The Secret of Ishango

On the helix structure of prime numbers

A mathematical essay

9 | 9 | 19 Christof Born

Abstract

The bones of Ishango were found in the 1950s by Belgian archaeologist Jean de Heinzelin near a Palaeolithic residence in Ishango, Africa. Inscriptions, which can be interpreted as numbers, make these bones the oldest mathematical find in human history. There are various scientific papers on the interpretation of the inscriptions. Interestingly, on one of the two bones, we also find the six consecutive prime numbers 5, 7, 11, 13, 17 and 19. Did Stone Age people already know the secret of the prime numbers? This question is explored in my mathematical essay "The Secret of Ishango": an adventurous journey around the world – from Basel in Switzerland to Erode in India. The presumed connection between the numbers on the bones of Ishango and the structure of the prime numbers is illustrated by a sketch at the end of the text.

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"We especially need imagination in science. It is not all mathematics, nor all logic, but is somewhat beauty and poetry."

Maria Mitchell, Astronomist (1818-1889)

Mesopotamia, 5500 years ago. Uruk was the largest urban centre in the Tigris-Euphrates river system and the centre of establishment of Sumerian culture. Counting was done using clay counters. Every object was assigned a specific kind of counter, for example, a sheep was represented by a plain disc with a cross on it. To count the number of sheep in a herd, the corresponding amount of counters were kept in a container made of clay. Locked containers prevented shepherds and sheep-owners to secretly change the amount of counters in each container. To know the contents of each sealed container, the pattern representing the amount of counters inside the container would also be scored on to its surface. Thus, it seemed obvious to abstain from writing on the containers themselves, and instead to simply score the patterns of the counters on a flat piece of clay, signalling the start of writing itself. Soon the idea arose that, instead of writing eight symbols for eight sheep, one could denote separate symbols for the number eight and for the sheep. This number was then independent of the objects which it described. This was the birth of abstract mathematics, whose history is told in the wonderful book "It All Adds Up: The Story of People and Mathematics" by French mathematician Mickael Launay.

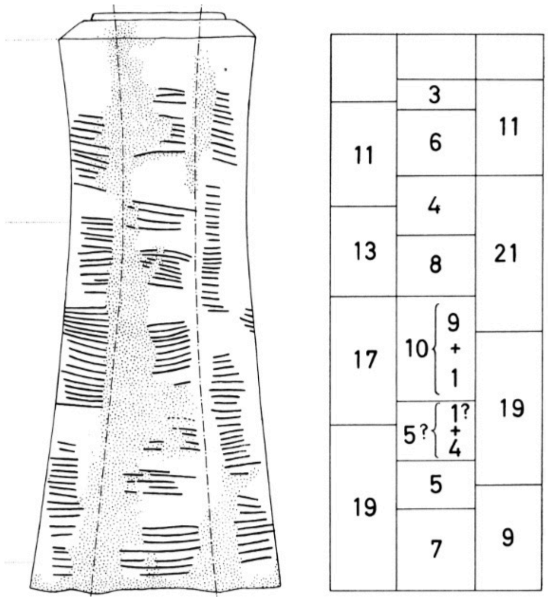
1. The Bones of Ishango

Let us flash back to the Old Stone Age. The village of Ishango, situated on the bank of Lake Eduard in the Democratic Republic of Congo, was buried in ash after a nearby volcanic eruption. It is here in the 1950s that the Belgian archaeologist Jean de Heinzelin de Braucourt lead the Ishango excavations. During the excavation of this prehistoric settlement, Heinzelin found two bones of 10 cm and 14 cm with groups of inscriptions, which themselves can be interpreted as numbers. These bones are considered to be the oldest mathematical find in human history, and are still on display today in the Royal Belgian Institute of Natural Sciences, Brussels.

