_j (v^{\otimes k})_{i_1\ldots i_k, j_1\ldots j_k}(\xi_\pi)_{j_1\ldots j_k} = (\xi_\pi)_{i_1\ldots i_k}, \forall i\\ &\iff \sum_j \delta_\pi(j_1,\ldots,j_k)v_{i_1j_1}\ldots v_{i_kj_k} = \delta_\pi(i_1,\ldots,i_k), \forall i\end{aligned}$$

Now let us recall that our corepresentation has the special form $v = diag(u,1)$. We conclude from this that for any index $a \in \{1,\ldots,k\}$, we must have:

$$i_a = N \implies j_a = N$$

With this observation in hand, if we denote by i', j' the multi-indices obtained from i, j obtained by erasing all the above $i_a = j_a = N$ values, and by $k' \leq k$ the common length of these new multi-indices, our condition becomes:

$$\sum_{j'} \delta_\pi(j_1,\ldots,j_k)(v^{\otimes k'})_{i'j'} = \delta_\pi(i_1,\ldots,i_k), \forall i$$

Here the index j is by definition obtained from j' by filling with N values. In order to finish now, we have two cases, depending on i, as follows:

Case 1. Assume that the index set $\{a|i_a = N\}$ corresponds to a certain subpartition $\pi' \subset \pi$. In this case, the N values will not matter, and our formula becomes:

$$\sum_{j'} \delta_\pi(j'_1,\ldots,j'_{k'})(v^{\otimes k'})_{i'j'} = \delta_\pi(i'_1,\ldots,i'_{k'})$$

Case 2. Assume now the opposite, namely that the set $\{a|i_a = N\}$ does not correspond to a subpartition $\pi' \subset \pi$. In this case the indices mix, and our formula reads:

$$0 = 0$$

Thus, we are led to $\xi_{\pi'} \in Fix(v^{\otimes k'})$, for any subpartition $\pi' \subset \pi$, as claimed.

Now with this claim in hand, the result follows from Tannakian duality. $\square$

At the level of the basic examples, from Proposition 9.1 above, the classical and free quantum groups are uniform, while the half-liberations are not. This can be seen either with categories of partitions, or with intersections, the point in the half-classical case being that the relations $abc = cba$, when applied to the coefficients of a matrix of type $v = diag(u,1)$, collapse with $c = 1$ to the usual commutation relations $ab = ba$.

For classification purposes the uniformity axiom is something very natural and useful, substantially cutting from complexity, and we have the following result, from [35]:

Theorem 9.3. *The classical and free uniform orthogonal easy quantum groups, with inclusions between them, are as follows:*

$$\begin{tikzcd}
 & H_N^+ \arrow[rr] & & O_N^+ \\
S_N^+ \arrow[ur] \arrow[rr] & & B_N^+ \arrow[ur] & \\
 & H_N \arrow[uu, dotted] \arrow[rr] & & O_N \arrow[uu, dotted] \\
S_N \arrow[uu, dotted] \arrow[ur] \arrow[rr] & & B_N \arrow[uu, dotted] \arrow[ur] &
\end{tikzcd}$$

Moreover, this is an intersection/easy generation diagram, in the sense that for any of its square subdiagrams $P \subset Q, R \subset S$ we have $P = Q \cap R$ and $\{Q, R\} = S$.

Proof. We know that the quantum groups in the statement are indeed easy and uniform, the corresponding categories of partitions being as follows:

$$\begin{tikzcd}
 & NC_{even} \arrow[dl] \arrow[dd, dotted] & & NC_2 \arrow[ll] \arrow[dl] \arrow[dd, dotted] \\
NC \arrow[dd, dotted] & & NC_{12} \arrow[ll] \arrow[dd, dotted] & \\
 & P_{even} \arrow[dl] & & P_2 \arrow[ll] \arrow[dl] \\
P & & P_{12} \arrow[ll] &
\end{tikzcd}$$

Since this latter diagram is an intersection and generation diagram, we conclude that we have an intersection and easy generation diagram of quantum groups, as stated.

Regarding now the classification, consider an easy quantum group $S_N \subset G_N \subset O_N$. This most come from a category $P_2 \subset D \subset P$, and by doing some combinatorics, we can see that D is uniquely determined by the subset $L \subset \mathbb{N}$ consisting of the sizes of the blocks of the partitions in D, with the admissible sets being as follows:

(1) $L = \{2\}$, producing O_N.
(2) $L = \{1, 2\}$, producing B_N.
(3) $L = \{2, 4, 6, \ldots\}$, producing H_N.
(4) $L = \{1, 2, 3, \ldots\}$, producing S_N.

In the free case, $S_N^+ \subset G_N \subset O_N^+$, the situation is quite similar, the admissible sets being once again the above ones, producing this time $O_N^+, B_N^+, H_N^+, S_N^+$. See [35]. $\square$

The above proof is of course quite brief, but we will not be really interested here in the classification of the uniform easy quantum groups, which is a quite technical topic. Let us mention, however, that there are two important known extensions of Theorem 9.3, concerning the general orthogonal case, and the classical/free unitary case. The results here come from [78], [83], by imposing the uniformity axiom, which makes a big cleanup, basically leading to cubes as above, with a few more objects added. See [5], [6].

The problem with all this, indeed, comes from the following negative result:

Proposition 9.4. *The cubic diagram from Theorem 9.3, and its unitary analogue,*

$$\begin{tikzcd}
& K_N^+ \arrow[rr] & & U_N^+ \\
S_N^+ \arrow[rr] \arrow[ur] & & C_N^+ \arrow[ur] & \\
& K_N \arrow[rr] \arrow[uu, dotted] & & U_N \arrow[uu, dotted] \\
S_N \arrow[rr] \arrow[uu, dotted] \arrow[ur] & & C_N \arrow[ur] \arrow[uu, dotted] &
\end{tikzcd}$$

cannot be merged, without degeneration, into a 4-dimensional cubic diagram.

Proof. All this is a bit philosophical, with the problem coming from the "taking the bistochastic version" operation, and more specifically, from the following equalities:

$$H_N \cap C_N = K_N \cap C_N = S_N$$

Indeed, these equalities do hold, and so the 3D cube obtained by merging the classical faces of the orthogonal and unitary cubes is something degenerate, as follows:

$$\begin{tikzcd}
& K_N \arrow[rr] & & U_N \\
S_N \arrow[rr] \arrow[ur] & & C_N \arrow[ur] & \\
& H_N \arrow[rr] \arrow[uu] & & O_N \arrow[uu] \\
S_N \arrow[rr] \arrow[uu] \arrow[ur] & & B_N \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

Thus, the 4D cube, having this 3D cube as one of its faces, is degenerate too. $\square$

Summarizing, when positioning ourselves at U_N^+, we have 4 natural directions to be followed, namely taking the classical, discrete, real and bistochastic versions. And the problem is that, while the first three operations are "good", the fourth one is "bad".

This is not very good news. In order to fix this, we will have to slash the bistochastic qauntum groups B_N, B_N^+, C_N, C_N^+, which after all is not a problem, but we will have to slash as well the quantum permutation groups S_N, S_N^+, that we definitely love.

This is life. In order to formulate now our second general axiom, doing the job, consider the cube $T_N = \mathbb{Z}_2^N$, regarded as diagonal torus of O_N. We have then:

Proposition 9.5. *For an easy quantum group $G = (G_N)$, coming from a category of partitions $D \subset P$, the following conditions are equivalent:*

1. $T_N \subset G_N$.
2. $H_N \subset G_N$.
3. $D \subset P_{even}$.

If these conditions are satisfied, we say that G_N is "twistable".

Proof. We use the general theory from section 5 above.

(1) $\iff$ (2) Here it is enough to check that the easy envelope T_N' of the cube equals the hyperoctahedral group H_N. But this follows from:

$$T_N' =< T_N, S_N >' = H_N' = H_N$$

(2) $\iff$ (3) This follows by functoriality, from the fact that H_N comes from the category of partitions P_{even}, that we know from section 8 above. $\square$

The teminology in the above result comes from the fact that, assuming $D \subset P_{even}$, we can indeed twist G_N, into a certain quizzy quantum group $\bar{G}_N$. We refer to section 5 above to full details regarding the construction $G_N \to \bar{G}_N$. In what follows we will not need this twisting procedure, and we will just use Proposition 9.5 as it is, as a statement providing us with a simple and natural condition to be imposed on G_N.

In practice now, imposing this second axiom leads to something nice, namely:

Theorem 9.6. *The basic quantum unitary and quantum reflection groups, from Proposition 9.1 above, which are uniform and twistable, are as follows,*

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
H_N \arrow[rr] \arrow[uu] \arrow[ur] & & O_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

and this is an intersection and easy generation diagram.

Proof. The first assertion comes from discussion after Proposition 9.2, telling us that the uniformity condition eliminates $O_N^*, U_N^*, H_N^*, K_N^*$, and from the fact that the twistability condition eliminates B_N, B_N^+, C_N, C_N^+ and S_N, S_N^+. Thus, we are left with the 8 quantum groups in the statement, which are indeed easy, coming from the following categories:

$$\begin{tikzcd}
 & \mathcal{NC}_{even} \arrow[dl] \arrow[dd] & & \mathcal{NC}_2 \arrow[ll] \arrow[dl] \arrow[dd] \\
NC_{even} \arrow[dd] & & NC_2 \arrow[ll] \arrow[dd] & \\
 & \mathcal{P}_{even} \arrow[dl] & & \mathcal{P}_2 \arrow[ll] \arrow[dl] \\
P_{even} & & P_2 \arrow[ll] &
\end{tikzcd}$$

Since this latter diagram is an intersection and generation diagram, we conclude that we have an intersection and easy generation diagram of quantum groups, as stated. □

Let us explore now the general case, where we have an arbitrary uniform and twistable easy quantum group. Such a quantum group appears by definition as follows:

$$H_N \subset G_N \subset U_N^+$$

Thus, our quantum group can be imagined as sitting inside the above cube. The point now is that, by using the operations $\cap$ and $\{\,,\}$, we can in principle "project" it on the faces and edges of the cube, and then use some kind of 3D orientation coming from this, in order to deduce some structure and classification results. This will be our plan.

In order to clarify now all this, let us start with the following definition:

Definition 9.7. *Associated to any easy quantum group $H_N \subset G_N \subset U_N^+$ are its classical, discrete and real versions, given by*

$$G_N^c = G_N \cap U_N \quad , \quad G_N^d = G_N \cap K_N^+ \quad , \quad G_N^r = G_N \cap O_N^+$$

as well as its free, smooth and unitary versions, given by

$$G_N^f = \{G_N, H_N^+\} \quad , \quad G_N^s = \{G_N, O_N\} \quad , \quad G_N^u = \{G_N, K_N\}$$

where $\cap$ and $\{\,,\}$ are respectively the intersection and easy generation operations.

In this definition the classical, real and unitary versions are something quite standard. Regarding now the discrete and smooth versions, here we have no abstract justification for our terminology, due to the fact that easy quantum groups do not have known differential geometry. However, in the classical case, where $G_N \subset U_N$, our constructions produce indeed discrete and smooth versions, and this is where our terminology comes from.

Finally, regarding the free version, this is something quite subtle. The various results that we have so far show that the liberation operation $G_N \to G_N^+$ usually appears via the formula $G_N^+ = \{G_N, S_N^+\}$, which expresses the fact that the category of partitions of G_N^+ is obtained from the one of G_N by removing the crossings. But in the twistable setting, where $H_N \subset G_N$, this is the same as setting $G_N^+ = \{G_N, H_N^+\}$. All this is of course a bit approximative, and this is why we use f, and keep $+$ for rock-solid liberations.

In relation now with our questions, and our 3D plan, we can now formulate:

Proposition 9.8. *Given an intermediate quantum group $H_N \subset G_N \subset U_N^+$, we have a diagram of closed subgroups of U_N^+, obtained by inserting*

$$\begin{tikzcd}
& G_N^f & \\
& & G_N^u \\
G_N^d \arrow[r] & G_N \arrow[uu] \arrow[ur] \arrow[r] & G_N^s \\
G_N^r \arrow[ur] & & \\
& G_N^c \arrow[uu] &
\end{tikzcd}
\quad \dashrightarrow \quad
\begin{tikzcd}
& K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[ur] \arrow[rr] & & O_N^+ \arrow[ur] & \\
& K_N \arrow[uu] \arrow[rr] & & U_N \arrow[uu] \\
H_N \arrow[uu] \arrow[ur] \arrow[rr] & & O_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

in the obvious way, with each G_N^x belonging to the main diagonal of each face.

Proof. The fact that we have indeed the diagram of inclusions on the left is clear from Definition 9.7. Regarding now the insertion procedure, consider any of the faces of the cube, $P \subset Q, R \subset S$. Our claim is that the corresponding quantum group $G = G_N^x$ can be inserted on the corresponding main diagonal $P \subset S$, as follows:

$$\begin{tikzcd}
Q \arrow[rr] & & S \\
& G \arrow[ur] & \\
P \arrow[uu] \arrow[ur] \arrow[rr] & & R \arrow[uu]
\end{tikzcd}$$

We have to check here a total of $6 \times 2 = 12$ inclusions. But, according to Definition 9.7, these inclusions to be checked are as follows:

(1) $H_N \subset G_N^c \subset U_N$, where $G_N^c = G_N \cap U_N$.
(2) $H_N \subset G_N^d \subset K_N^+$, where $G_N^d = G_N \cap K_N^+$.
(3) $H_N \subset G_N^r \subset O_N^+$, where $G_N^r = G_N \cap O_N^+$.
(4) $H_N^+ \subset G_N^f \subset U_N^+$, where $G_N^f = \{G_N, H_N^+\}$.
(5) $O_N \subset G_N^s \subset U_N^+$, where $G_N^s = \{G_N, O_N\}$.
(6) $K_N \subset G_N^u \subset U_N^+$, where $G_N^u = \{G_N, K_N\}$.

All these statements being trivial from the definition of $\cap$ and $\{\,,\}$, and from our assumption $H_N \subset G_N \subset U_N^+$, our insertion procedure works indeed, and we are done. $\square$

In order now to complete the diagram, we have to project as well G_N on the edges of the cube. For this purpose we can basically assume, by replacing G_N with each of its 6 projections on the faces, that G_N actually lies on one of the six faces.

The technical result that we will need here is as follows:

Proposition 9.9. *Given an intersection and easy generation diagram $P \subset Q, R \subset S$ and an intermediate easy quantum group $P \subset G \subset S$, we have a diagram as follows:*

$$\begin{tikzcd}
Q \arrow[r] & \{G,Q\} \arrow[r] & S \\
G\cap Q \arrow[u] \arrow[r] & G \arrow[u] \arrow[r] & \{G,R\} \arrow[u] \\
P \arrow[u] \arrow[r] & G\cap R \arrow[u] \arrow[r] & R \arrow[u]
\end{tikzcd}$$

In addition, G "slices the square", in the sense that this is an intersection and easy generation diagram, precisely when $G = \{G \cap Q, G \cap R\}$ and $G = \{G, Q\} \cap \{G, R\}$.

Proof. This is indeed clear from definitions, because the intersection and easy generation conditions are automatic for the upper left and lower right squares, and so are half of the intersection and easy generation conditions for the lower left and upper right squares. Thus, we are left with two conditions only, which are those in the statement. □

Now back to 3 dimensions, and to the cube, we have the following result:

Proposition 9.10. *Assuming that $H_N \subset G_N \subset U_N^+$ satisfies the conditions*

$$G_N^{cs} = G_N^{sc} \quad , \quad G_N^{cu} = G_N^{uc} \quad , \quad G_N^{df} = G_N^{fd}$$

$$G_N^{du} = G_N^{ud} \quad , \quad G_N^{rf} = G_N^{fr} \quad , \quad G_N^{rs} = G_N^{sr}$$

the diagram in Proposition 9.8 can be completed, via the construction in Proposition 9.9, into a diagram dividing the cube along the 3 coordinates axes, into 8 small cubes.

Proof. We have to prove that the 12 projections on the edges are well-defined, with the problem coming from the fact that each of these projections can be defined in 2 possible ways, depending on the face that we choose first. The verification goes as follows:

(1) Regarding the 3 edges emanating from H_N, the result here follows from the following formulae, which are all trivial, of type $(G \cap Q) \cap R = (G \cap R) \cap Q = G \cap P$:

$$G_N^{cd} = G_N^{dc} = G_N \cap K_N$$

$$G_N^{cr} = G_N^{rc} = G_N \cap O_N$$

$$G_N^{dr} = G_N^{rd} = G_N \cap H_N^+$$

(2) Regarding the 3 edges landing into U_N^+, the result here follows from the following formulae, which are again trivial, of type $\{\{G,Q\},R\}=\{\{G,R\},Q\}=\{G,S\}$:

$$G_N^{fs}=G_N^{sf}=\{G_N,O_N^+\}$$

$$G_N^{fu}=G_N^{uf}=\{G_N,K_N^+\}$$

$$G_N^{su}=G_N^{us}=\{G_N,U_N\}$$

(3) Finally, regarding the remaining 6 edges, not emanating from H_N or landing into U_N^+, here the result follows from our assumptions in the statement. □

We are not done yet (!) because nothing guarantees that we obtain in this way an intersection and easy generation diagram. So, we must add more axioms, as follows:

Theorem 9.11. *Assume that $H_N\subset G_N\subset U_N^+$ satisfies the following conditions, where by "intermediate" we mean in each case "parallel to its neighbors":*

1. *The 6 compatibility conditions in Proposition 9.10 above,*
2. *G_N^c,G_N,G_N^f slice the classical/intermediate/free faces,*
3. *G_N^d,G_N,G_N^s slice the discrete/intermediate/smooth faces,*
4. *G_N^r,G_N,G_N^u slice the real/intermediate/unitary faces,*

Then G_N "slices the cube", in the sense that the diagram obtained in Proposition 9.10 above is an intersection and easy generation diagram.

Proof. This follows indeed from Proposition 9.9 and Proposition 9.10 above. □

All this is of course quite theoretical, and might actually seem to be on the verge of insanity. Indeed, we are asking here for a total of $6\times 4=24$ conditions to be satisfied. For the moment, let us not bother with all this, and start the classification work.

It is quite clear that G_N can be reconstructed from its edge projections, so in order to do the classification, we first need a "coordinate system". Common sense would suggest to use the one emanating from H_N, or perhaps the one landing into U_N^+. However, technically speaking, best is to use the coordinate system based at O_N, highlighted below:

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[ur] \arrow[rr] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[uu] \arrow[rr] & & U_N \arrow[uu] \\
H_N \arrow[uu] \arrow[ur] \arrow[rr, Rightarrow] & & O_N \arrow[uu, Rightarrow] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

This choice comes from the fact that the classification result for $O_N\subset O_N^+$, explained below, is something very simple. And this is not the case with the results for $H_N\subset H_N^+$

and for $U_N\subset U_N^+$, from [71], [78] which are quite complicated, with uncountably many solutions, in the general non-uniform case. As for the result for $K_N\subset K_N^+$, this is not available yet, but it is known that there are uncountably many solutions here as well.

So, here is now the key result, from [37], dealing with the vertical direction:

Theorem 9.12. *There is only one proper intermediate easy quantum group*

$$O_N\subset G_N\subset O_N^+$$

namely the quantum group O_N^*, which is not uniform.

Proof. We must compute here the categories of pairings $NC_2\subset D\subset P_2$, and this can be done via some standard combinatorics, in three steps, as follows:

(1) Let $\pi\in P_2-NC_2$, having $s\geq 4$ strings. Our claim is that:

– If $\pi\in P_2-P_2^*$, there exists a semicircle capping $\pi'\in P_2-P_2^*$.

– If $\pi\in P_2^*-NC_2$, there exists a semicircle capping $\pi'\in P_2^*-NC_2$.

Indeed, both these assertions can be easily proved, by drawing pictures.

(2) Consider now a partition $\pi\in P_2(k,l)-NC_2(k,l)$. Our claim is that:

– If $\pi\in P_2(k,l)-P_2^*(k,l)$ then $<\pi>=P_2$.

– If $\pi\in P_2^*(k,l)-NC_2(k,l)$ then $<\pi>=P_2^*$.

This can be indeed proved by recurrence on the number of strings, $s=(k+l)/2$, by using (1), which provides us with a descent procedure $s\to s-1$, at any $s\geq 4$.

(3) Finally, assume that we are given an easy quantum group $O_N\subset G\subset O_N^+$, coming from certain sets of pairings $D(k,l)\subset P_2(k,l)$. We have three cases:

– If $D\not\subset P_2^*$, we obtain $G=O_N$.

– If $D\subset P_2,D\not\subset NC_2$, we obtain $G=O_N^*$.

– If $D\subset NC_2$, we obtain $G=O_N^+$.

Thus, we have proved the uniquess result. As for the non-uniformity of the unique solution, O_N^*, this is something that we already know, from Theorem 9.6 above. □

The above result is something quite remarkable, and it is actually believed that the result could still hold, without the easiness assumption. We refer here to [16].

As already mentioned, the related inclusions $H_N\subset H_N^+$ and $U_N\subset U_N^+$, studied in [71] and [78], are far from being maximal, having uncountably many intermediate objects, and the same is known to hold for $K_N\subset K_N^+$. There are many interesting open questions here. It is conjectured for instance that there should be a contravariant duality $H_N^\times\leftrightarrow U_N^\times$, mapping the family and series from [78] to the series and family from [83].