The two bones, which have been estimated with radiocarbon dating to be around 22,000 years old, are mysterious in several ways. The shorter and more well known of the two (see 3D model) is affixed with a sharp piece of quartz to one end, thus symbolising an engraving tool in a time and culture where there was no such concept of a written language. There are several interpretations of the numbers on this bone. In his work "The Roots of Civilisation", Alexander Marshack interprets them as a lunar calendar. This assumption is further supported by the mathematician Claudia Zaslavsky, who cites the female menstrual cycle as the reason for measuring time in the rhythm of the moon phases. Heinzelin himself viewed the inscriptions as a kind of elementary arithmetic game and calculation. Additionally, the mathematician Dirk Huylebrouck as well as the physicist Vladimir Pletser both support the hypothesis that the bone was used as a slide rule. "The inscriptions on the bones of Ishango incontestably possess a logical pattern. Perhaps the first signs of abstract mathematical thinking really did take place in Ishango" (Huylebrouck, in his report in "Ethnomathematik 2/2006" from "Spektrum der Wissenschaft").

We will now consider the shorter of the two Ishango bones (Fig. 1): Towards the bottom of the central column, we observe the numbers 5 and 7, in the left column the numbers 11, 13, 17 and 19, i.e. six prime numbers in a sequence. On the other hand, these numbers can also be interpreted as 6, 12 and 18 plus/minus 1 and thus as a system of numbers in base 6. Huylebrouck claims that "there isn't a sign that the unknown engraver of the bones of Ishango managed to spark any interest for prime numbers".

Fig. 1. and cover picture. The shorter of the two bones with the prime numbers from 5 to 19 in the central and left column. The cover picture shows the bones from four different sides. Photo from the Royal Belgian Institute of Natural Sciences



A recommended read is the featured article “Ethno-mathematik 2/2006” from “Spektrum der Wissenschaft” with fascinating articles about the findings and discoveries in Africa, China and India as well as the North-American Indians, the Mayans and the Incas, and additionally about the Arabs’ and the Islams’ mathematical poetry and magic squares.

2. Prime Numbers – The Atoms of Mathematics

Nowadays, it is known that the Chinese and the Greeks were involved in prime numbers from as early as 1000 B.C., and that Euclid had proven that there are infinitely many of them in 300 B.C. During 2000 years of radio silence, there were no profound discoveries in prime number theory, until the 15-year-old Carl Friedrich Gauss discovered in 1792, Germany, that the amount of prime numbers could be expressed by a logarithmic function. Whilst studying prime number tables, he noticed that the further you go along the number line, the fewer prime numbers there were, and that the number of prime numbers up to a certain x, for example, between 0 and 1000, could be modelled very precisely with the function x/ln(x). The larger we choose x, the better this approximation becomes. This formula, known as the Prime Number Theorem (PNT), is considered to be one of the most important theorems in the history of mathematics and was proven independently a century afterwards by Hadamard and Valleé Poussin.

Another breakthrough was made by Bernhard Riemann, a student of Gauss’, who hypothesised that the roots of the Riemann Zeta Function:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \dots$$

all lie on a straight line in the complex plane and recognised that these roots have a direct connection to the prime numbers. The proof of this hypothesis was placed on German mathematician David Hilbert’s list of 23 mathematical problems during his memorable speech at the International Mathematician’s Congress in 1900 in Paris. The Riemann Hypothesis is still considered nowadays as one of the most important unsolved problems in mathematics, and also belongs to the seven problems whose solutions are to be awarded $1 million by the Boston businessman Landon T. Clay. One of these problems, the Poincaré conjecture, has already been solved by the outstanding Russian mathematician Girgori Perelman in 2002, however, he refused the prize money. The Poincaré conjecture is related to the Poincaré-Dodecahedron, which we will get familiar with later.

Fig. 2. “The mathematicians have tried up to this day to find some sort of order in the sequence of prime numbers, all to no avail, and one tends to believe that this is a secret which the human spirit can never penetrate.” Leonhard Euler on the dated Swiss 10-Franc banknote. Photo from Wikipedia

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An important contribution to the prime numbers was provided by the Basler Leonhard Euler, one of the most famous and productive mathematicians of all time. With $e^{i\pi} + 1 = 0$, the Swiss found the most beautiful expression of mathematics which connects five fundamental constants. Euler proved that the series of the reciprocals of all prime numbers diverges to infinity, and he also proved that the Zeta function could be written as the Euler product over all prime numbers, which we will see in the following chapter. Nonetheless, Euler himself was pessimistic regarding the ability of mankind to recognise the ordering of the prime numbers (see caption to Fig. 2). This of course does not mean that this ordering does not exist.The human imagination tires before nature, or as the French mathematician Blaise Pascal so profoundly formulated:

“L’imagination se lassera plutôt de concevoir que la nature de fournir.”

Prime numbers are the atoms of mathematics. According to the Fundamental Law of Arithmetic, which also goes back to Euclid, every natural number can be expressed as a product of prime numbers. Can and do we want to believe that these fundamental numbers are simply randomly distributed? A further example from the world of physics: is there a formula which determines the numbers 656, 486, 434, 410 and 397, the wavelengths of the hydrogen atom? The Swiss writing and maths teacher Johann Jakob Balmer was genuinely convinced that there was a mathematical pattern hidden in these seemingly random numbers. On 25th June 1884, almost 60 years old, he presented the simple and equally beautiful formula:

$$\lambda = A \times m^2 / (m^2 - 2^2)$$

to calculate these numbers and provided the foundation for Niels Bohr's model of the atom. Balmer succeeded thanks to "his determined belief in the principle simplicity of the laws of nature, which Einstein metaphorically denoted *the insight into God's map*," as Rudolf Taschner wrote in his book "Der Zahlen gigantische Schatten". On the search for the structure of prime numbers, Balmer and the Chinese mathematician Yitang Zhang were always lighthouses for me in a vast sea of doubt.

Even if the prime numbers seem completely arbitrarily distributed, there are several hints which imply that the distribution is in fact not random. For example, the works of Soundarajan and Lemke Oliver from Stanford University show that the distribution of the last digits of prime numbers is not random. The Ulam-Spirals too have "brought a hint of fantasy in speculation of the secret connection of order and chaos to the distribution of prime numbers", as formulated by the legendary scientific journalist Martin Gardner. We also know from fractals that, despite their apparent complexity and haphazardness, they are often generated from very simple rules where iterations are executed consecutively (more in "The Fractal Geometry of Nature" by Benoît B. Mandelbrot). Let us follow Spinoza's motto for now: "Nothing in nature is by chance. Something appears to be chance only because of our lack of knowledge."

The comprehensive and excitingly written story of the primes can be found in "The Music of the Primes. Searching to Solve the Greatest Mystery in Mathematics" by Marcus du Sautoy.

3. The man who knew infinity

The pursuit for order is deeply rooted in human evolution. Douglas Hofstadter, who we will meet later, once wrote that "Intelligence loves order and shies away from chaos". Are there perhaps such patterns on the Ishango bones as well? Can the inscriptions of the sequence 5, 7, 11, 13, 17 and 19 indeed be the prime numbers, and do the rest of the inscriptions tell us more about the structure of the primes? What is intriguing is the column with the etchings 11, 21, 19 and 9 on the same bone as the six prime numbers. These can also be interpreted as the beginning of a sequence, or as a representation of squares:

11, 21, 31, 41... | 9, 19, 29, 39...

$11^2 - 21 = 100 = 19 + 9^2$

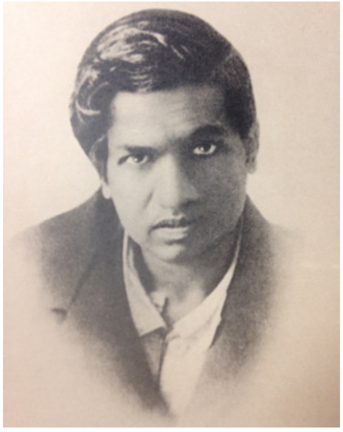


Fig. 3. Ramanujan, the artist under the mathematicians, found an extraordinary formula for π, wonderfully pictured by Florian Freistetter in his column "Freistetters Formelwelt" in "Spektrum der Wissenschaft". Photo by G.H. Hardy