Here is now another basic result that we will need, in order to perform our classification work here, dealing this time with the "discrete vs. continuous" direction:

Theorem 9.13. *There are no proper intermediate easy groups*

$$H_N\subset G_N\subset O_N$$

except for H_N,O_N themselves.

Proof. We must prove that there are no proper intermediate categories $P_2 \subset D \subset P_{even}$. But this can done via some combinatorics, in the spirit of the proof of Theorem 9.12, and with the idea of the proof of Theorem 9.3 in mind. For full details here, see [35]. $\square$

As a comment here, the inclusion $H_N^+ \subset O_N^+$ is maximal as well, as explained once again in [35]. As for the complex versions of these results, regarding the inclusions $K_N \subset U_N$ and $K_N^+ \subset U_N^+$, here the classification, in the non-uniform case, is available from [83].

Summarizing, we have here once again something very basic and fundamental, providing some evidence for a kind of general "discrete vs. continuous" dictotomy.

Finally, here is a third and last result that we will need, for our classification work here, regarding the missing direction, namely the "real vs. complex" one:

Theorem 9.14. *The proper intermediate easy groups*

$$O_N \subset G_N \subset U_N$$

are the groups $\mathbb{Z}_r O_N$ with $r \in \{2, 3, \ldots, \infty\}$, which are not uniform.

Proof. We must compute here the intermediate categories $\mathcal{P}_2 \subset D \subset P_2$. If we pick $\pi \in D$, assumed to be flat, we can first cap all mixed-colored semicircles, and then pair the black and white semicircles, as to assume that π consists only of black or white semicircles. But the number of these semicircles gives the parameter r. For details here, see [83]. $\square$

Once again, there are many comments that can be made here, with the whole subject in the easy case being generally covered by the classification results in [83]. As for the non-easy case, there are many interesting things here as well, as for instance the results in [16], stating that $PO_N \subset PU_N$, and $\mathbb{T}O_N \subset U_N$ as well, are maximal.

We can now formulate a nice classification result, as follows:

Theorem 9.15 (Ground zero)**.** *There are exactly eight closed subgroups $G_N \subset U_N^+$ having the following properties,*

(1) *Easiness,*
(2) *Uniformity,*
(3) *Twistability,*
(4) *Slicing property,*

namely the quantum groups O_N, U_N, H_N, K_N and $O_N^+, U_N^+, H_N^+, K_N^+$.

Proof. This follows indeed from Theorem 9.12, Theorem 9.13 and Theorem 9.14, which show that the edge projections of G_N must be among the vertices of the cube. By using the slicing axiom, we deduce from this that G_N itself must be a vertex of the cube. $\square$

All this is quite philosophical. Bluntly put, by piling up a number of very natural axioms, namely those of Woronowicz from [98], then our assumption $S^2 = id$, and then the easiness, uniformity, twistability, and slicing properties, we have managed to destroy everything, or almost. The casualities include lots of interesting finite and compact Lie groups, the duals of all finitely generated discrete groups, plus of course lots of interesting quantum groups, which appear not to be strong enough to survive our axioms.

With this job done, let us try now to build something new, and more powerful, on top of this. In order to do so, let us first examine Theorem 9.15, as it is. The easiness property there is definitely something quite heavy. But so is the slicing axiom too, at least in our formulation, from Theorem 9.11 above, which looks written a bit in a hurry.

In order to fix this, and reach to a more powerful 3D axiom, the idea will be that of looking at only 2 of the 6 cubes producing the slicing, namely the lower cube, based at H_N, and the upper cube, based at U_N^+. To be more precise, we have:

Definition 9.16. *An easy quantum group $H_N\subset G_N\subset U_N^+$ is called "bi-oriented" if*

$$\begin{tikzcd}
 & G_N^d \arrow[rr] & & G_N & & & G_N^{fu} \arrow[rr] & & U_N^+ \\
G_N^{dr} \arrow[ur] \arrow[rr] & & G_N^r \arrow[ur] & & & G_N^f \arrow[ur] \arrow[rr] & & G_N^{sf} \arrow[ur] & \\
 & G_N^{cd} \arrow[uu] \arrow[rr] & & G_N^c \arrow[uu] & : & & G_N^u \arrow[uu] \arrow[rr] & & G_N^{su} \arrow[uu] \\
H_N \arrow[uu] \arrow[ur] \arrow[rr] & & G_N^{cr} \arrow[uu] \arrow[ur] & & & G_N \arrow[uu] \arrow[ur] \arrow[rr] & & G_N^s \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

are both intersection and easy generation diagrams.

Observe that the diagram on the left is automatically an intersection diagram, and that the diagram on the right is automatically an easy generation diagram.

The question of replacing the slicing axiom in Theorem 9.15 with the bi-orientability condition makes sense. In fact, we can even talk about weaker axioms, as follows:

Definition 9.17. *An easy quantum group $H_N\subset G_N\subset U_N^+$ is called "oriented" if*

$$G_N=\{G_N^{cd},G_N^{cr},G_N^{dr}\}\quad,\quad G_N=G_N^{fs}\cap G_N^{fu}\cap G_N^{su}$$

and "weakly oriented" if the following weaker conditions hold,

$$G_N=\{G_N^c,G_N^d,G_N^r\}\quad,\quad G_N=G_N^f\cap G_N^s\cap G_N^u$$

where the various versions are those in Definition 9.7 above.

Here, and in Definition 9.16 as well, the terminology comes from the fact that any G_N not satisfying the assumptions looks a bit "disoriented" inside the cube.

In order to prove now the uniqueness result, in the bi-orientable case, we can still proceed as in the proof of Theorem 9.15, but we are no longer allowed to use the coordinate system there, based at O_N. To be more precise, we must use the 2 coordinate systems highlighted below, both taken in some weak sense, weaker than the slicing:

$$\begin{tikzcd}
 & K_N^+ \arrow[rr, Rightarrow] & & U_N^+ \\
H_N^+ \arrow[ur] \arrow[rr] & & O_N^+ \arrow[ur, Rightarrow] & \\
 & K_N \arrow[uu] \arrow[rr] & & U_N \arrow[uu, Rightarrow] \\
H_N \arrow[uu, Rightarrow] \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] & & O_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Skipping some details here, all this seems to be doable, by using the known "edge results" surveyed above, and with the key fact being that the quantum group $H_N^{[\infty]}$ from [78] has no orthogonal counterpart. Thus, we obtain in principle some improvements of Theorem 9.15, under the bi-orientability assumption, and more generally under the orientability assumption. As for the weak orientability assumption, the situation here is more tricky, because we would need "face results", which are not available yet.

Let us discuss now the general, non-easy case. This is in fact the point where we wanted to get, with all that follows being the main motivation for Theorem 9.15 above.

We must first find extensions of the notions of uniformity, twistability and orientability. Regarding the uniformity, the situation here is as follows:

Definition 9.18. *A series $G = (G_N)$ of closed subgroups $G_N \subset U_N^+$ is called:*

1. *Weakly uniform, if $G_{N-1} = G_N \cap U_{N-1}^+$ for any $N \in \mathbb{N}$, with respect to the embedding $U_{N-1}^+ \subset U_N^+$ given by $u \to diag(u, 1)$.*
2. *Uniform, if $G_{N-1} = G_N \cap U_{N-1}^+$ for any $N \in \mathbb{N}$, with respect to the N possible embeddings $U_{N-1}^+ \subset U_N^+$, of type $u \to diag(u, 1)$.*

Observe the difference with what happens in the easy case, from Proposition 9.2, where these two conditions are equivalent. In what follows we will use the condition (2) here, for classification purposes, but we will need as well (1) later on, in section 10 below.

Regarding now the examples, in the classical case we have substantially more examples than in the easy case, obtained by using the determinant, and its powers:

Proposition 9.19. *The following compact groups are uniform,*

1. *The complex reflection groups $H_N^{s,d} = \left\{g \in \mathbb{Z}_s \wr S_N \middle| (\det g)^d = 1\right\}$, for any values of the parameters $s \in \{1, 2, \ldots, \infty\}$ and $d \in \mathbb{N}$, $d|s$,*
2. *The orthogonal group O_N, the special orthogonal group SO_N, and the series of modified unitary groups $U_N^d = \left\{g \in U_N \middle| (\det g)^d = 1\right\}$, with $s \in \{1, 2, \ldots, \infty\}$,*

and so are the bistochastic versions of these groups.

Proof. Both these assertions are clear from definitions. Observe that the groups in (1), which are well-known objects in finite group theory, and more precisely form the series of complex reflection groups, generalize the groups H_N^s from section 8 above, which appear at $d = s$. See [80]. The groups in (2) are well-known as well, in compact Lie group theory, with U_N^1 being equal to SU_N, and with U_N^∞ being by definition U_N itself. $\square$

In the free case now, corresponding to the condition $S_N^+ \subset G_N \subset U_N^+$, it is widely believed that the only examples are the easy ones. A precise conjecture in this sense, which is a bit more general, valid for any $G_N \subset U_N^+$, states that we should have:

$$< G_N, S_N^+ >= \{G_N', S_N^+\}$$

Here G_N' denotes as usual the easy envelope of G_N, and $\{\, ,\}$ is an easy generation operation. This conjecture is probably something quite difficult.

Now back to our questions, we have definitely no new examples in the free case. So, the basic examples will be those that we previously met, namely:

Proposition 9.20. *The following free quantum groups are uniform,*

(1) *Liberations $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$ of the complex reflection groups $H_N^s = \mathbb{Z}_s \wr S_N$,*
(2) *Liberations O_N^+, U_N^+ of the continuous groups O_N, U_N,*

and so are the bistochastic versions of these quantum groups.

Proof. This is something that we basically know, with the uniformity check for H_N^{s+} being the same as for S_N^+, H_N^+, K_N^+, which appear at $s = 1, 2, \infty$. $\square$

In order to cut now a bit from complexity, we would need a second axiom, such as the twistability condition $T_N \subset G_N$. However, if we look at Proposition 9.19, and really like the series there, a condition of type $A_N \subset G_N$ would be more appropriate.

In order to comment on this dillema, let us recall from the discussion after Proposition 9.4 that "taking the bistochastic version" is a bad direction, geometrically speaking. But the operations "taking the diagonal torus" and "taking the special version", that we are currently discussing, are bad too. Thus, we have 3 bad directions, and so a cube:

Proposition 9.21. *We have the following diagram of finite groups,*

$$\begin{tikzcd}
& S_N \arrow[rr] & & H_N \\
A_N \arrow[rr] \arrow[ur] & & SH_N \arrow[ur] & \\
& \{1\} \arrow[rr] \arrow[uu] & & T_N \arrow[uu] \\
\{1\} \arrow[rr] \arrow[uu] \arrow[ur] & & ST_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

obtained from H_N by taking bistochastic, special and diagonal versions.

Proof. This is clear, with the operations of taking bistochastic versions, special versions and diagonal subgroups corresponding to going left, backwards, and downwards. □

Observe that the above cube is degenerate on the bottom left, but this is certainly not surprising, because what we are doing here is to combine 3 bad directions.

Now back to our classification questions, the vertices of the above cube are all interesting groups, and assuming that our quantum groups $G_N \subset U_N^+$ contain any of them is something quite natural. Let us just select here three such conditions, as follows:

Definition 9.22. *A closed subgroup $G_N \subset U_N^+$ is called:*

1. *Twistable, if $T_N \subset G_N$.*
2. *Homogeneous, if $S_N \subset G_N$.*
3. *Half-homogeneous, if $A_N \subset G_N$.*

Let us go ahead now, and formulate our third and last definition, regarding the orientability axiom. Things are quite tricky here, and we must start as follows:

Definition 9.23. *Associated to any closed subgroup $G_N \subset U_N^+$ are its classical, discrete and real versions, and mixes of those, given as usual by*

$$G_N^c = G_N \cap U_N \quad , \quad G_N^d = G_N \cap K_N^+ \quad , \quad G_N^r = G_N \cap O_N^+$$
$$G_N^{cd} = G_N \cap K_N \quad , \quad G_N^{cr} = G_N \cap O_N^+ \quad , \quad G_N^{dr} = G_N \cap H_N^+$$

as well as its free, smooth and unitary versions, and mixes of those, given by

$$G_N^f =< G_N, H_N^+ > \quad , \quad G_N^s =< G_N, O_N > \quad , \quad G_N^u =< G_N, K_N >$$
$$G_N^{fs} =< G_N, O_N^+ > \quad , \quad G_N^{fu} =< G_N, K_N^+ > \quad , \quad G_N^{us} =< G_N, U_N >$$

where $<,>$ is the usual (non-easy) topological generation operation.

Observe the difference, and notational clash, with some of the notions from Definition 9.7 and afterwards. As explained in section 5 above, it is believed that we should have $\{,\} =<,>$, but this is not clear at all, and the problem comes from this.

There is an extra issue as well with the mixed versions of the free, smooth and unitary versions, because we do not really know that these appear indeed by composing the basic f, s, u operations. Once again, we agree here to use Definition 9.23 as it is.

Now back to our orientation questions, the slicing and bi-orientability conditions lead us again into $\{,\}$ vs. $<,>$ troubles, and are therefore rather to be ignored. The orientability conditions from Definition 9.17, however, have the following analogue:

Definition 9.24. *A closed subgroup $G_N \subset U_N^+$ is called "oriented" if*

$$G_N =< G_N^{cd}, G_N^{cr}, G_N^{dr} > \quad , \quad G_N = G_N^{fs} \cap G_N^{fu} \cap G_N^{su}$$

and "weakly oriented" if the following conditions hold,

$$G_N =< G_N^c, G_N^d, G_N^r > \quad , \quad G_N = G_N^f \cap G_N^s \cap G_N^u$$

where the various versions are those in Definition 9.23 above.

With these notions, our claim is that some classification results are possible:

(1) In the classical case for instance, we believe that the uniform, half-homogeneous, oriented groups are those in Proposition 9.19, with some bistochastic versions excluded. This is of course something quite heavy, well beyond easiness, with the potential tools available for proving such things coming from advanced finite group theory and Lie algebra theory. Our uniformity axiom could play a key role here, when combined with [80], in order to exclude all the exceptional objects which might appear on the way.

(2) In the free case, under similar assumptions, we believe that the solutions should be those in Proposition 9.20, once again with some bistochastic versions excluded. This is something heavy, too, related to the above conjecture $< G_N, S_N^+ >= \{G_N', S_N^+\}$. Indeed, assuming that we would have such a formula, and perhaps some more formulae of the same type as well, we can in principle work out our way inside the cube, from the edge and face projections to G_N itself, and in this process G_N would become easy.

(3) In the group dual case, the orientability axiom simplifies, because the group duals are discrete in our sense. We believe that the uniform, twistable, oriented group duals should appear as combinations of certain abelian groups, which appear in the classical case, with duals of varieties of real reflection groups, which appear in the real case. All this looks quite elementary, and nice as well, related for instance to [78], and is probably the topic to start with, in this whole "orientable and non-easy" business.

Summarizing, we have many interesting questions here. As a philosophical conclusion, in view of Proposition 9.8 and of Proposition 9.21, and of [33], [59] and related papers as well, the classification problem in general looks like something quite highly dimensional, with some of the dimensions being good, some other bad, and some other ugly.

10. Toral subgroups

We have seen in the previous sections that the group dual subgroups $\widehat{\Lambda}\subset G$ play an important role in the theory. Our purpose here is to understand how the structure of a closed subgroup $G\subset U_N^+$ can be recovered from the knowledge of such subgroups.

Let us start with a basic statement, regarding the classical and group dual cases:

Proposition 10.1. *Let $G\subset U_N^+$ be a compact quantum group, and consider the group dual subgroups $\widehat{\Lambda}\subset G$, also called toral subgroups, or simply "tori".*

1. *In the classical case, where $G\subset U_N$ is a compact Lie group, these are the usual tori, where by torus we mean here closed abelian subgroup.*
2. *In the group dual case, $G=\widehat{\Gamma}$ with $\Gamma=<g_1,\ldots,g_N>$ being a discrete group, these are the duals of the various quotients $\Gamma\to\Lambda$.*

Proof. Both these assertions are elementary, as follows:

(1) This follows indeed from the fact that a closed subgroup $H\subset U_N^+$ is at the same time classical, and a group dual, precisely when it is classical and abelian.

(2) This follows from the general propreties of the Pontrjagin duality, and more precisely from the fact that the subgroups $\widehat{\Lambda}\subset\widehat{\Gamma}$ correspond to the quotients $\Gamma\to\Lambda$. $\square$

There are two motivations for the study of such subgroups. First, it is well-known that the fine structure of a compact Lie group $G\subset U_N$ is partly encoded by its maximal torus. Thus, in view of Proposition 10.1, the various tori $\widehat{\Lambda}\subset G$ encode interesting information about a quantum group $G\subset U_N^+$, both in the classical and in the group dual case.

As a second motivation, any action $G\curvearrowright X$ on some geometric object, such as a manifold, will produce actions of its tori on the same object, $\widehat{\Lambda}\curvearrowright X$. And, due to the fact that Λ are familiar objects, namely discrete groups, these latter actions are easier to study, and this can ultimately lead to results about the action $G\curvearrowright X$ itself.

At a more concrete level now, most of the tori that we met appear as diagonal tori, in the sense of Proposition 2.18. Let us first review this material. We first have:

Proposition 10.2. *The diagonal torus $T\subset G$, which appears via the formula*

$$C(T)=C(G)\Big/\Big\langle u_{ij}=0\Big|\forall i\neq j\Big\rangle$$

can be defined as well via the following intersection formula, inside U_N^+,

$$T=G\cap\mathbb{T}_N^+$$

where $\mathbb{T}_N^+\subset U_N^+$ is the dual of the free group $F_N=<g_1,\ldots,g_N>$, with $u=diag(g_i)$.

Proof. According to Theorem 2.17 above, the free torus $\mathbb{T}_N^+$ appears as follows:

$$C(\mathbb{T}_N^+)=C(U_N^+)\Big/\Big\langle u_{ij}=0\Big|\forall i\neq j\Big\rangle$$

Thus, by intersecting with G we obtain the diagonal torus of G. See [37]. $\square$

Most of our computations so far of diagonal tori concern various classes of easy quantum groups. In the general easy case, we have the following result:

Proposition 10.3. *For an easy quantum group $G\subset U_N^+$, coming from a category of partitions $D\subset P$, the associated diagonal torus is $T=\widehat{\Gamma}$, with:*

$$\Gamma=F_N\Big/\left\langle g_{i_1}\ldots g_{i_k}=g_{j_1}\ldots g_{j_l}\Big|\forall i,j,k,l,\exists\pi\in D(k,l),\delta_\pi\begin{pmatrix}i\\ j\end{pmatrix}\neq0\right\rangle$$

Moreover, we can just use partitions π which generate the category D.

Proof. If we denote by $g_i=u_{ii}$ the standard coordinates on the associated diagonal torus T, then we have, with $g=diag(g_1,\ldots,g_N)$:

$$\begin{aligned}C(T)&=\left[C(U_N^+)\Big/\left\langle T_\pi\in Hom(u^{\otimes k},u^{\otimes l})\Big|\forall\pi\in D\right\rangle\right]\Big/\left\langle u_{ij}=0\Big|\forall i\neq j\right\rangle\\&=\left[C(U_N^+)\Big/\left\langle u_{ij}=0\Big|\forall i\neq j\right\rangle\right]\Big/\left\langle T_\pi\in Hom(u^{\otimes k},u^{\otimes l})\Big|\forall\pi\in D\right\rangle\\&=C^*(F_N)\Big/\left\langle T_\pi\in Hom(g^{\otimes k},g^{\otimes l})\Big|\forall\pi\in D\right\rangle\end{aligned}$$

The associated discrete group, $\Gamma=\widehat{T}$, is therefore given by:

$$\Gamma=F_N\Big/\left\langle T_\pi\in Hom(g^{\otimes k},g^{\otimes l})\Big|\forall\pi\in D\right\rangle$$

Now observe that, with $g=diag(g_1,\ldots,g_N)$ as above, we have:

$$\begin{aligned}T_\pi g^{\otimes k}(e_{i_1}\otimes\ldots\otimes e_{i_k})&=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}\cdot g_{i_1}\ldots g_{i_k}\\ g^{\otimes l}T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})&=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}\cdot g_{j_1}\ldots g_{j_l}\end{aligned}$$

We conclude that the relation $T_\pi\in Hom(g^{\otimes k},g^{\otimes l})$ reformulates as follows:

$$\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}\neq0\implies g_{i_1}\ldots g_{i_k}=g_{j_1}\ldots g_{j_l}$$

Thus, we obtain the formula in the statement. Finally, the last assertion follows from Tannakian duality, because we can replace everywhere D by a generating subset. □

In practice now, in the continuous case we have the following result:

Theorem 10.4. *The diagonal tori of the basic unitary quantum groups, namely*

$$\begin{tikzcd} U_N \arrow[r] & U_N^* \arrow[r] & U_N^+ \\ O_N \arrow[r] \arrow[u] & O_N^* \arrow[r] \arrow[u] & O_N^+ \arrow[u] \end{tikzcd}$$

and of their $q=-1$ twists as well, are $T_N=\mathbb{Z}_2^N,\mathbb{T}_N=\mathbb{T}^N$ and their liberations:

$$\begin{tikzcd} \mathbb{T}_N \arrow[r] & \mathbb{T}_N^* \arrow[r] & \mathbb{T}_N^+ \\ T_N \arrow[r] \arrow[u] & T_N^* \arrow[r] \arrow[u] & T_N^+ \arrow[u] \end{tikzcd}$$

Also, for the quantum groups B_N,B_N^+,C_N,C_N^+, the diagonal torus collapses to $\{1\}$.

Proof. The main assertion, regarding the basic unitary quantum groups, is something that we already know, from Theorem 2.21 above, with the various liberations $T_N^\times,\mathbb{T}_N^\times$ of the basic tori $T_N,\mathbb{T}_N$ in the statement being by definition those appearing there.

Regarding the invariance under twisting, this is best seen by using Proposition 10.3. Indeed, the computation in the proof there applies in the same way to the general quizzy case, and shows that the diagonal torus is invariant under twisting.

Finally, in the bistochastic case the fundamental corepresentation $g=diag(g_1,\ldots,g_N)$ of the diagonal torus must be bistochastic, and so $g_1=\ldots=g_N=1$, as claimed. □

Regarding now the discrete case, the result is as follows:

Theorem 10.5. *The diagonal tori of the basic quantum reflection groups, namely*

$$\begin{tikzcd} K_N \arrow[r] & K_N^* \arrow[r] & K_N^+ \\ H_N \arrow[r] \arrow[u] & H_N^* \arrow[r] \arrow[u] & H_N^+ \arrow[u] \end{tikzcd}$$

are the same as those for $O_N^\times,U_N^\times$, given above. Also, for S_N,S_N^+ we have $T=\{1\}$.

Proof. The first assertion follows from the general fact that the diagonal torus of $G_N\subset U_N^+$ equals the diagonal torus of the discrete version $G_N^d=G_N\cap K_N^+$, which follows from definitions. As for the second assertion, this follows from $S_N\subset B_N$, $S_N^+\subset B_N^+$. □

As a conclusion, the diagonal torus $T\subset G$ is usually a quite interesting object, but for certain quantum groups like the bistochastic ones, or the quantum permutation group ones, this torus collapses to $\{1\}$, and so it cannot be of use in the study of G.

In order to deal with this issue, the idea, from [9], [32], will be that of using:

Proposition 10.6. *Given a closed subgroup $G\subset U_N^+$ and a matrix $Q\in U_N$, we let $T_Q\subset G$ be the diagonal torus of G, with fundamental representation spinned by Q:*

$$C(T_Q)=C(G)\Big/\left<(QuQ^*)_{ij}=0\Big|\forall i\neq j\right>$$

This torus is then a group dual, $T_Q=\widehat{\Lambda}_Q$, where $\Lambda_Q=<g_1,\ldots,g_N>$ is the discrete group generated by the elements $g_i=(QuQ^*)_{ii}$, which are unitaries inside $C(T_Q)$.

Proof. This follows indeed from Proposition 2.18, because, as said in the statement, T_Q is by definition a diagonal torus. Equivalently, since $v=QuQ^*$ is a unitary corepresentation, its diagonal entries $g_i=v_{ii}$, when regarded inside $C(T_Q)$, are unitaries, and satisfy:

$$\Delta(g_i)=g_i\otimes g_i$$

Thus $C(T_Q)$ is a group algebra, and more specifically we have $C(T_Q)=C^*(\Lambda_Q)$, where $\Lambda_Q=<g_1,\ldots,g_N>$ is the group in the statement, and this gives the result. □

Summarizing, associated to any closed subgroup $G\subset U_N^+$ is a whole family of tori, indexed by the unitaries $U\in U_N$. We use the following terminology:

Definition 10.7. *Let $G\subset U_N^+$ be a closed subgroup.*

1. *The tori $T_Q\subset G$ constructed above are called standard tori of G.*
2. *The collection of tori $T=\left\{T_Q\subset G\big|Q\in U_N\right\}$ is called skeleton of G.*

This might seem a bit awkward, but in view of various results, examples and counterexamples, to be presented below, this is perhaps the best terminology.

As a first general result regarding these tori, we have:

Theorem 10.8. *Any torus $T\subset G$ appears as follows, for a certain $Q\in U_N$:*

$$T\subset T_Q\subset G$$

In other words, any torus appears inside a standard torus.

Proof. Given a torus $T\subset G$, we have an inclusion $T\subset G\subset U_N^+$. On the other hand, we know from Proposition 3.24 that each torus $T=\widehat{\Lambda}\subset U_N^+$, coming from a discrete group $\Lambda=<g_1,\ldots,g_N>$, has a fundamental corepresentation as follows, with $Q\in U_N$:

$$u=Qdiag(g_1,\ldots,g_N)Q^*$$

But this shows that we have $T\subset T_Q$, and this gives the result. □

Let us do now some computations. In the classical case, the result is as follows:

Proposition 10.9. *For a closed subgroup $G\subset U_N$ we have*
$$T_Q=G\cap(Q^*\mathbb T^NQ)$$
where $\mathbb T^N\subset U_N$ is the group of diagonal unitary matrices.

Proof. This is indeed clear at $Q=1$, where Γ_1 appears by definition as the dual of the compact abelian group $G\cap\mathbb T^N$. In general, this follows by conjugating by Q. $\square$

In the group dual case now, we have the following result:

Proposition 10.10. *Given a discrete group $\Gamma=<g_1,\ldots,g_N>$, consider its dual compact quantum group $G=\widehat{\Gamma}$, diagonally embedded into U_N^+. We have then*
$$\Lambda_Q=\Gamma/<g_i=g_j|\exists k,Q_{ki}\neq0,Q_{kj}\neq0>$$
with the embedding $T_Q\subset G=\widehat{\Gamma}$ coming from the quotient map $\Gamma\to\Lambda_Q$.

Proof. Assume indeed that $\Gamma=<g_1,\ldots,g_N>$ is a discrete group, with $\widehat{\Gamma}\subset U_N^+$ coming via $u=diag(g_1,\ldots,g_N)$. With $v=QuQ^*$, we have:
$$\sum_s\bar Q_{si}v_{sk}=\sum_{st}\bar Q_{si}Q_{st}\bar Q_{kt}g_t=\sum_t\delta_{it}\bar Q_{kt}g_t=\bar Q_{ki}g_i$$

Thus $v_{ij}=0$ for $i\neq j$ gives $\bar Q_{ki}v_{kk}=\bar Q_{ki}g_i$, which is the same as saying that $Q_{ki}\neq0$ implies $g_i=v_{kk}$. But this latter equality reads:
$$g_i=\sum_j|Q_{kj}|^2g_j$$

We conclude from this that $Q_{ki}\neq0,Q_{kj}\neq0$ implies $g_i=g_j$, as desired. As for the converse, this is elementary to establish as well. $\square$

According to the above results, we can expect the skeleton T to encode various algebraic and analytic properties of G. We first have the following result:

Theorem 10.11. *The following results hold, both over the category of compact Lie groups, and over the category of duals of finitely generated discrete groups:*

1. *Injectivity: the construction $G\to T$ is injective, in the sense that $G\neq H$ implies $T_Q(G)\neq T_Q(H)$, for some $Q\in U_N$.*
2. *Monotony: the construction $G\to T$ is increasing, in the sense that passing to a subgroup $H\subset G$ decreases at least one of the tori T_Q.*
3. *Generation: any closed quantum subgroup $G\subset U_N^+$ has the generation property $G=<T_Q|Q\in U_N>$. In other words, G is generated by its tori.*

Proof. In the classical case, where $G\subset U_N$, the proof is elementary, based on standard facts from linear algebra, and goes as follows:

(1) Injectivity. This follows from the generation statement, explained below.

(2) Monotony. Once again, this follows from the generation statement.

(3) Generation. We use the following formula, established above:

$$T_Q = G \cap Q^*\mathbb{T}^N Q$$

Since any group element $U \in G$ is diagonalizable, $U = Q^*DQ$ with $Q \in U_N, D \in \mathbb{T}^N$, we have $U \in T_Q$ for this value of $Q \in U_N$, and this gives the result.

Regarding now the group duals, here everything is trivial. Indeed, when the group duals are diagonally embedded we can take $Q = 1$, and when the group duals are embedded by using a spinning matrix $Q \in U_N$, we can use precisely this matrix Q. □

Generally speaking, going beyond Theorem 10.11 is a difficult question, because the above properties are quite abstract. So, let us discuss now a number of more concrete results, relating the skeleton T to the various algebraic and analytic properties of G.

In the classical case, $G \subset U_N$, the use of the maximal tori traditionally requires G to be connected. So, we are in need of a quantum extension of this notion. We have:

Proposition 10.12. *For a closed subgroup $G \subset U_N^+$ the following conditions are equivalent, and if they are satisfied, we call G connected:*

1. *There is no finite quantum group quotient $G \to F \neq \{1\}$.*
2. *The algebra $< v_{ij} >$ is infinite dimensional, for any corepresentation $v \neq 1$.*

In the classical case, $G \subset U_N$, we recover in this way the usual notion of connectedness. For the group duals, $G = \widehat{\Gamma}$, this is the same as asking for Γ to have no torsion.

Proof. The above equivalence comes from the fact that a quotient $G \to F$ must correspond to an embedding $C(F) \subset C(G)$, which must be of the form $C(F) =< v_{ij} >$.

In the classical case, $G \subset U_N$, it is well-known that $F = G/G_1$ is a finite group, where G_1 is the connected component of the identity $1 \in G$, and this gives the result.

As for the group dual case, $G = \widehat{\Gamma}$, here the irreducible corepresentations are 1-dimensional, corresponding to the group elements $g \in \Gamma$, and this gives the result. □

We will be interested in conjectures about characters, amenability and growth. The notion of growth is something that we have not introduced yet, as follows:

Proposition 10.13. *Given a closed subgroup $G \subset U_N^+$, with $1 \in u = \bar{u}$, consider the series whose coefficients are the ball volumes on the corresponding Cayley graph,*

$$f(z) = \sum_k b_k z^k \quad , \quad b_k = \sum_{l(v) \leq k} \dim(v)^2$$

and call it growth series of $\widehat{G}$. Then, in the group dual case, $G = \widehat{\Gamma}$, we obtain in this way the usual growth series of Γ. Also, polynomial growth implies amenability.

Proof. We recall from Proposition 6.6 above that the Cayley graph of $\widehat{G}$ has by definition the elements of $Irr(G)$ as vertices, and the distance is as follows:

$$d(v, w) = \min\left\{k \in \mathbb{N}\Big| 1 \subset \bar{v} \otimes w \otimes u^{\otimes k}\right\}$$

By taking $w = 1$ and by using Frobenius reciprocity, the lenghts are given by:

$$l(v) = \min\left\{k \in \mathbb{N}\Big| v \subset u^{\otimes k}\right\}$$

Thus, we have indeed an extension of the usual notions of length, and growth.

Regarding now the last assertion, by Peter-Weyl we have a decomposition as follows, where B_k is the ball of radius k, and $m_k(v) \in \mathbb{N}$ are certain multiplicities:

$$u^{\otimes k} = \sum_{v \in B_k} m_k(v) \cdot v$$

By using Cauchy-Schwarz, we obtain the following inequality:

$$m_{2k}(1)b_k = \sum_{v \in B_k} m_k(v)^2 \sum_{v \in B_k} \dim(v)^2 \geq \left(\sum_{v \in B_k} m_k(v)\dim(v)\right)^2 = N^{2k}$$

But shows that if b_k has polynomial growth, then the following happens:

$$\limsup_{k \to \infty} m_{2k}(1)^{1/2k} \geq N$$

Thus, the Kesten type criterion applies, and gives the result. □

With the above conventions made, we have the following result, from [32]:

Theorem 10.14. *The following results hold, both over the category of compact Lie groups, and over the category of duals of finitely generated discrete groups:*

(1) *Characters: if G is connected, for any nonzero $P \in C(G)_{central}$ there exists $Q \in U_N$ such that P becomes nonzero, when mapped into $C(T_Q)$.*
(2) *Amenability: a closed subgroup $G \subset U_N^+$ is coamenable if and only if each of the tori T_Q is coamenable, in the usual discrete group sense.*
(3) *Growth: assuming $G \subset U_N^+$, the discrete quantum group $\widehat{G}$ has polynomial growth if and only if each the discrete groups $\widehat{T_Q}$ has polynomial growth.*

Proof. In the classical case, where $G \subset U_N$, the proof goes as follows:

(1) Characters. We can take here $Q \in U_N$ to be such that $QTQ^* \subset \mathbb{T}^N$, where $T \subset U_N$ is a maximal torus for G, and this gives the result.

(2) Amenability. This conjecture holds trivially in the classical case, $G \subset U_N$, due to the fact that these latter quantum groups are all coamenable.

(3) Growth. This is something nontrivial, well-known from the theory of compact Lie groups, and we refer here for instance to [53].

Regarding now the group duals, here everything is trivial. Indeed, when the group duals are diagonally embedded we can take $Q = 1$, and when the group duals are embedded by using a spinning matrix $Q \in U_N$, we can use precisely this matrix Q. □

The above result complements quite well Theorem 10.11, and so we have a beginning of theory here. As explained in [32], it is possible to go beyond the above verifications, notably with some results regarding the half-classical and the free cases. However, there is no serious idea so far, in order to deal with the general case. See [32].

Let us focus now on the generation property, from Theorem 10.11 (3), which is perhaps the most important. In order to discuss the general case, we will need:

Proposition 10.15. *Given a closed subgroup $G\subset U_N^+$ and a matrix $Q\in U_N$, the corresponding standard torus and its Tannakian category are given by*

$$T_Q=G\cap\mathbb{T}_Q\quad,\quad C_{T_Q}=<C_G,C_{\mathbb{T}_Q}>$$

where $\mathbb{T}_Q\subset U_N^+$ is the dual of the free group $F_N=<g_1,\ldots,g_N>$, with the fundamental corepresentation of $C(\mathbb{T}_Q)$ being the matrix $u=Qdiag(g_1,\ldots,g_N)Q^*$.

Proof. The first assertion comes from the well-known fact that given two closed subgroups $G,H\subset U_N^+$, the corresponding quotient algebra $C(U_N^+)\to C(G\cap H)$ appears by dividing by the kernels of both the quotient maps $C(U_N^+)\to C(G)$ and $C(U_N^+)\to C(H)$.

Indeed, the construction of T_Q from Proposition 10.6 amounts precisely in performing this operation, with $H=\mathbb{T}_Q$, and so we obtain $T_Q=G\cap\mathbb{T}_Q$, as claimed.

As for the Tannakian category formula, this follows from this, and from the general duality formula $C_{G\cap H}=<C_G,C_H>$ from section 5 above. □

We have the following Tannakian reformulation of the toral generation property:

Theorem 10.16. *Given a closed subgroup $G\subset U_N^+$, the subgroup $G'=<T_Q|Q\in U_N>$ generated by its standard tori has the following Tannakian category:*

$$C_{G'}=\bigcap_{Q\in U_N}<C_G,C_{\mathbb{T}_Q}>$$

In particular we have $G=G'$ when this intersection reduces to C_G.

Proof. Consider indeed the subgroup $G'\subset G$ constructed in the statement. We have:

$$C_{G'}=\bigcap_{Q\in U_N}C_{T_Q}$$

Together with the formula in Proposition 10.15, this gives the result. □

The above result can be used for investigating the toral generation conjecture, but the combinatorics is quite difficult, and there are no results yet, along these lines.

Let us further discuss now the toral generation property, with some modest results, regarding its behaviour with respect to product operations. We first have:

Proposition 10.17. *Given two closed subgroups $G,H\subset U_N^+$, and $Q\in U_N$, we have:*

$$<T_Q(G),T_Q(H)>\subset T_Q(<G,H>)$$

Also, the toral generation property is stable under the operation $<,>$.

Proof. The first assertion can be proved either by using Theorem 10.16, or directly. For the direct proof, which is perhaps the simplest, we have:

$$T_Q(G) = G \cap \mathbb{T}_Q \subset < G, H > \cap \mathbb{T}_Q = T_Q(< G, H >)$$

$$T_Q(H) = H \cap \mathbb{T}_Q \subset < G, H > \cap \mathbb{T}_Q = T_Q(< G, H >)$$

Now since $A, B \subset C$ implies $< A, B > \subset C$, this gives the result.

Regarding now the second assertion, we have the following computation:

$$\begin{aligned} < G, H > &= << T_Q(G)|Q \in U_N >, < T_Q(H)|Q \in U_N >> \\ &= < T_Q(G), T_Q(H)|Q \in U_N > \\ &= << T_Q(G), T_Q(H) > |Q \in U_N > \\ &\subset < T_Q(< G, H >)|Q \in U_N > \end{aligned}$$

Thus the quantum group $< G, H >$ is generated by its tori, as claimed. □

We have as well the following result:

Proposition 10.18. *We have the following formula, for any G, H and R, S:*

$$T_{R\otimes S}(G \times H) = T_R(G) \times T_S(H)$$

Also, the toral generation property is stable under usual products $\times$.

Proof. The product formula in the statement is clear from definitions. Regarding now the second assertion, we have the following computation:

$$\begin{aligned} < T_Q(G \times H)|Q \in U_{MN} > &\supset < T_{R\otimes S}(G \times H)|R \in U_M, S \in U_N > \\ &= < T_R(G) \times T_S(H)|R \in U_M, S \in U_N > \\ &= < T_R(G) \times \{1\}, \{1\} \times T_S(H)|R \in U_M, S \in U_N > \\ &= < T_R(G)|R \in U_M > \times < T_G(H)|H \in U_N > \\ &= G \times H \end{aligned}$$

Thus the quantum group $G \times H$ is generated by its tori, as claimed. □

In order to get beyond this, let us discuss now some weaker versions of the generation property, which are partly related to the classification program from section 9:

Definition 10.19. *A closed subgroup $G_N \subset U_N^+$, with classical version G_N^c, is called:*

(1) *Weakly generated by its tori, when $G_N =< G_N^c, (T_Q)_{Q\in U_N} >$.*
(2) *A diagonal liberation of G_N^c, when $G_N =< G_N^c, T_1 >$.*

According to our various results above, the first property is satisfied for the groups, for the group duals, and is stable under generations, and direct products.

Regarding now the second property, this is something quite interesting, which takes us away from our original generation questions. The idea here, from [48] and subsequent papers, is that such things can be usually proved by recurrence on $N \in \mathbb{N}$.

In order to discuss this, let us start with:

Proposition 10.20. *Assume that $G = (G_N)$ is weakly uniform, let $n \in \{2, 3, \ldots, \infty\}$ be minimal such that G_n is not classical, and consider the following conditions:*

1. *Strong generation: $G_N =< G_N^c, G_n >$, for any $N > n$.*
2. *Usual generation: $G_N =< G_N^c, G_{N-1} >$, for any $N > n$.*
3. *Initial step generation: $G_{n+1} =< G_{n+1}^c, G_n >$.*

We have then $(1) \iff (2) \implies (3)$, and (3) is in general strictly weaker.

Proof. All the implications and non-implications are elementary, as follows:

$(1) \implies (2)$ This follows from $G_n \subset G_{N-1}$ for $N > n$, coming from uniformity.

$(2) \implies (1)$ By using twice the usual generation, and then the uniformity, we have:

$$G_N =< G_N^c, G_{N-1} >=< G_N^c, G_{N-1}^c, G_{N-2} >=< G_N^c, G_{N-2} >$$

Thus we have a descent method, and we end up with the strong generation condition.

$(2) \implies (3)$ This is clear, because (2) at $N = n + 1$ is precisely (3).

$(3) \not\Longrightarrow (2)$ In order to construct counterexamples here, simplest is to use group duals. Indeed, with $G_N = \widehat{\Gamma_N}$ and $\Gamma_N =< g_1, \ldots, g_N >$, the uniformity condition tells us that we must be in a projective limit situation, as follows:

$$\Gamma_1 \leftarrow \Gamma_2 \leftarrow \Gamma_3 \leftarrow \Gamma_4 \leftarrow \ldots \quad , \quad \Gamma_{N-1} = \Gamma_N / < g_N = 1 >$$

Now by assuming for instance that Γ_2 is given and not abelian, there are many ways of completing the sequence, and so the uniqueness coming from (2) can only fail. □

Let us introduce now a few more notions, as follows:

Proposition 10.21. *Assume that $G = (G_N)$ is weakly uniform, let $n \in \{2, 3, \ldots, \infty\}$ be as above, and consider the following conditions, where $I_N \subset G_N$ is the diagonal torus:*

1. *Strong diagonal liberation: $G_N =< G_N^c, I_n >$, for any $N \geq n$.*
2. *Technical condition: $G_N =< G_N^c, I_{N-1} >$ for any $N > n$, and $G_n =< G_n^c, I_n >$.*
3. *Diagonal liberation: $G_N =< G_N^c, I_N >$, for any N.*
4. *Initial step diagonal liberation: $G_n =< G_n^c, I_n >$.*

We have then $(1) \implies (2) \implies (3) \implies (4)$.