As a matter of fact, both interpretations will prove to be very useful during our investigation. Perhaps there is a relation between the prime numbers and the numbers 9 and 11 respectively 9^2 and 11^2? To investigate this further, we will head to Erode, India, a city south-west of Madras. It was here in 1887 on the 9th day of the 9th Indian month, Margarisha, that the great and extraordinary mathematician Srinivasa Ramanujan was born. Ramanujan was a master of infinite series and continued fractions. "It was reported that he imagined numbers as continued fractions and that this partially explains his astonishing results," Jean-Paul Delahaye writes in "π – die Story". Ramanujan also dealt with prime numbers intensively and he wanted, like many other mathematicians, to find the magic formula of the prime numbers. Ramanujan devised a large number of fantastic formulae in his life, including one which calculates π very efficiently (every further iteration gives exactly eight further decimal places). The main term of this mysterious formula is $9801 / 2\sqrt{2}$ respectively $(9^2 \text{x} 11^2) / \sqrt{8}$ i.e. we notice a connection between π and the squares 9^2 and 11^2.

Since the 18th century, we have also known that there is also a connection between π and the aforementioned Zeta function, for π comes up in many of its function values like $\zeta(2) = \pi^2 / 6$. This beautiful result was found by Euler in 1735, thirty years after the Basler brothers Jakob and Johann Bernouilli intensively albeit unsuccessfully attempted to find a solution themselves (hence why the problem is also known as the "Basler problem"). The Bernoulli family had produced many fine mathematicians and occupied the chair of mathematics at the University of Basel over 105 years. "If we compared the Bernoullis to the Bach family, then Leonhard Euler is unquestionably the Mozart of mathematics" Eli Maor writes in "e: The Story of a Number". The preeminent personalities in Swiss mathematics also include the unfortunately almost forgotten watch-maker, astronomist and mathematician Jost Bürgi, one of the discoverers of the logarithm.

We can summarise that both products 9^2 and 11^2 are related to π by Ramanujan's formula; π is also found by the function values of the Zeta function (Euler), and the Zeta function leads us to the prime numbers (over its roots or the Euler product). Creatively, we can write this as

$$9^2 \mid 11^2 = \pi = \text{Zeta} = \text{Prim}$$

Thanks to Ramanujan's and Euler's formulas, we can build a bridge from the products of 9^2 and 11^2 to the prime numbers and assume a mathematical con-nection, as presented on the shorter of the Ishango bones. For me, this uncertain relation was the start-ing point and the motivation of three years' work with the bones of Ishango.

For the time being, let us forget all of our knowl-edge of the history of numbers. Let us assume that the bones of Ishango are actually the timeless holy grail of mathematics and we start on our quest for the prime number formula. A quest in the style of Marcus de Sautoy: "Maybe we have become so hung up on looking at the primes from Gauss's and Riemann's perspective that what we are missing is simply a different way to understand these enigmat-ic numbers", and of course also a quest without fearing error. "It is in fact that one only comes to the truth through failure" once said the great Alberto Giacometti. And we also listen to the astrophysicist Avi Loeb: "Deciding what's likely ahead of time limits the possibilities."

We will end this chapter with another literature recommendation. How Ramanujan's great mind worked and how he found his inspiring and adven-turous formulas remains a mystery. He once claimed that his family godmother Namagiri inspired him in his dreams and tirelessly turned to metaphysical speculations. Robert Kanigel wrote about this cryp-tic prodigy in his book "The Man Who Knew Infini-ty: A Life of the Genius Ramanujan". Also worth watching is the documentary "Letters from an Indi-an Clerk" as well as Matthew Brown's film "The Man Who Knew Infinity".

4. The double Lottie

The majority of reports and interpretations about the discovery at Ishango are based on the shorter of the two bones with the prime numbers. The second bone (see 3D model) was unknown by many authors, for Heinzelin only manifested this find forty years after the excavations, shortly before his death. The two bones, also known as Ishango Bone I and Ishango Bone II, however, can only be understood together. The shorter bone with the prime numbers shows us the ingredients of the "recipe" and shows us what we can cook with them, that is to say the prime numbers. On the second, more complicated bone, we see how we can use these said ingredients.