Proof. Our claim is that when assuming that $G = (G_N)$ is weakly uniform, so is the family of diagonal tori $I = (I_N)$. Indeed, we have the following computation:

$$\begin{aligned} I_N \cap U_{N-1}^+ &= (G_N \cap \mathbb{T}_N^+) \cap U_{N-1}^+ \\ &= (G_N \cap U_{N-1}^+) \cap (\mathbb{T}_N^+ \cap U_{N-1}^+) \\ &= G_{N-1} \cap \mathbb{T}_{N-1}^+ \\ &= I_{N-1} \end{aligned}$$

Thus our claim is proved, and this gives the various implications in the statement. □

We can now formulate a key theoretical observation, as follows:

Theorem 10.22. *Assuming that $G = (G_N)$ is weakly uniform, and with $n \in \{2, 3, \ldots, \infty\}$ being as above, the following conditions are equivalent, modulo their initial steps:*

1. *Generation: $G_N =< G_N^c, G_{N-1} >$, for any $N > n$.*
2. *Strong generation: $G_N =< G_N^c, G_n >$, for any $N > n$.*
3. *Diagonal liberation: $G_N =< G_N^c, I_N >$, for any $N \geq n$.*
4. *Strong diagonal liberation: $G_N =< G_N^c, I_n >$, for any $N \geq n$.*

Proof. Our first claim is that generation plus initial step diagonal liberation imply the technical diagonal liberation condition. Indeed, the recurrence step goes as follows:

$$G_N =< G_N^c, G_{N-1} >=< G_N^c, G_{N-1}^c, I_{N-1} >=< G_N^c, I_{N-1} >$$

In order to pass now from the technical diagonal liberation condition to the strong diagonal liberation condition itself, observe that we have:

$$G_N =< G_N^c, G_{N-1} >=< G_N^c, G_{N-1}^c, I_{N-1} >=< G_N^c, I_{N-1} >$$

With this condition in hand, we have then as well:

$$G_N =< G_N^c, G_{N-1} >=< G_N^c, G_{N-1}^c, I_{N-2} >=< G_N^c, I_{N-2} >$$

This procedure can be of course be continued. Thus we have a descent method, and we end up with the strong diagonal liberation condition.

In the other sense now, we want to prove that we have $G_N =< G_N^c, G_{N-1} >$ at $N \geq n$. At $N = n + 1$ this is something that we already have. At $N = n + 2$ now, we have:

$$G_{n+2} =< G_{n+2}^c, I_n >=< G_{n+2}^c, G_{n+1}^c, I_n >=< G_{n+2}^c, G_{n+1} >$$

This procedure can be of course be continued. Thus, we have a descent method, and we end up with the strong generation condition. □

It is possible to prove that many interesting quantum groups have the above properties, and hence appear as diagonal liberations, but the whole subject is quite technical. Here is however a statement, collecting most of the known results on the subject:

Theorem 10.23. *The basic quantum unitary and reflection groups are as follows:*

1. *O_N^*, U_N^* appear via diagonal liberation.*
2. *O_N^+, U_N^+ appear via diagonal liberation.*
3. *H_N^*, K_N^* appear via diagonal liberation.*
4. *H_N^+, K_N^+ do not appear via diagonal liberation.*

In addition, B_N^+, C_N^+, S_N^+ do not appear either via diagonal liberation.

Proof. All this is very technical, and is a matter of ongoing research. However, since the result is very important, in connection with the other considerations in this book, we have preffered to state is as a "theorem", as above, the idea being as follows:

(1) The quantum groups O_N^*,U_N^* are not uniform, and cannot be investigated with the above techniques. However, these quantum groups can be studied by using the technology in [14], [43], [44], which will be briefly discussed in section 12 below, and this leads to $O_N^*=<O_N,T_N^*>$, as well as to $U_N^*=<U_N,T_N^*>$, which implies $U_N^*=<U_N,\mathbb{T}_N^*>$.

(2) The quantum groups O_N^+,U_N^+ are uniform, and a quite technical computation, from [45], [46], [48], [49], shows that the generation conditions from Theorem 10.22 are satisfied for O_N^+. Thus we obtain $O_N^+=<O_N,T_N^+>$, and from this we can deduce via the results in [16] that we have $U_N^+=<U_N,T_N^+>$, which implies $U_N^+=<U_N,\mathbb{T}_N^+>$. See [49].

(3) The situation for H_N^*,K_N^* is quite similar to the one for O_N^*,U_N^*, explained above. Indeed, the technology in [14], [43], [44] applies, and this leads to $H_N^*=<H_N,T_N^*>$, as well as to $K_N^*=<K_N,T_N^*>$, which implies $K_N^*=<K_N,\mathbb{T}_N^*>$. In fact, these results are stronger than the above ones for O_N^*,U_N^*, via some standard generation formulae.

(4) This is something subtle as well, coming from the quantum groups $H_N^{[\infty]},K_N^{[\infty]}$ from [78], discussed in Theorem 8.19. Indeed, since the relations $g_ig_ig_j=g_jg_ig_i$ are trivially satisfied for real reflections, the diagonal tori of these quantum groups coincide with those for H_N^+,K_N^+. Thus, the diagonal liberation procedure "stops" at $H_N^{[\infty]},K_N^{[\infty]}$.

Finally, regarding the last assertion, here B_N^+,C_N^+,S_N^+ do not appear indeed via diagonal liberation, and this because of a trivial reason, namely $T=\{1\}$. □

Summarizing, all this is extremely technical, and working out a full proof of Theorem 10.23, or rather of the positive results there, which is uniform in nature, say based on computations with partitions, remains an open problem, subject to ongoing research.

Regardless of these technical difficulties, and of the various positive results on the subject, the notion of diagonal liberation is, obviously, not exactly the good one.

In order to fix this problem, and come up with a better notion of "toral liberation", let us first discuss the quantum permutation groups. Following [42], we have:

Proposition 10.24. *Given a closed subgroup $G\subset S_N^+$, with standard coordinates denoted $u_{ij}\in C(G)$, the following defines an equivalence relation on $\{1,\ldots,N\}$,*

$$i\sim j\iff u_{ij}\neq 0$$

that we call orbit decomposition associated to the action $G\curvearrowright\{1,\ldots,N\}$. In the classical case, $G\subset S_N$, this is the usual orbit equivalence coming from the action of G.

Proof. We first check the fact that we have indeed an equivalence relation:

(1) $i\sim i$ follows from $\varepsilon(u_{ij})=\delta_{ij}$, which gives $\varepsilon(u_{ii})=1$, and so $u_{ii}\neq 0$, for any i.

(2) $i\sim j\implies j\sim i$ follows from $S(u_{ij})=u_{ji}$, which gives $u_{ij}\neq 0\implies u_{ji}\neq 0$.

(3) $i\sim j,j\sim k\implies i\sim k$ follows from $\Delta(u_{ik})=\sum_j u_{ij}\otimes u_{jk}$. Indeed, in this formula, the right-hand side is a sum of projections, so assuming $u_{ij}\neq 0,u_{jk}\neq 0$ for a certain index j, we have $u_{ij}\otimes u_{jk}>0$, and so $\Delta(u_{ik})>0$, which gives $u_{ik}\neq 0$, as desired.

In the classical case now, $G\subset S_N$, the standard coordinates are the characteristic functions $u_{ij}=\chi(\sigma\in G|\sigma(j)=i)$. Thus the condition $u_{ij}\neq 0$ is equivalent to the existence of an element $\sigma \in G$ such that $\sigma(j) = i$, and this means precisely that i, j must be in the same orbit under the action of G, as claimed. $\square$

Generally speaking, the theory from the classical case extends well to the quantum group setting, and we have in particular the following result, of analytic flavor:

Proposition 10.25. *For a closed subgroup $G \subset S_N^+$, the following are equivalent:*

(1) *G is transitive.*
(2) *$Fix(u) = \mathbb{C}\xi$, where ξ is the all-one vector.*
(3) *$\int_G u_{ij} = \frac{1}{N}$, for any i, j.*

Proof. This is well-known in the classical case. In general, the proof is as follows:

(1) $\iff$ (2). We use the fact that $Fix(u)$ is the fixed point algebra of the standard coaction $\alpha : \mathbb{C}^N \to C(G) \otimes \mathbb{C}^N$, given by $\alpha(\delta_i) = \sum_j u_{ij} \otimes \delta_j$, in the sense that:

$$Fix(u) = \left\{\xi \in \mathbb{C}^N \middle| \alpha(\xi) = 1 \otimes \xi\right\}$$

On the other hand, as explained for instance in [42], via the standard identification $\mathbb{C}^N = C(1, \ldots, N)$, this latter fixed point algebra can be written as:

$$Fix(u) = \left\{\xi \in C(1, \ldots, N) \middle| i \sim j \implies \xi(i) = \xi(j)\right\}$$

In particular, the transitivity condition corresponds to $Fix(u) = \mathbb{C}\xi$, as stated.

(2) $\iff$ (3) This is clear from the general properties of the Haar integration. $\square$

As a comment here, we should mention that the whole theory of quantum group orbits and transitivity, originally developed in [42], has an interesting extension into a theory of quantum group orbitals and 2-transitivity, recently developed in [69].

Now back to the tori, we have the following key result, from [42]:

Theorem 10.26. *Consider a quotient group as follows, with $N = N_1 + \ldots + N_k$:*

$$\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$$

We have then $\widehat{\Gamma} \subset S_N^+$, and any group dual subgroup of S_N^+ appears in this way.

Proof. The fact that we have a subgroup as in the statement follows from:

$$\begin{aligned} \widehat{\Gamma} &\subset \widehat{\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k}} = \widehat{\mathbb{Z}_{N_1}} \,\hat{*} \ldots \hat{*}\, \widehat{\mathbb{Z}_{N_k}} \\ &\simeq \mathbb{Z}_{N_1} \,\hat{*} \ldots \hat{*}\, \mathbb{Z}_{N_k} \subset S_{N_1} \,\hat{*} \ldots \hat{*}\, S_{N_k} \\ &\subset S_{N_1}^+ \,\hat{*} \ldots \hat{*}\, S_{N_k}^+ \subset S_N^+ \end{aligned}$$

Conversely, assume that we have a group dual subgroup $\widehat{\Gamma} \subset S_N^+$. By Theorem 10.8, the corresponding magic unitary must be of the following form, with $U \in U_N$:

$$u = U diag(g_1, \ldots, g_N) U^*$$

Now if we denote by $N = N_1 + \ldots + N_k$ the orbit decomposition for $\widehat{\Gamma} \subset S_N^+$, coming from Proposition 10.24, we conclude that u has a $N = N_1 + \ldots + N_k$ block-diagonal pattern, and so that U has as well this $N = N_1 + \ldots + N_k$ block-diagonal pattern.

But this discussion reduces our problem to its $k = 1$ particular case, with the statement here being that the cyclic group $\mathbb{Z}_N$ is the only transitive group dual $\widehat{\Gamma} \subset S_N^+$. The proof of this latter fact being elementary, we obtain the result. See [42]. □

Here is a related result, from [9], which is useful for our purposes:

Theorem 10.27. *For the quantum permutation group S_N^+, we have:*

(1) *Given $Q \in U_N$, the quotient $F_N \to \Lambda_Q$ comes from the following relations:*

$$\begin{cases} g_i = 1 & \text{if } \sum_l Q_{il} \neq 0 \\ g_i g_j = 1 & \text{if } \sum_l Q_{il} Q_{jl} \neq 0 \\ g_i g_j g_k = 1 & \text{if } \sum_l Q_{il} Q_{jl} Q_{kl} \neq 0 \end{cases}$$

(2) *Given a decomposition $N = N_1 + \ldots + N_k$, for the matrix $Q = diag(F_{N_1}, \ldots, F_{N_k})$, where $F_N = \frac{1}{\sqrt{N}}(\xi^{ij})_{ij}$ with $\xi = e^{2\pi i/N}$ is the Fourier matrix, we obtain:*

$$\Lambda_Q = \mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k}$$

(3) *Given an arbitrary matrix $Q \in U_N$, there exists a decomposition $N = N_1 + \ldots + N_k$, such that Λ_Q appears as quotient of $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k}$.*

Proof. This is something more or less equivalent to Theorem 10.26, and the proof can be deduced either from Theorem 10.26, or from some direct computations. See [9]. □

Summarizing, in the quantum permutation group case, the standard tori parametrized by Fourier matrices play a special role. Now let us recall from section 5 that in what regards the bistochastic groups, which are our second class of examples where the diagonal liberation procedure does not apply, the Fourier matrices appear there as well.

All this discussion suggests formulating the following definition:

Definition 10.28. *Consider a closed subgroup $G \subset U_N^+$.*

(1) *Its standard tori T_F, with $F = F_{N_1} \otimes \ldots \otimes F_{N_k}$, and $N = N_1 + \ldots + N_k$ being regarded as a partition, are called Fourier tori.*
(2) *In the case where we have $G_N =< G_N^c, (T_F)_F >$, we say that G_N appears as a Fourier liberation of its classical version G_N^c.*

We believe that the easy quantum groups should appear as Fourier liberations. With respect to Theorem 10.23 above, the situation in the free case is as follows:

(1) O_N^+, U_N^+ are diagonal liberations, so they are Fourier liberations as well.
(2) B_N^+, C_N^+ are Fourier liberations too, by using Proposition 5.23 above.
(3) S_N^+ is a Fourier liberation too, being generated by its tori [45], [49].
(4) H_N^+, K_N^+ remain to be investigated, by using the general theory in [78].

Finally, as a word of warning here, observe that an arbitrary classical group $G_N \subset U_N$ is not necessarily generated by its Fourier tori, and nor is an arbitrary discrete group dual, with spinned embedding. Thus, the Fourier tori, and the related notion of Fourier liberation, remain something quite technical, in connection with the easy case.

As an application of all this, let us go back to quantum permutation groups, and more specifically to the quantum symmetry groups of finite graphs, from section 8 above. One interesting question is whether $G^+(X)$ appears as a Fourier liberation of $G(X)$.

Generally speaking, this is something quite difficult, because for the empty graph itself we are in need of the above-mentioned technical results from [45], [49].

In order to discuss however this question, let us begin with:

Proposition 10.29. *The Fourier tori of $G^+(X)$ are the biggest quotients*

$$\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$$

whose duals act on the graph, $\widehat{\Gamma} \curvearrowright X$.

Proof. We have indeed the following computation, at $F = 1$:

$$\begin{aligned} C(T_1(G^+(X))) &= C(G^+(X))/ < u_{ij} = 0, \forall i \neq j > \\ &= [C(S_N^+)/ < [d,u] = 0 >]/ < u_{ij} = 0, \forall i \neq j > \\ &= [C(S_N^+)/ < u_{ij} = 0, \forall i \neq j >]/ < [d,u] = 0 > \\ &= C(T_1(S_N^+))// < [d,u] = 0 > \end{aligned}$$

Thus, we obtain the result, with the remark that the quotient that we are interested in appears via relations of type $d_{ij} = 1 \implies g_i = g_j$. The proof in general is similar. □

An interesting question is whether the "non quantum symmetry" property can be seen at the level of Fourier tori. In order to comment on this, let us start with:

Proposition 10.30. *Consider the following conditions:*

(1) *We have $G(X) = G^+(X)$.*
(2) *$G(X) \subset G^+(X)$ is a Fourier liberation.*
(3) *$\widehat{\Gamma} \curvearrowright X$ implies that Γ is abelian.*

We have then $(1) \iff (2) + (3)$.

Proof. This is something elementary, the proof being as follows:

$(1) \implies (2,3)$ Here both the implications are trivial.

$(2,3) \implies (1)$ Assuming $G(X) \neq G^+(X)$, from (2) we know that $G^+(X)$ has at least one non-classical Fourier torus, and this contradicts (3). □

With this observation in hand, our question is whether $(3) \implies (1)$ holds.

In other words, our conjecture would be that a graph X has no quantum symmetry if and only if any action $\widehat{\Gamma} \curvearrowright X$ of a quotient $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$ must come from an abelian quotient $\mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k} \to \Gamma$. This would be of course something very useful.

We have the following result, regarding the torus coactions on finite graphs:

Proposition 10.31. *For a quotient group $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$, and a graph X having $N = N_1 + \ldots + N_k$ vertices, the condition $\widehat{\Gamma} \curvearrowright X$ is equivalent to*

$$(F^*dF)_{ij} \neq 0 \implies I_i = I_j$$

where $F = diag(F_{N_1}, \ldots, F_{N_k})$, and where $I = diag(I_1, \ldots, I_k)$ is the diagonal matrix formed by the elements of the images of $\mathbb{Z}_{N_1}, \ldots, \mathbb{Z}_{N_k}$.

Proof. We know that with F, I being as in the statement, we have $u = FIF^*$. Now with this formula in hand, we have the following equivalences:

$$\begin{aligned}\widehat{\Gamma} \curvearrowright X \quad &\iff \quad du = ud\\ &\iff \quad dFIF^* = FIF^*d\\ &\iff \quad [F^*dF, I] = 0\end{aligned}$$

Also, since the matrix I is diagonal, with $M = F^*dF$ have:

$$\begin{aligned}MI = IM \quad &\iff \quad (MI)_{ij} = (IM)_{ij}\\ &\iff \quad M_{ij}I_j = I_iM_{ij}\\ &\iff \quad [M_{ij} \neq 0 \implies I_i = I_j]\end{aligned}$$

Thus, we obtain the condition in the statement. $\square$

Observe now that in the cyclic case, where $F = F_N$ is a usual Fourier matrix, associated to a cyclic group $\mathbb{Z}_N$, we have the following formula, with $w = e^{2\pi i/N}$:

$$(F^*dF)_{ij} = \sum_{kl}(F^*)_{ik}d_{kl}F_{lj} = \sum_{kl} w^{lj-ik}d_{kl} = \sum_{k\sim l} w^{lj-ik}$$

All this suggests that the random graphs should be "weakly rigid", in the sense that there are no group dual actions on them. Indeed, this should follow in principle from the observation that if $d \in M_N(0,1)$ is random, then we will have $(F^*dF)_{ij} \neq 0$ almost everywhere, and so we will obtain $I_i = I_j$ almost everywhere, and so abelianity.

This remains of course to be worked out, and would be a nice complement to the general work in [79], and to asymptotic no quantum symmetry results from [69].

As a conclusion, the theory of toral subgroups $T \subset G$ is probably as interesting in the compact quantum group case as it is in the compact Lie group case. Among the possible applications, we have a potential Fourier liberation picture for the easy quantum groups, which is related to the classification program explained in section 9 above, as well as various applications to the quantum automorphism groups of finite graphs.

11. Homogeneous spaces

We have seen that the closed subgroups $G\subset U_N^+$ can be investigated with a variety of techniques, for the most belonging to algebraic geometry and probability theory.

Our purpose here is to extend some of these results to certain classes of "quantum homogeneous spaces". This is somehow the first step into extending what we have into a theory of noncommutative geometry, of algebraic and probabilistic nature.

This can be done at several levels of generality, and there has been quite some work here, starting with [29], [34], then going further with [6], and even further with [7].

In what follows we discuss the formalism in [6], which is quite broad, while remaining not very abstract. We will study the spaces of the following type:

$$X=(G_M\times G_N)\big/(G_L\times G_{M-L}\times G_{N-L})$$

These spaces cover indeed the quantum groups and the spheres. And also, they are quite concrete and useful objects, consisting of certain classes of "partial isometries".

Our main result will be a verification of the Bercovici-Pata liberation criterion, for certain variables associated $\chi\in C(X)$, in a suitable $L,M,N\to\infty$ limit.

We begin with a study in the classical case. Our starting point will be:

Definition 11.1. *Associated to any integers $L\leq M,N$ are the spaces*

$$O_{MN}^L=\left\{T:E\to F\text{ isometry}\Big|E\subset\mathbb R^N,F\subset\mathbb R^M,\dim_{\mathbb R}E=L\right\}$$

$$U_{MN}^L=\left\{T:E\to F\text{ isometry}\Big|E\subset\mathbb C^N,F\subset\mathbb C^M,\dim_{\mathbb C}E=L\right\}$$

where the notion of isometry is with respect to the usual real/complex scalar products.

As a first observation, at $L=M=N$ we obtain the groups O_N,U_N:

$$O_{NN}^N=O_N\quad,\quad U_{NN}^N=U_N$$

Another interesting specialization is $L=M=1$. Here the elements of O_{1N}^1 are the isometries $T:E\to\mathbb R$, with $E\subset\mathbb R^N$ one-dimensional, and such an isometry is uniquely determined by the element $T^{-1}(1)\in\mathbb R^N$, which must belong to the sphere $S^{N-1}_{\mathbb R}$. Thus, we have $O_{1N}^1=S^{N-1}_{\mathbb R}$. Similarly, in the complex case we have $U_{1N}^1=S^{N-1}_{\mathbb C}$:

$$O_{1N}^1=S^{N-1}_{\mathbb R}\quad,\quad U_{1N}^1=S^{N-1}_{\mathbb C}$$

Yet another interesting specialization is $L=N=1$. Here the elements of O_{1N}^1 are the isometries $T:\mathbb R\to F$, with $F\subset\mathbb R^M$ one-dimensional, and such an isometry is uniquely determined by the element $T(1)\in\mathbb R^M$, which must belong to the sphere $S^{M-1}_{\mathbb R}$. Thus, we have $O_{M1}^1=S^{M-1}_{\mathbb R}$. Similarly, in the complex case we have $U_{M1}^1=S^{M-1}_{\mathbb C}$:

$$O_{M1}^1=S^{M-1}_{\mathbb R}\quad,\quad U_{M1}^1=S^{M-1}_{\mathbb C}$$

Summarizing, our formalism so far covers well the unitary groups, and the spheres.

In general, the most convenient is to view the elements of O_{MN}^L,U_{MN}^L as rectangular matrices, and to use matrix calculus for their study. We have indeed:

Proposition 11.2. *We have identifications of compact spaces*

$$O_{MN}^L\simeq\left\{U\in M_{M\times N}(\mathbb R)\Big|UU^t=\text{projection of trace }L\right\}$$
$$U_{MN}^L\simeq\left\{U\in M_{M\times N}(\mathbb C)\Big|UU^*=\text{projection of trace }L\right\}$$

with each partial isometry being identified with the corresponding rectangular matrix.

Proof. We can indeed identify the partial isometries $T:E\to F$ with their corresponding extensions $U:\mathbb R^N\to\mathbb R^M$, $U:\mathbb C^N\to\mathbb C^M$, obtained by setting $U_{E^\perp}=0$, and then identify these latter linear maps U with the corresponding rectangular matrices. □

As an illustration, at $L=M=N$ we recover in this way the usual matrix description of O_N,U_N. Also, at $L=M=1$ we obtain the usual description of $S^{N-1}_\mathbb R,S^{N-1}_\mathbb C$, as row spaces over the corresponding groups O_N,U_N. Finally, at $L=N=1$ we obtain the usual description of $S^{N-1}_\mathbb R,S^{N-1}_\mathbb C$, as column spaces over the corresponding groups O_N,U_N.

Now back to the general case, observe that the isometries $T:E\to F$, or rather their extensions $U:\mathbb K^N\to\mathbb K^M$, with $\mathbb K=\mathbb R,\mathbb C$, obtained by setting $U_{E^\perp}=0$, can be composed with the isometries of $\mathbb K^M,\mathbb K^N$, according to the following scheme:

$$\begin{tikzcd}
\mathbb K^N \arrow[r, "B^*"] & \mathbb K^N \arrow[r, dotted, "U"] & \mathbb K^M \arrow[r, "A"] & \mathbb K^M \\
B(E) \arrow[u] \arrow[r, dotted] & E \arrow[u] \arrow[r, "T"] & F \arrow[u] \arrow[r, dotted] & A(F) \arrow[u]
\end{tikzcd}$$

In other words, the groups $O_M\times O_N,U_M\times U_N$ act respectively on O_{MN}^L,U_{MN}^L.

With the identifications in Proposition 11.2 made, the statement here is:

Proposition 11.3. *We have action maps as follows, which are transitive,*

$$O_M\times O_N\curvearrowright O_{MN}^L\quad:\quad (A,B)U=AUB^t$$
$$U_M\times U_N\curvearrowright U_{MN}^L\quad:\quad (A,B)U=AUB^*$$

whose stabilizers are respectively $O_L\times O_{M-L}\times O_{N-L}$ and $U_L\times U_{M-L}\times U_{N-L}$.

Proof. We have indeed action maps as in the statement, which are transitive. Let us compute now the stabilizer G of the point $U=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. Since the elements $(A,B)\in G$ satisfy $AU=UB$, their components must be of the following form:

$$A=\begin{pmatrix}x&*\\0&a\end{pmatrix}\quad,\quad B=\begin{pmatrix}x&0\\ *&b\end{pmatrix}$$

Now since A, B are both unitaries, these matrices follow to be block-diagonal, and so:

$$G = \left\{ (A,B) \middle| A = \begin{pmatrix} x & 0 \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} x & 0 \\ 0 & b \end{pmatrix} \right\}$$

We conclude that the stabilizer of $U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is parametrized by triples (x,a,b) belonging respectively to $O_L \times O_{M-L} \times O_{N-L}$ and $U_L \times U_{M-L} \times U_{N-L}$, as claimed. $\square$

Finally, let us work out the quotient space description of O_{MN}^L, U_{MN}^L:

Theorem 11.4. *We have isomorphisms of homogeneous spaces as follows,*

$$\begin{aligned} O_{MN}^L &= (O_M \times O_N)/(O_L \times O_{M-L} \times O_{N-L}) \\ U_{MN}^L &= (U_M \times U_N)/(U_L \times U_{M-L} \times U_{N-L}) \end{aligned}$$

with the quotient maps being given by $(A,B) \to AUB^*$, where $U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Proof. This is just a reformulation of Proposition 11.3 above, by taking into account the fact that the fixed point used in the proof there was $U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. $\square$

Once again, the basic examples here come from the cases $L = M = N$ and $L = M = 1$, where the quotient spaces at right are respectively O_N, U_N and $O_N/O_{N-1}, U_N/U_{N-1}$. In fact, in the general $L = M$ case we obtain the following spaces, considered in [34]:

$$\begin{aligned} O_{MN}^M &= (O_M \times O_N)/(O_M \times O_{N-M}) = O_N/O_{N-M} \\ U_{MN}^M &= (U_M \times U_N)/(U_M \times U_{N-M}) = U_N/U_{N-M} \end{aligned}$$

Similarly, the examples coming from the cases $L = M = N$ and $L = N = 1$ are particular cases of the general $L = N$ case, where we obtain the following spaces:

$$\begin{aligned} O_{MN}^N &= (O_M \times O_N)/(O_M \times O_{M-N}) = O_N/O_{M-N} \\ U_{MN}^N &= (U_M \times U_N)/(U_M \times U_{M-N}) = U_N/U_{M-N} \end{aligned}$$

For some further information on these spaces, we refer to [6], [34].

We can liberate the spaces O_{MN}^L, U_{MN}^L, as follows:

Definition 11.5. *Associated to any integers $L \leq M, N$ are the algebras*

$$\begin{aligned} C(O_{MN}^{L+}) &= C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N} \middle| u = \bar{u}, uu^t = \text{projection of trace } L \right) \\ C(U_{MN}^{L+}) &= C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N} \middle| uu^*, \bar{u}u^t = \text{projections of trace } L \right) \end{aligned}$$

with the trace being by definition the sum of the diagonal entries.

Observe that the above universal algebras are indeed well-defined, as it was previously the case for the free spheres, and this due to the trace conditions, which read:

$$\sum_{ij} u_{ij} u_{ij}^* = \sum_{ij} u_{ij}^* u_{ij} = L$$

Indeed, these conditions show that we have $||u_{ij}||\leq\sqrt{L}$, for any i,j.

We have inclusions between the various spaces constructed so far, as follows:

$$\begin{array}{ccc}O_{MN}^{L+}&\rightarrow&U_{MN}^{L+}\\ \uparrow&&\uparrow\\ O_{MN}^{L}&\rightarrow&U_{MN}^{L}\end{array}$$

At the level of basic examples now, we first have the following result:

Proposition 11.6. *At $L=M=1$ and $L=N=1$ we obtain the diagrams*

$$\begin{array}{ccc}S^{N-1}_{\mathbb{R},+}&\rightarrow&S^{N-1}_{\mathbb{C},+}\\ \uparrow&&\uparrow\\ S^{N-1}_{\mathbb{R}}&\rightarrow&S^{N-1}_{\mathbb{C}}\end{array}\qquad\qquad\begin{array}{ccc}S^{M-1}_{\mathbb{R},+}&\rightarrow&S^{M-1}_{\mathbb{C},+}\\ \uparrow&&\uparrow\\ S^{M-1}_{\mathbb{R}}&\rightarrow&S^{M-1}_{\mathbb{C}}\end{array}$$

via some standard identifications.

Proof. We recall that the various spheres involved are constructed as follows, with the symbol $\times$ standing for "commutative" and "free", respectively:

$$\begin{aligned}C(S^{N-1}_{\mathbb{R},\times})&=&C^*_\times\left(z_1,\ldots,z_N\Big|z_i=z_i^*,\sum_iz_i^2=1\right)\\ C(S^{N-1}_{\mathbb{C},\times})&=&C^*_\times\left(z_1,\ldots,z_N\Big|\sum_iz_iz_i^*=\sum_iz_i^*z_i=1\right)\end{aligned}$$

Now by comparing with the definition of $O_{1N}^{1\times},U_{1N}^{1\times}$, this proves our first claim.

As for the proof of the second claim, this is similar, via standard identifications. □

We have as well the following result:

Proposition 11.7. *At $L=M=N$ we obtain the diagram*

$$\begin{array}{ccc}O_N^+&\rightarrow&U_N^+\\ \uparrow&&\uparrow\\ O_N&\rightarrow&U_N\end{array}$$

consisting of the groups O_N,U_N, and their liberations.

Proof. We recall that the various quantum groups are constructed as follows, with the symbol $\times$ standing once again for "commutative" and "free":

$$\begin{aligned} C(O_N^\times) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},uu^t=u^tu=1\right)\\ C(U_N^\times) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*=u^*u=1,\bar{u}u^t=u^t\bar{u}=1\right)\end{aligned}$$

On the other hand, according to Proposition 11.2 and to Definition 11.5 above, we have the following presentation results:

$$\begin{aligned} C(O_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},uu^t=\text{projection of trace }N\right)\\ C(U_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*,\bar{u}u^t=\text{projections of trace }N\right)\end{aligned}$$

We use now the standard fact that if $p=aa^*$ is a projection then $q=a^*a$ is a projection too. Together with $Tr(uu^*)=Tr(u^t\bar{u})$ and $Tr(\bar{u}u^t)=Tr(u^*u)$, this gives:

$$\begin{aligned} C(O_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},\ uu^t,u^tu=\text{projections of trace }N\right)\\ C(U_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*,u^*u,\bar{u}u^t,u^t\bar{u}=\text{projections of trace }N\right)\end{aligned}$$

Now observe that, in tensor product notation, and by using the normalized trace, the conditions at right are all of the form $(tr\otimes id)p=1$, with $p=uu^*,u^*u,\bar{u}u^t,u^t\bar{u}$. We therefore obtain, for any faithful state φ:

$$(tr\otimes\varphi)(1-p)=0$$

It follows from this that the projections $p=uu^*,u^*u,\bar{u}u^t,u^t\bar{u}$ must be all equal to the identity, as desired, and this finishes the proof. $\square$

Regarding now the homogeneous space structure of $O_{MN}^{L\times},U_{MN}^{L\times}$, the situation here is more complicated in the free case than in the classical case. We have:

Proposition 11.8. *The spaces $U_{MN}^{L\times}$ have the following properties:*

(1) *We have an action $U_M^\times\times U_N^\times\curvearrowright U_{MN}^{L\times}$, given by $u_{ij}\to\sum_{kl}a_{ik}\otimes b_{jl}^*\otimes u_{kl}$.*
(2) *We have a map $U_M^\times\times U_N^\times\to U_{MN}^{L\times}$, given by $u_{ij}\to\sum_{l\leq L}a_{il}\otimes b_{jl}^*$.*

Similar results hold for the spaces $O_{MN}^{L\times}$, with all the $*$ exponents removed.

Proof. In the classical case, the transpose of the action map $U_M\times U_N\curvearrowright U_{MN}^L$ and of the quotient map $U_M\times U_N\to U_{MN}^L$ are as follows, where $J=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$:

$$\begin{aligned}\varphi &\to ((A,B,U)\to\varphi(AUB^*))\\ \varphi &\to ((A,B)\to\varphi(AJB^*))\end{aligned}$$

But with $\varphi=u_{ij}$ we obtain precisely the formulae in the statement. The proof in the orthogonal case is similar. Regarding now the free case, the proof goes as follows:

(1) Assuming $uu^*u=u$, with $U_{ij}=\sum_{kl}a_{ik}\otimes b_{jl}^*\otimes u_{kl}$ we have:

$$\begin{aligned}(UU^*U)_{ij} &= \sum_{pq}\sum_{klmnst}a_{ik}a_{qm}^*a_{qs}\otimes b_{pl}^*b_{pn}b_{jt}^*\otimes u_{kl}u_{mn}^*u_{st}\\ &= \sum_{klmt}a_{ik}\otimes b_{jt}^*\otimes u_{kl}u_{ml}^*u_{mt}\\ &= \sum_{kt}a_{ik}\otimes b_{jt}^*\otimes u_{kt}\\ &= U_{ij}\end{aligned}$$

Also, assuming that we have $\sum_{ij}u_{ij}u_{ij}^*=L$, we obtain:

$$\begin{aligned}\sum_{ij}U_{ij}U_{ij}^* &= \sum_{ij}\sum_{klst}a_{ik}a_{is}^*\otimes b_{jl}^*b_{jt}\otimes u_{kl}u_{st}^*\\ &= \sum_{kl}1\otimes 1\otimes u_{kl}u_{kl}^*\\ &= L\end{aligned}$$

(2) Assuming $uu^*u=u$, with $V_{ij}=\sum_{l\leq L}a_{il}\otimes b_{jl}^*$ we have:

$$\begin{aligned}(VV^*V)_{ij} &= \sum_{pq}\sum_{x,y,z\leq L}a_{ix}a_{qy}^*a_{qz}\otimes b_{px}^*b_{py}b_{jz}^*\\ &= \sum_{x\leq L}a_{ix}\otimes b_{jx}^*\\ &= V_{ij}\end{aligned}$$

Also, assuming that we have $\sum_{ij}u_{ij}u_{ij}^*=L$, we obtain:

$$\begin{aligned}\sum_{ij}V_{ij}V_{ij}^* &= \sum_{ij}\sum_{l,s\leq L}a_{il}a_{is}^*\otimes b_{jl}^*b_{js}\\ &= \sum_{l\leq L}1\\ &= L\end{aligned}$$

By removing all the $*$ exponents, we obtain as well the orthogonal results. □

Let us examine now the relation between the above maps. In the classical case, given a quotient space $X=G/H$, the associated action and quotient maps are given by:

$$\begin{cases}a:G\times X\to X &:\quad (g,g'H)\to gg'H\\ p:G\to X &:\quad g\to gH\end{cases}$$

Thus we have $a(g,p(g'))=p(gg')$. In our context, a similar result holds:

Theorem 11.9. *With $G = G_M \times G_N$ and $X = G_{MN}^L$, where $G_N = O_N^\times, U_N^\times$, we have*

$$\begin{tikzcd} G\times G \arrow[r, "m"] \arrow[d, "id\times p"'] & G \arrow[d, "p"] \\ G\times X \arrow[r, "a"] & X \end{tikzcd}$$

where a, p are the action map and the map constructed in Proposition 11.8.

Proof. At the level of the associated algebras of functions, we must prove that the following diagram commutes, where Φ, π are morphisms of algebras induced by a, p:

$$\begin{tikzcd} C(X) \arrow[r, "\Phi"] \arrow[d, "\pi"'] & C(G\times X) \arrow[d, "id\otimes\pi"] \\ C(G) \arrow[r, "\Delta"] & C(G\times G) \end{tikzcd}$$

When going right, and then down, the composition is as follows:

$$\begin{aligned}(id\otimes\pi)\Phi(u_{ij}) &= (id\otimes\pi)\sum_{kl} a_{ik}\otimes b_{jl}^*\otimes u_{kl}\\ &= \sum_{kl}\sum_{s\leq L} a_{ik}\otimes b_{jl}^*\otimes a_{ks}\otimes b_{ls}^*\end{aligned}$$

On the other hand, when going down, and then right, the composition is as follows, where F_{23} is the flip between the second and the third components:

$$\begin{aligned}\Delta\pi(u_{ij}) &= F_{23}(\Delta\otimes\Delta)\sum_{s\leq L} a_{is}\otimes b_{js}^*\\ &= F_{23}\left(\sum_{s\leq L}\sum_{kl} a_{ik}\otimes a_{ks}\otimes b_{jl}^*\otimes b_{ls}^*\right)\end{aligned}$$

Thus the above diagram commutes indeed, and this gives the result. □

In general, going beyond Theorem 11.9 leads to some non-trivial questions. A first issue comes from the fact that the inclusions $G_L \times G_{M-L} \times G_{N-L} \subset G_M \times G_N$ are not well-defined, in the free case. There are as well some analytic issues, coming from the fact that the maps in Proposition 11.8 (2) are in general not surjective. See [34].

Let us discuss now some extensions of the above constructions, by using other classes of quantum groups. We will be mostly interested in the quantum reflection groups, so let us first discuss, with full details, the case of the quantum groups H_N^s, H_N^{s+}.

We use the following notion:

Definition 11.10. *Associated to any partial permutation, $\sigma : I \simeq J$ with $I \subset \{1,\ldots,N\}$ and $J \subset \{1,\ldots,M\}$, is the real/complex partial isometry*

$$T_\sigma : span\left(e_i\Big|i \in I\right) \to span\left(e_j\Big|j \in J\right)$$

given on the standard basis elements by $T_\sigma(e_i) = e_{\sigma(i)}$.

We denote by S_{MN}^L the set of partial permutations $\sigma : I \simeq J$ as above, with range $I \subset \{1,\ldots,N\}$ and target $J \subset \{1,\ldots,M\}$, and with $L = |I| = |J|$.

In analogy with the decomposition result $H_N^s = \mathbb{Z}_s \wr S_N$, we have:

Proposition 11.11. *The space of partial permutations signed by elements of $\mathbb{Z}_s$,*

$$H_{MN}^{sL} = \left\{T(e_i) = w_i e_{\sigma(i)}\Big|\sigma \in S_{MN}^L, w_i \in \mathbb{Z}_s\right\}$$

is isomorphic to the quotient space

$$(H_M^s \times H_N^s)/(H_L^s \times H_{M-L}^s \times H_{N-L}^s)$$

via a standard isomorphism.

Proof. This follows by adapting the computations in the proof of Proposition 11.3 above. Indeed, we have an action map as follows, which is transitive:

$$H_M^s \times H_N^s \curvearrowright H_{MN}^{sL} \quad : \quad (A,B)U = AUB^*$$

The stabilizer of the point $U = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ follows to be the group $H_L^s \times H_{M-L}^s \times H_{N-L}^s$, embedded via $(x,a,b) \to [\left(\begin{smallmatrix}x&0\\0&a\end{smallmatrix}\right), \left(\begin{smallmatrix}x&0\\0&b\end{smallmatrix}\right)]$, and this gives the result. $\square$

In the free case now, the idea is similar, by using inspiration from the construction of the quantum group $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$ in [8]. The result here is as follows:

Proposition 11.12. *The noncommutative space H_{MN}^{sL+} associated to the algebra*

$$C(H_{MN}^{sL+}) = C(U_{MN}^{L+}) \Big/ \left\langle u_{ij}u_{ij}^* = u_{ij}^*u_{ij} = p_{ij} = \text{projections}, u_{ij}^s = p_{ij}\right\rangle$$

has an action map, and is the target of a quotient map, as in Theorem 11.9 above.

Proof. We must show that if the variables u_{ij} satisfy the relations in the statement, then these relations are satisfied as well for the following variables:

$$U_{ij} = \sum_{kl} a_{ik} \otimes b_{jl}^* \otimes u_{kl}$$

$$V_{ij} = \sum_{l \le L} a_{il} \otimes b_{jl}^*$$

Since the standard coordinates a_{ij}, b_{ij} on the quantum groups H_M^{s+}, H_N^{s+} satisfy the relations $xy = xy^* = 0$, for any $x \neq y$ on the same row or column of a, b, we obtain:

$$\begin{aligned} U_{ij}U_{ij}^* &= \sum_{klmn} a_{ik}a_{im}^* \otimes b_{jl}^* b_{jm} \otimes u_{kl}u_{mn}^* \\ &= \sum_{kl} a_{ik}a_{ik}^* \otimes b_{jl}^* b_{jl} \otimes u_{kl}u_{kl}^* \end{aligned}$$

We have as well the following formula:

$$\begin{aligned} V_{ij}V_{ij}^* &= \sum_{l,r\leq L} a_{il}a_{ir}^* \otimes b_{jl}^* b_{jr} \\ &= \sum_{l\leq L} a_{il}a_{il}^* \otimes b_{jl}^* b_{jl} \end{aligned}$$

Thus, in terms of the projections $x_{ij} = a_{ij}a_{ij}^*, y_{ij} = b_{ij}b_{ij}^*, p_{ij} = u_{ij}u_{ij}^*$, we have:

$$U_{ij}U_{ij}^* = \sum_{kl} x_{ik} \otimes y_{jl} \otimes p_{kl}$$

$$V_{ij}V_{ij}^* = \sum_{l\leq L} x_{il} \otimes y_{jl}$$

By repeating the computation, we conclude that these elements are projections. Also, a similar computation shows that $U_{ij}^*U_{ij}, V_{ij}^*V_{ij}$ are given by the same formulæ.

Finally, once again by using the relations of type $xy = xy^* = 0$, we have:

$$\begin{aligned} U_{ij}^s &= \sum_{k_r l_r} a_{ik_1} \ldots a_{ik_s} \otimes b_{jl_1}^* \ldots b_{jl_s}^* \otimes u_{k_1 l_1} \ldots u_{k_s l_s} \\ &= \sum_{kl} a_{ik}^s \otimes (b_{jl}^*)^s \otimes u_{kl}^s \end{aligned}$$

We have as well the following formula:

$$\begin{aligned} V_{ij}^s &= \sum_{l_r\leq L} a_{il_1} \ldots a_{il_s} \otimes b_{jl_1}^* \ldots b_{jl_s}^* \\ &= \sum_{l\leq L} a_{il}^s \otimes (b_{jl}^*)^s \end{aligned}$$

Thus the conditions of type $u_{ij}^s = p_{ij}$ are satisfied as well, and we are done. □

Let us discuss now the general case. We have the following result:

Proposition 11.13. *The various spaces G_{MN}^L constructed so far appear by imposing to the standard coordinates of U_{MN}^{L+} the relations*

$$\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=L^{|\pi\vee\sigma|}$$

with $s=(e_1,\ldots,e_s)$ ranging over all the colored integers, and with $\pi,\sigma\in D(0,s)$.