Let us first consider the shorter bone (Fig. 1). In the central column, we can see the numbers 3 and 6 as well as 4 and 8 on the top, which, when summed, can also be read as 9 and 12. The two following inscriptions can be interpreted as 10 and 5. However with the 10, we effectively notice 9 strokes and a much shorter one, which can be interpreted as 9 x 10 = 90, and analogously the following number is just 4 strokes. The fifth stroke adorned with a question mark was designed to maintain the principle of doubling the first four numbers. By multiplying these numbers, we get:

36	=	3 x 12 or 4 x 9
72	=	6 x 12 or 8 x 9
108	=	9 x 12
360	=	4 x 90

The resulting numbers have a significant meaning as the angles contained in the two golden triangles, and we will shortly see that we can calculate the prime numbers 5 and 7 on the bottom of the central col-umn with them. I will denominate this part of the bone as the "Section d'Or" in reference to the cub-ism art community of the same name, who them-selves studied the golden ratio and the fourth dimen-sion intensively. We will now switch to the longer of the two bones (Fig. 4) and observe the first of the three columns with 20 inscriptions (6 of which are short), 6 inscriptions and 18 inscriptions. We will multiply these together, whereby we will use num-bers twice that have two strokes which are closer to each other. This results once again in the numbers of the "Section d'Or":

360	=	18 x 20
108	=	6 x 18
36	=	6 x 6

On this bone, we can additionally see how we can use these numbers. Heinzelin assumed that the shorter strokes could have meant either division or subtraction. Furthermore, the numbers in the lower part of the bone belong to the denominator. There-fore, we obtain the two handsome formulas for the two prime numbers:

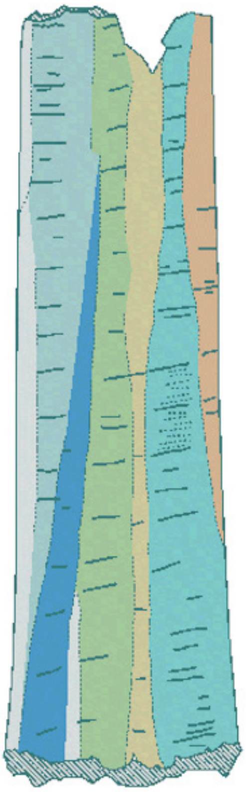


Fig. 4. The six columns arranged around the long Ishango Bone, drawn next to each other (In the first column, one short stroke was forgotten in the drawing). Photo from the Royal Belgian Institute of Natural Sciences

5 = 360 / (108 – 36)
7 = (360 – 108) / 36

Cancelling the numerator and denominator by 9:

5 = 40 / (12 – 4)
7 = (40 – 12) / 4

What is not so trivial to recognise is the correlation between the numbers 12 and 4 as well as the two primes: 12 = 5 + 7 and 4 = 5 – 1. Substituting and rearranging gives us

40 = 5 x (7 + 1)

This shows us that the number 40 defines the two primes 5 and 7. We will define these pairs of prime numbers as "The double Lottie" after the novel by Erich Kästner, in which two separated twins find each other again. If we know the number 40, but not the two primes, we would only have the equation 40 = x (y + 1) with two unknowns. The sought-after primes 5 and 7 which solve this equation cannot be determined analytically, yet they are relatively simple to find, either by factorising 40 or by starting from $\sqrt{40}$ and attempting to find the first integer solution. Hence, we can hypothesise that the two prime numbers can be characterised by a single number, which I will denote in the following as an ordinal number. This means the other ordinal numbers we need to identify are:

154 = 11 x (13 + 1)
340 = 17 x (19 + 1)
690 = 23 x (29 + 1)
1178 = 31 x (37 + 1)
⋮

If we succeed in finding a formula to calculate the ordinal numbers, then we will have a structure which comes very close to the primes themselves. This shows us that they are far less randomly distributed than we have presumed up to now.