Proof. According to the various constructions above, the relations defining G_{MN}^L can be written as follows, with σ ranging over a family of generators, with no upper legs, of the corresponding category of partitions D:

$$\sum_{j_1\ldots j_s}\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=\delta_\sigma(i)$$

We therefore obtain the relations in the statement, as follows:

$$\begin{aligned}\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}&=\sum_{i_1\ldots i_s}\delta_\pi(i)\sum_{j_1\ldots j_s}\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}\\&=\sum_{i_1\ldots i_s}\delta_\pi(i)\delta_\sigma(i)\\&=L^{|\pi\vee\sigma|}\end{aligned}$$

As for the converse, this follows by using the relations in the statement, by keeping π fixed, and by making σ vary over all the partitions in the category. □

In the general case now, where $G=(G_N)$ is an arbitary uniform easy quantum group, we can construct spaces G_{MN}^L by using the above relations, and we have:

Theorem 11.14. *The spaces $G_{MN}^L\subset U_{MN}^{L+}$ constructed by imposing the relations*

$$\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=L^{|\pi\vee\sigma|}$$

with π,σ ranging over all the partitions in the associated category, having no upper legs, are subject to an action map/quotient map diagram, as in Theorem 11.9.

Proof. We proceed as in the proof of Proposition 11.8. We must prove that, if the variables u_{ij} satisfy the relations in the statement, then so do the following variables:

$$U_{ij}=\sum_{kl}a_{ik}\otimes b_{jl}^*\otimes u_{kl}\quad,\quad V_{ij}=\sum_{l\leq L}a_{il}\otimes b_{jl}^*$$

Regarding the variables U_{ij}, the computation here goes as follows:

$$\begin{aligned}
&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)U_{i_1j_1}^{e_1}\ldots U_{i_sj_s}^{e_s}\\
=&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\delta_\pi(i)\delta_\sigma(j)a_{i_1k_1}^{e_1}\ldots a_{i_sk_s}^{e_s}\otimes(b_{j_sl_s}^{e_s}\ldots b_{j_1l_1}^{e_1})^*\otimes u_{k_1l_1}^{e_1}\ldots u_{k_sl_s}^{e_s}\\
=&\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\delta_\pi(k)\delta_\sigma(l)u_{k_1l_1}^{e_1}\ldots u_{k_sl_s}^{e_s}=L^{|\pi\vee\sigma|}
\end{aligned}$$

For the variables V_{ij} the proof is similar, as follows:

$$\begin{aligned}
&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)V_{i_1j_1}^{e_1}\ldots V_{i_sj_s}^{e_s}\\
=&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\sum_{l_1,\ldots,l_s\leq L}\delta_\pi(i)\delta_\sigma(j)a_{i_1l_1}^{e_1}\ldots a_{i_sl_s}^{e_s}\otimes(b_{j_sl_s}^{e_s}\ldots b_{j_1l_1}^{e_1})^*\\
=&\sum_{l_1,\ldots,l_s\leq L}\delta_\pi(l)\delta_\sigma(l)=L^{|\pi\vee\sigma|}
\end{aligned}$$

Thus we have constructed an action map, and a quotient map, as in Proposition 11.8 above, and the commutation of the diagram in Theorem 11.9 is then trivial. □

Let us discuss now the integration over G_{MN}^L, with a number of explicit formulae. Our main result will be the fact that the operations of type $G_{MN}^L\to G_{MN}^{L+}$ are indeed "liberations", in the sense of the Bercovici-Pata bijection [38].

The integration over G_{MN}^L is best introduced as follows:

Definition 11.15. *The integration functional of G_{MN}^L is the composition*

$$tr:C(G_{MN}^L)\to C(G_M\times G_N)\to\mathbb{C}$$

of the representation $u_{ij}\to\sum_{l\leq L}a_{il}\otimes b_{jl}^*$ with the Haar functional of $G_M\times G_N$.

Observe that in the case $L=M=N$ we obtain the integration over G_N. Also, at $L=M=1$, or at $L=N=1$, we obtain the integration over the sphere.

In the general case now, we first have the following result:

Proposition 11.16. *The integration functional tr has the invariance property*

$$(id\otimes tr)\Phi(x)=tr(x)1$$

with respect to the coaction map given by $\Phi(u_{ij})=\sum_{kl}a_{ik}\otimes b_{jl}^*\otimes u_{kl}$.

Proof. We restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must check the following formula:

$$(id\otimes tr)\Phi(u_{i_1j_1}\ldots u_{i_sj_s})=tr(u_{i_1j_1}\ldots u_{i_sj_s})$$

Let us compute the left term. This is given by:

$$\begin{aligned}
X &= (id\otimes tr)\sum_{k_rl_r}a_{i_1k_1}\ldots a_{i_sk_s}\otimes b_{j_1l_1}^*\ldots b_{j_sl_s}^*\otimes u_{k_1l_1}\ldots u_{k_sl_s}\\
&= \sum_{k_rl_r}\sum_{m_r\leq L}a_{i_1k_1}\ldots a_{i_sk_s}\otimes b_{j_1l_1}^*\ldots b_{j_sl_s}^*\int_{G_M}a_{k_1m_1}\ldots a_{k_sm_s}\int_{G_N}b_{l_1m_1}^*\ldots b_{l_sm_s}^*\\
&= \sum_{m_r\leq L}\sum_{k_r}a_{i_1k_1}\ldots a_{i_sk_s}\int_{G_M}a_{k_1m_1}\ldots a_{k_sm_s}\otimes\sum_{l_r}b_{j_1l_1}^*\ldots b_{j_sl_s}^*\int_{G_N}b_{l_1m_1}^*\ldots b_{l_sm_s}^*
\end{aligned}$$

By using now the invariance property of the Haar functionals of G_M,G_N, we obtain:

$$\begin{aligned}
X &= \sum_{m_r\leq L}\left(id\otimes\int_{G_M}\right)\Delta(a_{i_1m_1}\ldots a_{i_sm_s})\otimes\left(id\otimes\int_{G_N}\right)\Delta(b_{j_1m_1}^*\ldots b_{j_sm_s}^*)\\
&= \sum_{m_r\leq L}\int_{G_M}a_{i_1m_1}\ldots a_{i_sm_s}\otimes\int_{G_N}b_{j_1m_1}^*\ldots b_{j_sm_s}^*\\
&= \left(\int_{G_M}\otimes\int_{G_N}\right)\sum_{m_r\leq L}a_{i_1m_1}\ldots a_{i_sm_s}\otimes b_{j_1m_1}^*\ldots b_{j_sm_s}^*
\end{aligned}$$

But this gives the formula in the statement, and we are done. □

We will prove now that tr is in fact the unique positive unital invariant trace on $C(G_{MN}^L)$. For this purpose, we will need the Weingarten formula.

The integration formula is as follows:

Theorem 11.17. *We have the Weingarten type formula*

$$\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}=\sum_{\pi\sigma\tau\nu}L^{|\sigma\vee\nu|}\delta_\pi(i)\delta_\tau(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

where $W_{sM}=G_{sM}^{-1}$, with $G_{sM}(\pi,\sigma)=M^{|\pi\vee\sigma|}$.

Proof. We make use of the usual quantum group Weingarten formula, for which we refer to [20], [21], [35]. By using this formula for G_M,G_N, we obtain:

$$\begin{aligned}
\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s} &= \sum_{l_1\ldots l_s\leq L}\int_{G_M}a_{i_1l_1}\ldots a_{i_sl_s}\int_{G_N}b_{j_1l_1}^*\ldots b_{j_sl_s}^*\\
&= \sum_{l_1\ldots l_s\leq L}\sum_{\pi\sigma}\delta_\pi(i)\delta_\sigma(l)W_{sM}(\pi,\sigma)\sum_{\tau\nu}\delta_\tau(j)\delta_\nu(l)W_{sN}(\tau,\nu)\\
&= \sum_{\pi\sigma\tau\nu}\left(\sum_{l_1\ldots l_s\leq L}\delta_\sigma(l)\delta_\nu(l)\right)\delta_\pi(i)\delta_\tau(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)
\end{aligned}$$

The coefficient being $L^{|\sigma\vee\nu|}$, we obtain the formula in the statement. □

We can now derive an abstract characterization of tr, as follows:

Proposition 11.18. *The integration functional tr constructed above is the unique positive unital C^*-algebra trace*

$$C(G_{MN}^L)\to\mathbb C$$

which is invariant under the action of $G_M\times G_N$.

Proof. We use the method in [29], [34], the point being to show that tr has the following ergodicity property:

$$\left(\int_{G_M}\otimes\int_{G_N}\otimes id\right)\Phi=tr(.)1$$

We restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must verify that the following holds:

$$\left(\int_{G_M}\otimes\int_{G_N}\otimes id\right)\Phi(u_{i_1j_1}\ldots u_{i_kj_k})=tr(u_{i_1j_1}\ldots u_{i_kj_k})1$$

By using the Weingarten formula, the left term can be written as follows:

$$\begin{aligned}
X &= \sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\int_{G_M}a_{i_1k_1}\ldots a_{i_sk_s}\int_{G_N}b_{j_1l_1}\ldots b_{j_sl_s}\cdot u_{k_1l_1}\ldots u_{k_sl_s}\\
&= \sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\sum_{\pi\sigma}\delta_\pi(i)\delta_\sigma(k)W_{sM}(\pi,\sigma)\sum_{\tau\nu}\delta_\tau(j)\delta_\nu(l)W_{sN}(\tau,\nu)\cdot u_{k_1l_1}\ldots u_{k_sl_s}\\
&= \sum_{\pi\sigma\tau\nu}\delta_\pi(i)\delta_\tau(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\delta_\sigma(k)\delta_\nu(l)u_{k_1l_1}\ldots u_{k_sl_s}
\end{aligned}$$

By using now the formula in Theorem 11.17 above, we obtain:

$$X=\sum_{\pi\sigma\tau\nu}L^{|\sigma\vee\nu|}\delta_\pi(i)\delta_\tau(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

Now by comparing with the usual Weingarten formula, this proves our claim.

Assume now that $\tau:C(G_{MN}^L)\to\mathbb C$ satisfies the invariance condition. We have:

$$\begin{aligned}
\tau\left(\int_{G_M}\otimes\int_{G_N}\otimes id\right)\Phi(x) &= \left(\int_{G_M}\otimes\int_{G_N}\otimes\tau\right)\Phi(x)\\
&= \left(\int_{G_M}\otimes\int_{G_N}\right)(id\otimes\tau)\Phi(x)\\
&= \left(\int_{G_M}\otimes\int_{G_N}\right)(\tau(x)1)\\
&= \tau(x)
\end{aligned}$$

On the other hand, according to the formula established above, we have as well:

$$\tau\left(\int_{G_M}\otimes\int_{G_N}\otimes id\right)\Phi(x)=\tau(tr(x)1)=tr(x)$$

Thus we obtain $\tau=tr$, and this finishes the proof. □

We discuss now the precise computation of the laws of certain linear combinations of coordinates. A set of coordinates $\{u_{ij}\}$ is called "non-overlapping" if each horizontal index i and each vertical index j appears at most once. With this convention, we have:

Proposition 11.19. *For a sum of non-overlapping coordinates, of type*

$$\chi_E=\sum_{(ij)\in E}u_{ij}$$

we have the moment formula

$$\int_{G_{MN}^L}\chi_E^s=\sum_{\pi\sigma\tau\nu}K^{|\pi\vee\tau|}L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

where $K=|E|$ is the cardinality of the indexing set.

Proof. In terms of $K=|E|$, we can write $E=\{(\alpha(i),\beta(i))\}$, for certain embeddings $\alpha:\{1,\ldots,K\}\subset\{1,\ldots,M\}$ and $\beta:\{1,\ldots,K\}\subset\{1,\ldots,N\}$. In terms of these maps α,β, the moment in the statement is given by:

$$M_s=\int_{G_{MN}^L}\left(\sum_{i\leq K}u_{\alpha(i)\beta(i)}\right)^s$$

By using the Weingarten formula, we can write this quantity as follows:

$$\begin{aligned}M_s&=\int_{G_{MN}^L}\sum_{i_1\ldots i_s\leq K}u_{\alpha(i_1)\beta(i_1)}\ldots u_{\alpha(i_s)\beta(i_s)}\\&=\sum_{i_1\ldots i_s\leq K}\sum_{\pi\sigma\tau\nu}L^{|\sigma\vee\nu|}\delta_\pi(\alpha(i_1),\ldots,\alpha(i_s))\delta_\tau(\beta(i_1),\ldots,\beta(i_s))W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\\&=\sum_{\pi\sigma\tau\nu}\left(\sum_{i_1\ldots i_s\leq K}\delta_\pi(i)\delta_\tau(i)\right)L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\end{aligned}$$

But, as explained before, the coefficient on the left in the last formula equals $K^{|\pi\vee\tau|}$. We therefore obtain the formula in the statement. □

We can further advance in the classical/twisted and free cases, where the Weingarten theory for the corresponding quantum groups is available from [8], [20], [21], [35]. The result here, which justifies our various "liberation" claims, is as follows:

Theorem 11.20. *In the context of the liberation operations $O_{MN}^L\to O_{MN}^{L+}$, $U_{MN}^L\to U_{MN}^{L+}$, $H_{MN}^{sL}\to H_{MN}^{sL+}$, the laws of the sums of non-overlapping coordinates,*

$$\chi_E=\sum_{(ij)\in E}u_{ij}$$

are in Bercovici-Pata bijection, in the $|E|=\kappa N,L=\lambda N,M=\mu N,N\to\infty$ limit.

Proof. We use the general theory in [8], [20], [21], [35]. According to Proposition 11.19, in terms of $K=|E|$, the moments of the variables in the statement are given by:

$$M_s=\sum_{\pi\sigma\tau\nu}K^{|\pi\vee\tau|}L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

We use now two standard facts, namely the fact that in the $N\to\infty$ limit the Weingarten matrix W_{sN} is concentrated on the diagonal, and the fact that we have $|\pi\vee\sigma|\leq\frac{|\pi|+|\sigma|}{2}$, with equality precisely when $\pi=\sigma$. See [20]. In the regime $K=\kappa N,L=\lambda N,M=\mu N,N\to\infty$ from the statement, we therefore obtain:

$$\begin{aligned}M_s&\simeq\sum_{\pi\tau}K^{|\pi\vee\tau|}L^{|\pi\vee\tau|}M^{-|\pi|}N^{-|\tau|}\\&\simeq\sum_{\pi}K^{|\pi|}L^{|\pi|}M^{-|\pi|}N^{-|\pi|}\\&=\sum_{\pi}\left(\frac{\kappa\lambda}{\mu}\right)^{|\pi|}\end{aligned}$$

In order to interpret this formula, we use general theory from [8], [20], [21]:

(1) For $G_N=O_N,\bar{O}_N/O_N^+$, the above variables χ_E follow to be asymptotically Gaussian/semicircular, of parameter $\frac{\kappa\lambda}{\mu}$, and hence in Bercovici-Pata bijection.

(2) For $G_N=U_N,\bar{U}_N/U_N^+$ the situation is similar, with χ_E being asymptotically complex Gaussian/circular, of parameter $\frac{\kappa\lambda}{\mu}$, and in Bercovici-Pata bijection.

(3) Finally, for $G_N=H_N^s/H_N^{s+}$, the variables χ_E are asymptotically Bessel/free Bessel of parameter $\frac{\kappa\lambda}{\mu}$, and once again in Bercovici-Pata bijection. □

As a first comment, there are several possible extensions of the above result, for instance by using quantum reflection groups instead of unitary quantum groups, and by using twisting operations as well. We refer here to [6], and to [34] as well, for a number of supplementary results, which can be obtained by using the stronger formalism there.

Let us also mention that the above formalism, from [6], can be further extended, to a certain class of "affine homogeneous spaces", which appear as certain special submanifolds $X\subset S^{N-1}_{\mathbb{C},+}$. However, there is still no known axiomatization of the class of noncommutative algebraic manifolds that we can obtain in this way, at this level. See [7].

In relation with all this, and with noncommutative geometry in general, there are some interesting projective geometry aspects as well. Indeed, in analogy with the isomorphism $PO_N^+ = PU_N^+$, discussed in section 5 above, one can prove that there exists a "free projective space", which is unique, at the same time real and complex, $P^{N-1}_{\mathbb{R},+} = P^{N-1}_{\mathbb{C},+}$. This is quite interesting, and the further study here is not developed yet. See [30].

As a very general comment now, the compact quantum groups are meant to act of course, but not only on geometric objects. Besides the work regarding the actions of the compact quantum groups on compact Riemannian manifolds [9], [39], [40], [63], [64], there has been a lot of work as well concerning the actions on abstract von Neumann algebras and subfactors [56], and abstract sequences of random variables [27], [68]. There is as well an emerging differential geometry theory for the compact quantum groups and related homogeneous spaces, and we refer here to [29], [50], [54], [55], [85].

Getting back now to noncommutative geometry, we believe that the examples of noncommutative manifolds studied here, which obviously have some Riemannian features, are not very far from the manifolds of Connes [52], and could eventually fit into an extension of his theory. To be more precise, one theoretical downside of Connes' theory is the lack of an analogue of the Nash embedding theorem [74]. Assuming that this question will be solved one day, and with the target of the "generalized Nash embeddings" being the free sphere $S^{N-1}_{\mathbb{C},+}$, the unification problem would be probably solvable.

In short, we believe in the existence of a "Nash-Connes Geometry", covering most of the interesting examples of noncommutative manifolds known so far.

12. Modelling questions

One interesting method for the study of the closed subgroups $G \subset U_N^+$, that we have not tried yet, consists in modelling the standard coordinates $u_{ij} \in C(G)$ by concrete variables $U_{ij} \in B$. Indeed, assuming that the model is faithful in some suitable sense, that the algebra B is something quite familiar, and that the variables U_{ij} are not too complicated, all questions about G would correspond in this way to routine questions inside B.

Regarding the choice of B, some very convenient algebras are the random matrix ones, $B = M_K(C(T))$, with $K \in \mathbb{N}$, and with T being a compact space. These algebras generalize indeed the most familiar algebras that we know, namely the matrix ones $M_K(\mathbb{C})$, and the commutative ones $C(T)$. We are led in this way to:

Definition 12.1. *A matrix model for $G \subset U_N^+$ is a morphism of C^*-algebras*

$$\pi : C(G) \to M_K(C(T))$$

where T is a compact space, and $K \geq 1$ is an integer.

More generally, we can try to model in this way the standard coordinates $x_i \in C(X)$ of the various algebraic manifolds $X \subset S^{N-1}_{\mathbb{C},+}$. Indeed, these manifolds generalize the compact matrix quantum groups, which appear as $G \subset U_N^+ \subset S^{N^2-1}_{\mathbb{C},+}$, and we have many other interesting examples, such as the homogeneous spaces discussed in section 11.

However, at this level of generality, not much general theory is available. It is elementary to show that, under the technical assumption $X^c \neq \emptyset$, there exists a universal $K \times K$ model, which factorizes as follows, with $X^{(K)} \subset X$ being a certain submanifold:

$$\pi_K : C(X) \to C(X^{(K)}) \subset M_K(C(T_K))$$

To be more precise, T_K appears by imposing to the complex $K \times K$ matrices the relations defining X, and the algebra $C(X^{(K)})$ is by definition the image of π_K. By setting as well $X^{(\infty)} = \cup_{K \in \mathbb{N}} X^{(K)}$, we are led in this way to a filtration of X, as follows:

$$X^c = X^{(1)} \subset X^{(2)} \subset X^{(3)} \subset \ldots\ldots \subset X^{(\infty)} \subset X$$

It is possible to say a few non-trivial things about these manifolds $X^{(K)}$, by using algebraic and functional analytic techniques, and we refer here to [14], [48].

In the compact quantum group case, however, that we are mainly interested in here, the matrix truncations $G^{(K)} \subset G$ are generically not subgroups at $K \geq 2$, and so this theory is a priori not very useful, at least in its basic form presented here.

In order to reach, however, to some results, let us introduce as well:

Definition 12.2. *A matrix model $\pi : C(G) \to M_K(C(T))$ is called stationary when*

$$\int_G = \left(tr \otimes \int_T \right) \pi$$

where $\int_T$ is the integration with respect to a given probability measure on T.

Observe that this definition can be extended as well to the algebraic manifold case, $X\subset S^{N-1}_{\mathbb{C},+}$, provided that our manifolds have certain integration functionals $\int_X$. This is the case for instance with the homogeneous spaces discussed in section 11, where $\int_X$ appears as the unique G-invariant trace, with respect to the underlying quantum group G. However, the axiomatization of such manifolds being not available yet, we will keep this as a remark, and get back in what follows, until the end, to the quantum groups.

So, back to Definition 12.2, as it is, our first comment concerns the terminology. The term "stationary" comes from a functional analytic interpretation of all this, with a certain Cesàro limit being needed to be stationary, and this will be explained later on. Yet another explanation comes from a certain relation with the lattice models, but this relation is rather something folklore, not axiomatized yet. We will be back to this later.

As a first result now, the stationarity property implies the faithfulness:

Theorem 12.3. *Assuming that $G\subset U_N^+$ has a stationary model,*

$$\pi:C(G)\to M_K(C(T))\quad,\quad\int_G=\left(tr\otimes\int_T\right)\pi$$

it follows that G must be coamenable, and that the model is faithful.

Proof. Assume that we have a stationary model, as in the statement. By performing the GNS construction with respect to $\int_G$, we obtain a factorization as follows, which commutes with the respective canonical integration functionals:

$$\pi:C(G)\to C(G)_{red}\subset M_K(C(T))$$

Thus, in what regards the coamenability question, we can assume that π is faithful. With this assumption made, observe that we have embeddings as follows:

$$C^\infty(G)\subset C(G)\subset M_K(C(T))$$

An idea here would be that of lifting the counit of $C^\infty(G)$ into a linear form over $M_K(C(T))$, which would restrict into a counit for $C(G)$. However, this is not obvious.

In short, we must ask for help a fellow mathematician. And our colleague will point out that the GNS construction gives in fact a better embedding, as follows:

$$L^\infty(G)\subset M_K(L^\infty(T))$$

Now since the von Neumann algebra on the right is of type I, so must be its subalgebra $A=L^\infty(G)$. This means that, when writing the center of this latter algebra as $Z(A)=L^\infty(X)$, the whole algebra decomposes over X, as an integral of type I factors:

$$L^\infty(G)=\int_X M_{K_x}(\mathbb{C})\,dx$$

In particular, we can see from this that $C^\infty(G)\subset L^\infty(G)$ has a unique C^*-norm, and so G is coamenable. Thus we have proved our first assertion, and the second assertion follows as well, because our factorization of π consists of the identity, and of an inclusion. □

All this might seem a bit mysterious, but we really need the above result, and have no simple proof for it. For some background on these questions, which are quite beautiful, we recommend [90] and the other papers of von Neumann, which are a must-read.

Regarding now the examples of stationary models, we first have:

Proposition 12.4. *The following have stationary models:*
(1) *The compact Lie groups.*
(2) *The finite quantum groups.*

Proof. Both these assertions are elementary, with the proofs being as follows:

(1) This is clear, because we can use here the identity map:

$$id:C(G)\to M_1(C(G))$$

(2) This is clear as well, because we can use here the regular representation:

$$\lambda:C(G)\to M_{|G|}(\mathbb{C})$$

To be more precise, if we endow the linear space $H=C(G)$ with the scalar product $<a,b>=\int_G ab^*$, we have a representation $\lambda:C(G)\to B(H)$ given by $a\to[b\to ab]$. Now since we have $H\simeq\mathbb{C}^{|G|}$ with $|G|=\dim A$, we can view λ as a matrix model map, as above, and the stationarity axiom $\int_G=tr\circ\lambda$ is satisfied, as desired. □

In order to discuss now the group duals, let us recall that, according to the general theory of group algebras, the matrix models $\pi:C^*(\Gamma)\to M_K(C(T))$ must come from group representations $\rho:\Gamma\to C(T,U_K)$. With this identification made, we have:

Proposition 12.5. *An matrix model $\rho:\Gamma\subset C(T,U_K)$ is stationary when:*

$$\int_T tr(g^x)dx=0,\forall g\neq 1$$

Moreover, the examples include all the abelian groups, and all finite groups.

Proof. Consider indeed a group embedding $\rho:\Gamma\subset C(T,U_K)$, which produces by linearity a matrix model $\pi:C^*(\Gamma)\to M_K(C(T))$. By linearity and continuity, it is enough to formulate the stationarity condition on the group elements $g\in C^*(\Gamma)$. With the notation $\rho(g)=(x\to g^x)$, this stationarity condition reads:

$$\int_T tr(g^x)dx=\delta_{g,1}$$

Since this equality is trivially satisfied at $g=1$, where by unitality of our representation we must have $g^x=1$ for any $x\in T$, we are led to the condition in the statement.

Regarding now the examples, these are both clear. More precisely:

(1) When Γ is abelian we can use the following trivial embedding:

$$\Gamma\subset C(\widehat{\Gamma},U_1)\quad:\quad g\to[\chi\to\chi(g)]$$

(2) When Γ is finite we can use the left regular representation:

$$\Gamma\subset\mathcal L(\mathbb C\Gamma)\quad:\quad g\to[h\to gh]$$

Indeed, in both cases, the stationarity condition is trivially satisfied. $\square$

In order to further advance, and to come up with some tools for discussing the non-stationary case as well, let us keep looking at the group duals $G=\widehat{\Gamma}$. We know that a model $\pi:C^*(\Gamma)\to M_K(C(T))$ must come from a group representation $\rho:\Gamma\to C(T,U_K)$. Now observe that when ρ is faithful, the representation π is in general not faithful, for instance because when $T=\{.\}$ its target algebra is finite dimensional. On the other hand, this representation "reminds" Γ, and so can be used in order to fully understand Γ.

Summarizing, we have a new idea here, basically saying that, for practical purposes, the faithfuless property can be replaced with something much weaker. This weaker notion is called "inner faithfulness", and the theory here, from [12], is as follows:

Definition 12.6. *Let $\pi:C(G)\to M_K(C(T))$ be a matrix model.*

1. *The Hopf image of π is the smallest quotient Hopf C^*-algebra $C(G)\to C(H)$ producing a factorization of type $\pi:C(G)\to C(H)\to M_K(C(T))$.*
2. *When the inclusion $H\subset G$ is an isomorphism, i.e. when there is no non-trivial factorization as above, we say that π is inner faithful.*

These constructions work in fact for any C^*-algebra representation $\pi:C(G)\to B$, but here we will be only interested in the random matrix case, $B=M_K(C(T))$.

In the case where $G=\widehat{\Gamma}$ is a group dual, π must come from a group representation $\rho:\Gamma\to C(T,U_K)$, and the above factorization is simply the one obtained by taking the image, $\rho:\Gamma\to\Lambda\subset C(T,U_K)$. Thus π is inner faithful when $\Gamma\subset C(T,U_K)$.

Also, given a compact group G, and elements $g_1,\ldots,g_K\in G$, we have a representation $\pi:C(G)\to\mathbb C^K$, given by $f\to(f(g_1),\ldots,f(g_K))$. The minimal factorization of π is then via $C(H)$, with $H=\overline{<g_1,\ldots,g_K>}$, and π is inner faithful when $G=H$.

In general, the existence and uniqueness of the Hopf image comes from dividing $C(G)$ by a suitable ideal, as explained in [12]. In Tannakian terms, we have:

Theorem 12.7. *Assuming $G\subset U_N^+$, with fundamental corepresentation $u=(u_{ij})$, the Hopf image of $\pi:C(G)\to M_K(C(T))$ comes from the Tannakian category*

$$C_{kl}=Hom(U^{\otimes k},U^{\otimes l})$$

where $U_{ij}=\pi(u_{ij})$, and where the spaces on the right are taken in a formal sense.

Proof. Since the morphisms increase the intertwining spaces, when defined either in a representation theory sense, or just formally, we have inclusions as follows:

$$Hom(u^{\otimes k},u^{\otimes l})\subset Hom(U^{\otimes k},U^{\otimes l})$$

More generally, we have such inclusions when replacing (G, u) with any pair producing a factorization of π. Thus, by Tannakian duality, the Hopf image must be given by the fact that the intertwining spaces must be the biggest, subject to the above inclusions.

On the other hand, since u is biunitary, so is U, and it follows that the spaces on the right form a Tannakian category. Thus, we have a quantum group (H, v) given by:

$$Hom(v^{\otimes k}, v^{\otimes l}) = Hom(U^{\otimes k}, U^{\otimes l})$$

By the above discussion, $C(H)$ follows to be the Hopf image of π, as claimed. $\square$

The inner faithful models $\pi : C(G) \to M_K(C(T))$ are a very interesting notion, because they are not subject to the coamenability condition on G, as it was the case with the stationary models, as explained in Theorem 12.3. In fact, there are no known restrictions on the class of closed subgroups $G \subset U_N^+$ which can be modelled in an inner faithful way. Thus, our modelling theory applies a priori to any compact quantum group.

Regarding now the study of the inner faithful models, a key problem is that of computing the Haar integration functional. The result here, from [93], is as follows:

Theorem 12.8. *Given an inner faithful model $\pi : C(G) \to M_K(C(T))$, we have*

$$\int_G = \lim_{k \to \infty} \frac{1}{k} \sum_{r=1}^{k} \int_G^r$$

where $\int_G^r = (\varphi \circ \pi)^{*r}$, with $\varphi = tr \otimes \int_T$ being the random matrix trace.

Proof. As a first observation, there is an obvious similarity here with the Woronowicz construction of the Haar measure, explained in section 3 above. In fact, the above result holds more generally for any model $\pi : C(G) \to B$, with $\varphi \in B^*$ being a faithful trace. With this picture in hand, the Woronowicz construction simply corresponds to the case $\pi = id$, and the result itself is therefore a generalization of Woronowicz's result.

In order to prove now the result, we can proceed as in section 3. If we denote by $\int_G'$ the limit in the statement, we must prove that this limit converges, and that we have $\int_G' = \int_G$. It is enough to check this on the coefficients of corepresentations, and if we let $v = u^{\otimes k}$ be one of the Peter-Weyl corepresentations, we must prove that we have:

$$\left(id \otimes \int_G' \right) v = \left(id \otimes \int_G \right) v$$

We already know, from Theorem 3.18 above, that the matrix on the right is the orthogonal projection onto $Fix(v)$. Regarding now the matrix on the left, Proposition 3.16 applied to the linear form $\varphi\pi$ tells us that this is the orthogonal projection onto the 1-eigenspace of $(id \otimes \varphi\pi)v$. Now observe that, if we set $V_{ij} = \pi(v_{ij})$, we have:

$$(id \otimes \varphi\pi)v = (id \otimes \varphi)V$$

Thus, we can apply Proposition 3.17, or rather use the same computation as there, which is only based on the biunitarity condition, and we conclude that the 1-eigenspace that we are interested in equals $Fix(V)$. But, according to Theorem 12.7, we have:

$$Fix(V) = Fix(v)$$

Thus, we have proved that we have $\int_G' = \int_G$, as desired. $\square$

Getting back now to the stationary models, we have the following result:

Theorem 12.9. *For $\pi : C(G) \to M_K(C(T))$, the following are equivalent:*

(1) *$Im(\pi)$ is a Hopf algebra, and $(tr \otimes \int_T)\pi$ is the Haar integration on it.*
(2) *$\psi = (tr \otimes \int_X)\pi$ satisfies the idempotent state property $\psi * \psi = \psi$.*
(3) *$T_e^2 = T_e$, $\forall p \in \mathbb{N}$, $\forall e \in \{1, *\}^p$, where $(T_e)_{i_1 \ldots i_p, j_1 \ldots j_p} = (tr \otimes \int_T)(U_{i_1 j_1}^{e_1} \ldots U_{i_p j_p}^{e_p})$.*

If these conditions are satisfied, we say that π is stationary on its image.

Proof. Given a matrix model $\pi : C(G) \to M_K(C(T))$ as in the statement, we can factorize it via its Hopf image, as in Definition 12.6 above:

$$\pi : C(G) \to C(H) \to M_K(C(T))$$

Now observe that the conditions (1,2,3) in the statement depend only on the factorized representation $\nu : C(H) \to M_K(C(T))$. Thus, we can assume in practice that we have $G = H$, which means that we can assume that π is inner faithful.

With this assumption made, the general integration formula from Theorem 12.8 applies to our situation, and the proof of the equivalences goes as follows:

(1) $\implies$ (2) This is clear from definitions, because the Haar integration on any compact quantum group satisfies the idempotent equation $\psi * \psi = \psi$.

(2) $\implies$ (1) Assuming $\psi * \psi = \psi$, we have $\psi^{*r} = \psi$ for any $r \in \mathbb{N}$, and Theorem 12.8 gives $\int_G = \psi$. By using now Theorem 12.3, we obtain the result.

In order to establish now (2) $\iff$ (3), we use the following elementary formula, which comes from the definition of the convolution operation:

$$\psi^{*r}(u_{i_1 j_1}^{e_1} \ldots u_{i_p j_p}^{e_p}) = (T_e^r)_{i_1 \ldots i_p, j_1 \ldots j_p}$$

(2) $\implies$ (3) Assuming $\psi * \psi = \psi$, by using the above formula at $r = 1, 2$ we obtain that the matrices T_e and T_e^2 have the same coefficients, and so they are equal.

(3) $\implies$ (2) Assuming $T_e^2 = T_e$, by using the above formula at $r = 1, 2$ we obtain that the linear forms ψ and $\psi * \psi$ coincide on any product of coefficients $u_{i_1 j_1}^{e_1} \ldots u_{i_p j_p}^{e_p}$. Now since these coefficients span a dense subalgebra of $C(G)$, this gives the result. $\square$

As a conclusion, we have now a purely computational criterion for verifying the stationarity property, which is potentially quite powerful. As a first illustration, we will apply this criterion to certain models for the quantum groups O_N^*, U_N^*. We first have:

Proposition 12.10. *We have a matrix model as follows,*

$$C(O_N^*)\to M_2(C(U_N))\quad,\quad u_{ij}\to\begin{pmatrix}0&v_{ij}\\ \bar{v}_{ij}&0\end{pmatrix}$$

where v is the fundamental corepresentation of $C(U_N)$, as well as a model as follows,

$$C(U_N^*)\to M_2(C(U_N\times U_N))\quad,\quad u_{ij}\to\begin{pmatrix}0&v_{ij}\\ w_{ij}&0\end{pmatrix}$$

where v,w are the fundamental corepresentations of the two copies of $C(U_N)$.

Proof. It is routine to check that the matrices on the right are indeed biunitaries, and since the first matrix is also self-adjoint, we obtain in this way models as follows:

$$C(O_N^+)\to M_2(C(U_N))\quad,\quad C(U_N^+)\to M_2(C(U_N\times U_N))$$

Regarding now the half-commutation relations, this comes from something general, regarding the antidiagonal 2×2 matrices. Consider indeed matrices as follows:

$$X_i=\begin{pmatrix}0&x_i\\ y_i&0\end{pmatrix}$$

We have then the following computation:

$$X_iX_jX_k=\begin{pmatrix}0&x_i\\ y_i&0\end{pmatrix}\begin{pmatrix}0&x_j\\ y_j&0\end{pmatrix}\begin{pmatrix}0&x_k\\ y_k&0\end{pmatrix}=\begin{pmatrix}0&x_iy_jx_k\\ y_ix_jy_k&0\end{pmatrix}$$

Since this quantity is symmetric in i,k, we obtain $X_iX_jX_k=X_kX_jX_i$. Thus, the antidiagonal 2×2 matrices half-commute, and so our models factorize as claimed. $\square$

We can now formulate our first concrete modelling theorem, as folllows:

Theorem 12.11. *The above antidiagonal models, namely*

$$C(O_N^*)\to M_2(C(U_N))\quad,\quad C(U_N^*)\to M_2(C(U_N\times U_N))$$

are both stationary.

Proof. Let us first discuss the case of O_N^*. We will use the stationarity criterion in Theorem 12.9 (3) above. Since the fundamental representation is self-adjoint, the various matrices T_e with $e\in\{1,*\}^p$ are all equal. We denote this common matrix by T_p.

According to the definition of T_p, this matrix is given by:

$$(T_p)_{i_1\ldots i_p,j_1\ldots j_p}=\left(tr\otimes\int_H\right)\left[\begin{pmatrix}0&v_{i_1j_1}\\ \bar{v}_{i_1j_1}&0\end{pmatrix}\ldots\ldots\begin{pmatrix}0&v_{i_pj_p}\\ \bar{v}_{i_pj_p}&0\end{pmatrix}\right]$$

Since when multipliying an odd number of antidiagonal matrices we obtain an atidiagonal matrix, we have $T_p=0$ for p odd. Also, when p is even, we have:

$$\begin{aligned}
(T_p)_{i_1\dots i_p,j_1\dots j_p} &= \left(tr\otimes\int_H\right)\begin{pmatrix}v_{i_1j_1}\dots\bar{v}_{i_pj_p}&0\\0&\bar{v}_{i_1j_1}\dots v_{i_pj_p}\end{pmatrix}\\
&= \frac{1}{2}\left(\int_H v_{i_1j_1}\dots\bar{v}_{i_pj_p}+\int_H\bar{v}_{i_1j_1}\dots v_{i_pj_p}\right)\\
&= \int_H Re(v_{i_1j_1}\dots\bar{v}_{i_pj_p})
\end{aligned}$$

We have $T_p^2=T_p=0$ when p is odd, so we are left with proving that we have $T_p^2=T_p$, when p is even. For this purpose, we use the following formula:

$$Re(x)Re(y)=\frac{1}{2}\left(Re(xy)+Re(x\bar{y})\right)$$

By using this identity for each of the terms which appear in the product, and multi-index notations in order to simplify the writing, we obtain:

$$\begin{aligned}
(T_p^2)_{ij} &= \sum_{k_1\dots k_p}(T_p)_{i_1\dots i_p,k_1\dots k_p}(T_p)_{k_1\dots k_p,j_1\dots j_p}\\
&= \int_H\int_H\sum_{k_1\dots k_p}Re(v_{i_1k_1}\dots\bar{v}_{i_pk_p})Re(w_{k_1j_1}\dots\bar{w}_{k_pj_p})dvdw\\
&= \frac{1}{2}\int_H\int_H\sum_{k_1\dots k_p}Re(v_{i_1k_1}w_{k_1j_1}\dots\bar{v}_{i_pk_p}\bar{w}_{k_pj_p})+Re(v_{i_1k_1}\bar{w}_{k_1j_1}\dots\bar{v}_{i_pk_p}w_{k_pj_p})dvdw\\
&= \frac{1}{2}\int_H\int_H Re((vw)_{i_1j_1}\dots(\bar{v}\bar{w})_{i_pj_p})+Re((v\bar{w})_{i_1j_1}\dots(\bar{v}w)_{i_pj_p})dvdw
\end{aligned}$$

Now since $vw\in H$ is uniformly distributed when $v,w\in H$ are uniformly distributed, the quantity on the left integrates up to $(T_p)_{ij}$. Also, since H is conjugation-stable, $\bar{w}\in H$ is uniformly distributed when $w\in H$ is uniformly distributed, so the quantity on the right integrates up to the same quantity, namely $(T_p)_{ij}$. Thus, we have:

$$(T_p^2)_{ij}=\frac{1}{2}\Big((T_p)_{ij}+(T_p)_{ij}\Big)=(T_p)_{ij}$$

Summarizing, we have obtained that for any p, the condition $T_p^2=T_p$ is satisfied. Thus Theorem 12.9 applies, and shows that our model is stationary, as claimed.

As for the proof of the stationarity for the model for U_N^*, this is similar. See [21]. □

Summarizing, our notion of stationarity, and the various tools that we developed here, have some non-trivial applications. We should mention the Theorem 12.11 has many extensions, to more general half-classical quantum groups, or manifolds, and involving higher versions of the relations $abc=cba$ as well. We refer here to [14], [21], [43], [44].

Following [31], let us discuss now some more subtle examples of stationary models, related to the Pauli matrices, and Weyl matrices, and physics. We first have:

Definition 12.12. *Given a finite abelian group H, the associated Weyl matrices are*

$$W_{ia}:e_b\to<i,b>e_{a+b}$$

where $i\in H$, $a,b\in\widehat{H}$, and where $(i,b)\to<i,b>$ is the Fourier coupling $H\times\widehat{H}\to\mathbb{T}$.

As a basic example, consider the cyclic group $H=\mathbb{Z}_2=\{0,1\}$. Here the Fourier coupling is given by $<i,b>=(-1)^{ib}$, and so the Weyl matrices act via $W_{00}:e_b\to e_b$, $W_{10}:e_b\to(-1)^b e_b$, $W_{11}:e_b\to(-1)^b e_{b+1}$, $W_{01}:e_b\to e_{b+1}$. Thus, we have:

$$W_{00}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\ ,\ W_{10}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\ ,\ W_{11}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\ ,\ W_{01}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

We recognize here, up to some multiplicative factors, the four Pauli matrices.

Now back to the general case, we have the following well-known result:

Proposition 12.13. *The Weyl matrices are unitaries, and satisfy:*

1. $W_{ia}^*=<i,a>W_{-i,-a}$.
2. $W_{ia}W_{jb}=<i,b>W_{i+j,a+b}$.
3. $W_{ia}W_{jb}^*=<j-i,b>W_{i-j,a-b}$.
4. $W_{ia}^*W_{jb}=<i,a-b>W_{j-i,b-a}$.

Proof. The unitary follows from (3,4), and the rest of the proof goes as follows:

(1) We have indeed the following computation:

$$\begin{aligned}W_{ia}^*&=\left(\sum_b<i,b>E_{a+b,b}\right)^*=\sum_b<-i,b>E_{b,a+b}\\&=\sum_b<-i,b-a>E_{b-a,b}=<i,a>W_{-i,-a}\end{aligned}$$

(2) Here the verification goes as follows:

$$\begin{aligned}W_{ia}W_{jb}&=\left(\sum_d<i,b+d>E_{a+b+d,b+d}\right)\left(\sum_d<j,d>E_{b+d,d}\right)\\&=\sum_d<i,b><i+j,d>E_{a+b+d,d}=<i,b>W_{i+j,a+b}\end{aligned}$$

(3,4) By combining the above two formulae, we obtain:

$$\begin{aligned}W_{ia}W_{jb}^*&=<j,b>W_{ia}W_{-j,-b}=<j,b><i,-b>W_{i-j,a-b}\\W_{ia}^*W_{jb}&=<i,a>W_{-i,-a}W_{jb}=<i,a><-i,b>W_{j-i,b-a}\end{aligned}$$

But this gives the formulae in the statement, and we are done. □

Observe that, with $n = |H|$, we can use an isomorphism $l^2(\widehat{H}) \simeq \mathbb{C}^n$ as to view each W_{ia} as a usual matrix, $W_{ia} \in M_n(\mathbb{C})$, and hence as a usual unitary, $W_{ia} \in U_n$.

Given a vector ξ, we denote by $Proj(\xi)$ the orthogonal projection onto $\mathbb{C}\xi$.

Now let $N = n^2$, and consider Wang's quantum permutation algebra $C(S_N^+)$, with standard generators denoted $w_{ia,jb}$, using double indices. We have:

Proposition 12.14. *Given a closed subgroup $E \subset U_n$, we have a representation*

$$\pi_H : C(S_N^+) \to M_N(C(E)) \quad : \quad w_{ia,jb} \to [U \to Proj(W_{ia}UW_{jb}^*)]$$

where $n = |H|, N = n^2$, and where W_{ia} are the Weyl matrices associated to H.

Proof. The Weyl matrices being given by $W_{ia} : e_b \to < i, b > e_{a+b}$, we have:

$$tr(W_{ia}) = \begin{cases} 1 & \text{if } (i,a) = (0,0) \\ 0 & \text{if } (i,a) \neq (0,0) \end{cases}$$

Together with the formulae in Proposition 12.13, this shows that the Weyl matrices are pairwise orthogonal with respect to the scalar product $< x, y >= tr(x^*y)$ on $M_n(\mathbb{C})$. Thus, these matrices form an orthogonal basis of $M_n(\mathbb{C})$, consisting of unitaries:

$$W = \left\{ W_{ia} \Big| i \in H, a \in \widehat{H} \right\}$$

Thus, each row and each column of the matrix $\xi_{ia,jb} = W_{ia}UW_{jb}^*$ is an orthogonal basis of $M_n(\mathbb{C})$, and so the corresponding projections form a magic unitary, as claimed. $\square$

We will need the following well-known result:

Proposition 12.15. *With $T = Proj(x_1) \ldots Proj(x_p)$ and $||x_i|| = 1$ we have*

$$< \xi, T\eta >=< \xi, x_1 >< x_1, x_2 > \ldots < x_{p-1}, x_p >< x_p, \eta >$$

for any ξ, η. In particular, $Tr(T) =< x_1, x_2 >< x_2, x_3 > \ldots < x_p, x_1 >$.

Proof. For $||x|| = 1$ we have $Proj(x)\eta = x < x, \eta >$, and this gives:

$$\begin{aligned}
T\eta &= Proj(x_1) \ldots Proj(x_p)\eta \\
&= Proj(x_1) \ldots Proj(x_{p-1})x_p < x_p, \eta > \\
&= Proj(x_1) \ldots Proj(x_{p-2})x_{p-2} < x_{p-1}, x_p >< x_p, \eta > \\
&= \ldots \\
&= x_1 < x_1, x_2 > \ldots < x_{p-1}, x_p >< x_p, \eta >
\end{aligned}$$

Now by taking the scalar product with ξ, this gives the first assertion. As for the second assertion, this follows from the first assertion, by summing over $\xi = \eta = e_i$. $\square$

Now back to the Weyl matrix models, let us first compute T_p. We have:

Proposition 12.16. *We have the formula*

$$\begin{aligned}(T_p)_{ia,jb} &= \frac{1}{N} < i_1, a_1 - a_2 > \ldots < i_p, a_p - a_1 >< j_2, b_2 - b_1 > \ldots < j_1, b_1 - b_p > \\ &\int_E tr(W_{i_2-i_1,a_2-a_1} U W_{j_1-j_2,b_1-b_2} U^*) \ldots tr(W_{i_1-i_p,a_1-a_p} U W_{j_p-j_1,b_p-b_1} U^*) dU\end{aligned}$$

with all the indices varying in a cyclic way.

Proof. By using the trace formula in Proposition 12.15 above, we obtain:

$$\begin{aligned}(T_p)_{ia,jb} &= \left(tr \otimes \int_E\right) \left(Proj(W_{i_1a_1} U W^*_{j_1b_1}) \ldots Proj(W_{i_pa_p} U W^*_{j_pb_p})\right) \\ &= \frac{1}{N} \int_E < W_{i_1a_1} U W^*_{j_1b_1}, W_{i_2a_2} U W^*_{j_2b_2} > \ldots < W_{i_pa_p} U W^*_{j_pb_p}, W_{i_1a_1} U W^*_{j_1b_1} > dU\end{aligned}$$

In order to compute now the scalar products, observe that we have:

$$\begin{aligned}< W_{ia} U W^*_{jb}, W_{kc} U W^*_{ld} > &= tr(W_{jb} U^* W^*_{ia} W_{kc} U W^*_{ld}) \\ &= tr(W^*_{ia} W_{kc} U W^*_{ld} W_{jb} U^*) \\ &= < i, a - c >< l, d - b > tr(W_{k-i,c-a} U W_{j-l,b-d} U^*)\end{aligned}$$

By plugging these quantities into the formula of T_p, we obtain the result. □

Consider now the Weyl group $W = \{W_{ia}\} \subset U_n$, that we already met in the proof of Proposition 12.14 above. We have the following result, from [31]:

Theorem 12.17. *For any compact group $W \subset E \subset U_n$, the model*

$$\pi_H : C(S_N^+) \to M_N(C(E)) \quad : \quad w_{ia,jb} \to [U \to Proj(W_{ia} U W^*_{jb})]$$

constructed above is stationary on its image.

Proof. We must prove that we have $T_p^2 = T_p$. We have:

$$\begin{aligned}(T_p^2)_{ia,jb} &= \sum_{kc} (T_p)_{ia,kc} (T_p)_{kc,jb} \\ &= \frac{1}{N^2} \sum_{kc} < i_1, a_1 - a_2 > \ldots < i_p, a_p - a_1 >< k_2, c_2 - c_1 > \ldots < k_1, c_1 - c_p > \\ &< k_1, c_1 - c_2 > \ldots < k_p, c_p - c_1 >< j_2, b_2 - b_1 > \ldots < j_1, b_1 - b_p > \\ &\int_E tr(W_{i_2-i_1,a_2-a_1} U W_{k_1-k_2,c_1-c_2} U^*) \ldots tr(W_{i_1-i_p,a_1-a_p} U W_{k_p-k_1,c_p-c_1} U^*) dU \\ &\int_E tr(W_{k_2-k_1,c_2-c_1} V W_{j_1-j_2,b_1-b_2} V^*) \ldots tr(W_{k_1-k_p,c_1-c_p} V W_{j_p-j_1,b_p-b_1} V^*) dV\end{aligned}$$

By rearranging the terms, this formula becomes:

$$\begin{aligned}(T_p^2)_{ia,jb} &= \frac{1}{N^2} < i_1, a_1 - a_2 > \ldots < i_p, a_p - a_1 >< j_2, b_2 - b_1 > \ldots < j_1, b_1 - b_p > \\ & \int_E \int_E \sum_{kc} < k_1 - k_2, c_1 - c_2 > \ldots < k_p - k_1, c_p - c_1 > \\ & tr(W_{i_2-i_1,a_2-a_1} U W_{k_1-k_2,c_1-c_2} U^*) tr(W_{k_2-k_1,c_2-c_1} V W_{j_1-j_2,b_1-b_2} V^*) \\ & \ldots\ldots \\ & tr(W_{i_1-i_p,a_1-a_p} U W_{k_p-k_1,c_p-c_1} U^*) tr(W_{k_1-k_p,c_1-c_p} V W_{j_p-j_1,b_p-b_1} V^*) dU dV\end{aligned}$$

Let us denote by I the above double integral. By using $W_{kc}^* =< k, c > W_{-k,-c}$ for each of the couplings, and by moving as well all the U^* variables to the left, we obtain:

$$\begin{aligned}I &= \int_E \int_E \sum_{kc} tr(U^* W_{i_2-i_1,a_2-a_1} U W_{k_1-k_2,c_1-c_2}) tr(W^*_{k_1-k_2,c_1-c_2} V W_{j_1-j_2,b_1-b_2} V^*) \\ & \ldots\ldots \\ & tr(U^* W_{i_1-i_p,a_1-a_p} U W_{k_p-k_1,c_p-c_1}) tr(W^*_{k_p-k_1,c_p-c_1} V W_{j_p-j_1,b_p-b_1} V^*) dU dV\end{aligned}$$

In order to perform now the sums, we use the following formula:

$$\begin{aligned}tr(AW_{kc}) tr(W_{kc}^* B) &= \frac{1}{N} \sum_{qrst} A_{qr} (W_{kc})_{rq} (W_{kc}^*)_{st} B_{ts} \\ &= \frac{1}{N} \sum_{qrst} A_{qr} < k, q > \delta_{r-q,c} < k, -s > \delta_{t-s,c} B_{ts} \\ &= \frac{1}{N} \sum_{qs} < k, q - s > A_{q,q+c} B_{s+c,s}\end{aligned}$$

If we denote by A_x, B_x the variables which appear in the formula of I, we have:

$$\begin{aligned}I &= \frac{1}{N^p} \int_E \int_E \sum_{kcqs} < k_1 - k_2, q_1 - s_1 > \ldots < k_p - k_1, q_p - s_p > \\ & (A_1)_{q_1,q_1+c_1-c_2} (B_1)_{s_1+c_1-c_2,s_1} \ldots (A_p)_{q_p,q_p+c_p-c_1} (B_p)_{s_p+c_p-c_1,s_p} \\ &= \frac{1}{N^p} \int_E \int_E \sum_{kcqs} < k_1, q_1 - s_1 - q_p + s_p > \ldots < k_p, q_p - s_p - q_{p-1} + s_{p-1} > \\ & (A_1)_{q_1,q_1+c_1-c_2} (B_1)_{s_1+c_1-c_2,s_1} \ldots (A_p)_{q_p,q_p+c_p-c_1} (B_p)_{s_p+c_p-c_1,s_p}\end{aligned}$$

Now observe that we can perform the sums over $k_1, \ldots, k_p$. We obtain in this way a multiplicative factor n^p, along with the condition $q_1 - s_1 = \ldots = q_p - s_p$. Thus we must have $q_x = s_x + a$ for a certain a, and the above formula becomes:

$$I = \frac{1}{n^p} \int_E \int_E \sum_{csa} (A_1)_{s_1+a,s_1+c_1-c_2+a} (B_1)_{s_1+c_1-c_2,s_1} \ldots (A_p)_{s_p+a,s_p+c_p-c_1+a} (B_p)_{s_p+c_p-c_1,s_p}$$

Consider now the variables $r_x = c_x - c_{x+1}$, which altogether range over the set Z of multi-indices having sum 0. By replacing the sum over c_x with the sum over r_x, which creates a multiplicative n factor, we obtain the following formula:

$$I = \frac{1}{n^{p-1}} \int_E \int_E \sum_{r \in Z} \sum_{sa} (A_1)_{s_1+a, s_1+r_1+a} (B_1)_{s_1+r_1, s_1} \ldots (A_p)_{s_p+a, s_p+r_p+a} (B_p)_{s_p+r_p, s_p}$$

Since for an arbitrary multi-index r we have $\delta_{\sum_i r_i, 0} = \frac{1}{n} \sum_i < i, r_1 > \ldots < i, r_p >$, we can replace the sum over $r \in Z$ by a full sum, as follows:

$$\begin{aligned} I &= \frac{1}{n^p} \int_E \int_E \sum_{rsia} < i, r_1 > (A_1)_{s_1+a, s_1+r_1+a} (B_1)_{s_1+r_1, s_1} \\ & \qquad \ldots\ldots \\ & \qquad < i, r_p > (A_p)_{s_p+a, s_p+r_p+a} (B_p)_{s_p+r_p, s_p} \end{aligned}$$

In order to "absorb" now the indices i, a, we can use the following formula:

$$\begin{aligned} W_{ia}^* A W_{ia} &= \left(\sum_b < i, -b > E_{b, a+b} \right) \left(\sum_{bc} E_{a+b, a+c} A_{a+b, a+c} \right) \left(\sum_c < i, c > E_{a+c, c} \right) \\ &= \sum_{bc} < i, c-b > E_{bc} A_{a+b, a+c} \end{aligned}$$

Thus we have $(W_{ia}^* A W_{ia})_{bc} = < i, c-b > A_{a+b, a+c}$, and our formula becomes:

$$\begin{aligned} I &= \frac{1}{n^p} \int_E \int_E \sum_{rsia} (W_{ia}^* A_1 W_{ia})_{s_1, s_1+r_1} (B_1)_{s_1+r_1, s_1} \ldots (W_{ia}^* A_p W_{ia})_{s_p, s_p+r_p} (B_p)_{s_p+r_p, s_p} \\ &= \int_E \int_E \sum_{ia} tr(W_{ia}^* A_1 W_{ia} B_1) \ldots\ldots tr(W_{ia}^* A_p W_{ia} B_p) \end{aligned}$$

Now by replacing A_x, B_x with their respective values, we obtain:

$$\begin{aligned} I &= \int_E \int_E \sum_{ia} tr(W_{ia}^* U^* W_{i_2-i_1, a_2-a_1} U W_{ia} V W_{j_1-j_2, b_1-b_2} V^*) \\ & \qquad \ldots\ldots \\ & \qquad tr(W_{ia}^* U^* W_{i_1-i_p, a_1-a_p} U W_{ia} V W_{j_p-j_1, b_p-b_1} V^*) dU dV \end{aligned}$$

By moving the $W_{ia}^* U^*$ variables at right, we obtain, with $S_{ia} = U W_{ia} V$:

$$\begin{aligned} I &= \sum_{ia} \int_E \int_E tr(W_{i_2-i_1, a_2-a_1} S_{ia} W_{j_1-j_2, b_1-b_2} S_{ia}^*) \\ & \qquad \ldots\ldots \\ & \qquad tr(W_{i_1-i_p, a_1-a_p} S_{ia} W_{j_p-j_1, b_p-b_1} S_{ia}^*) dU dV \end{aligned}$$

Now since S_{ia} is Haar distributed when U, V are Haar distributed, we obtain:

$$I = N \int_E \int_E tr(W_{i_2-i_1, a_2-a_1} U W_{j_1-j_2, b_1-b_2} U^*) \ldots tr(W_{i_1-i_p, a_1-a_p} U W_{j_p-j_1, b_p-b_1} U^*) dU$$

But this is exactly N times the integral in the formula of $(T_p)_{ia,jb}$, from Proposition 12.16 above. Since the N factor cancels with one of the two N factors that we found in the beginning of the proof, when first computing $(T_p^2)_{ia,jb}$, we are done. □

As an illustration for the above result, going back to [22], we have:

Theorem 12.18. *We have a stationary matrix model*

$$\pi : C(S_4^+) \subset M_4(C(SU_2))$$

given on the standard coordinates by the formula

$$\pi(u_{ij}) = [x \to Proj(c_i x c_j)]$$

where $x \in SU_2$, and c_1, c_2, c_3, c_4 are the Pauli matrices.

Proof. As already explained in the comments following Definition 12.12, the Pauli matrices appear as particular cases of the Weyl matrices. By working out the details, we conclude that Theorem 12.17 produces in this case the model in the statement. □

Observe that, since the matrix $Proj(c_i x c_j)$ depends only on the image of x in the quotient $SU_2 \to SO_3$, we can replace the model space SU_2 by the smaller space SO_3, if we want to. This is something that can be used in conjunction with the isomorphism $S_4^+ \simeq SO_3^{-1}$ from section 7 above, and as explained in [11], our model becomes in this way something quite conceptual, algebrically speaking, as follows:

$$\pi : C(SO_3^{-1}) \subset M_4(C(SO_3))$$

As a somewhat philosophical conclusion, to this and to some previous findings as well, no matter what we do, we always end up getting back to SU_2, SO_3. Thus, we are probably doing some physics here. This is indeed the case, the above computations being closely related to the standard computations for the Ising and Potts models. The general relation, however, between quantum permutations and lattice models, is not axiomatixed yet.

Getting back now to mathematical questions, we have seen so far only examples of stationary models. Going beyond stationarity is a difficult task, and among the results here, let us mention the universal modelling questions for quantum permutations and quantum reflections [31], [45], various results on the flat models for the discrete groups [19], [28], questions regarding the Hadamard matrix models [12], [18], and the related fine analytic study on the compact and discrete quantum groups [46], [60], [84], [85].

In what follows we will only discuss the Hadamard models, which are of particular importance. Let us start with the following well-known definition:

Definition 12.19. *A complex Hadamard matrix is a square matrix*

$$H \in M_N(\mathbb{C})$$

whose entries are on the unit circle, and whose rows are pairwise orthogonal.

Observe that the orthogonality condition tells us that the rescaled matrix $U = H/\sqrt{N}$ must be unitary. Thus, these matrices form a real algebraic manifold, given by:

$$X_N = M_N(\mathbb{T}) \cap \sqrt{N} U_N$$

The basic example is the Fourier matrix, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$. More generally, we have as example the Fourier coupling of any finite abelian group G, regarded via the isomorphism $G \simeq \widehat{G}$ as a square matrix, $F_G \in M_G(\mathbb{C})$:

$$F_G = < i, j >_{i \in G, j \in \widehat{G}}$$

Observe that for the cyclic group $G = \mathbb{Z}_N$ we obtain in this way the above standard Fourier matrix F_N. In general, we obtain a tensor product of Fourier matrices F_N.

There are many other examples of Hadamard matrices, some being elementary, some other fairly exotic, appearing in various branches of mathematics and physics. The idea is that the complex Hadamard matrices can be though of as being "generalized Fourier matrices", and this is where the interest in these matrices comes from. See [81].

In relation with the quantum groups, the starting observation is as follows:

Proposition 12.20. *If $H \in M_N(\mathbb{C})$ is Hadamard, the rank one projections*

$$P_{ij} = Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1, \ldots, H_N \in \mathbb{T}^N$ are the rows of H, form a magic unitary.

Proof. This is clear, the verification for the rows being as follows:

$$\left\langle \frac{H_i}{H_j}, \frac{H_i}{H_k} \right\rangle = \sum_l \frac{H_{il}}{H_{jl}} \cdot \frac{H_{kl}}{H_{il}} = \sum_l \frac{H_{kl}}{H_{jl}} = N\delta_{jk}$$

The verification for the columns is similar. □

We can proceed now in the same way as we did with the Weyl matrices, namely by constructing a model of $C(S_N^+)$, and performing the Hopf image construction:

Definition 12.21. *To any Hadamard matrix $H \in M_N(\mathbb{C})$ we associate the quantum permutation group $G \subset S_N^+$ given by the fact that $C(G)$ is the Hopf image of*

$$\pi : C(S_N^+) \to M_N(\mathbb{C}) \quad , \quad u_{ij} \to Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1, \ldots, H_N \in \mathbb{T}^N$ are the rows of H.

Summarizing, we have a construction $H \to G$, and our claim is that this construction is something really useful, with G encoding the combinatorics of H. To be more precise, our claim is that "H can be thought of as being a kind of Fourier matrix for G".

This is of course quite interesting, philosophically speaking. There are several results supporting this, with the main evidence coming from the following result, which collects the basic known results regarding the construction:

Theorem 12.22. *The construction $H \to G$ has the following properties:*

(1) *For a Fourier matrix $H = F_G$ we obtain the group G itself, acting on itself.*
(2) *For $H \notin \{F_G\}$, the quantum group G is not classical, nor a group dual.*
(3) *For a tensor product $H = H' \otimes H''$ we obtain a product, $G = G' \times G''$.*

Proof. All this material is standard, and elementary, as follows:

(1) In the cyclic group case, $H = F_N$, all the objects involved in the construction $H \to G$ have an obvious cyclic structure, and we obtain from this $G = \mathbb{Z}_N$. We can pass then to the general case by using (3), whose proof is independent of this.

(2) This is something more tricky, needing some general study of the representations whose Hopf images are commutative, or cocommutative. For details here, along with a number of supplementary facts on the construction $H \to G$, we refer to [18].

(3) This is elementary, the idea being that the tensor products of matrix models are matrix models, and that at the level of corresponding Hopf images, under suitable assumptions, the compatibility holds as well. Once again, we refer here to [18]. □

Going beyond the above result is an interesting question, and we refer here to [13], and to follow-up papers. There are several computations available here, for the most regarding the deformations of the Fourier models. We believe that the unification of all this with the Weyl matrix models is a very good question, related to many interesting things.

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