5. The Prime Number Constant

We were able to calculate the first two prime numbers 5 and 7 using the numbers from the first three columns of the longer of the bones, and thus the first double lottie. Coincidence? If we consider the fifth column with the 31 inscriptions, most likely not. Unfortunately, in figure 4, we are not able to tell that there are wider and narrower inscriptions. Heinzelin, however, has described these narrower inscriptions in more detail: Five of these are situated in the upper half and seven in the lower half. The first two primes, which we have already determined, will be used here for further calculation. The starting number comes from the 19 wider inscriptions, one adds the 5 and 7 inscriptions to obtain the 31 inscriptions in the fifth column:

19 + 5 + 7 = 31

Moreover, in the middle of the fifth column, there is a band of 13 further, almost invisible inscriptions. This number can be calculated in the following way:

$$13 = \frac{31x9-19}{20} = \frac{260}{20}$$

We will expand more on this formula's structure in the next chapter on the double helix. At the moment, we are only interested that the number 13 can be calculated using the previous prime numbers 5 and 7. Does the number 13 perhaps lead us on to the next ordinal number 154, which then defines the two primes 11 and 13? On the long bone, we can in fact find all the elements of a certain constant P, which, when multiplied with 13, gives the ordinal number 154:

$$P = \frac{8}{6}\pi\sqrt{8} = \frac{4}{3}\pi\sqrt{8}$$

Where do these numbers come from? Due to the two inscriptions which are closer to each other, we will use the farthest right column twice: once as 8 and once as $\sqrt{8}$ (the square root is beautifully represented by the two shortened strokes). For the division by 6, the 6 inscriptions are to be found on the bottom of the fourth column. But where do we find π? One of the 31 strokes in the fifth column lies in the middle of the band of 13 strokes. This means that we can take the total number of 44 strokes in this column and divide through the 14 strokes and obtain π. The fraction 44/14, or rather 22/7, was used by the Greeks as an approximation for π. Archimedes used the method of exhaustion to restrict the value of π to be between 223/71 and 22/7. If we then simplify 8/6 to 4/3, we realise that this is simply the formula for the volume of a sphere, i.e. the prime number constant conforms to the volume of a sphere with the radius $\sqrt{2}$.

The fact that π appears in the prime number constant is not surprising, considering, as we will see in the next chapter, that the prime numbers have a spiral-like structure. The derivation of this constant may seem somewhat adventurous, nevertheless, the result is beautiful. "Beauty is the first test: there is no permanent place in the world for ugly mathematics", the English mathematician Godfrey Harold Hardy once wrote. As a child, Hardy often decomposed the numbers of hymns in their prime factors at church, and later became well-known as the mentor of Srinivasa Ramanujan. To summarise: with the help of the first two prime numbers 5 and 7, we can calculate the number 13, which, when multiplied with the prime constant P, gives us the ordinal number 154, the very ordinal number which provides us the next two primes, 11 and 13. In the search for the formula which calculates the ordinal numbers, we have to consider that we obtain these through multiplication of a constant. For the first four ordinal numbers, we obtain the following sought-after values:

$$13 \times P = 154 = 11 \times (13 + 1)$$
$$28.7 \times P = 340 = 17 \times (19 + 1)$$
$$58.24 \times P = 690 = 23 \times (29 + 1)$$
$$99.43 \times P = 1178 = 31 \times (37 + 1)$$

6. The Prime Number Double Helix

"Whenever, by moving upwards (or downwards) through levels of some hierarchical system, we unexpectedly find ourselves right back where we started." This was how Douglas R. Hofstadter described his self-invented term "Strange Loop" in his Pulitzer-Prize-winning masterpiece "Gödel, Escher, Bach: An Eternal Golden Braid". Such self-referential loops allow a system to become self-conscious and are the core of our conscience. Hofstadter's book is an attempt to explain how living objects form from non-living matter. Coming from the DNA double helix, Hofstadter shows the, in his words almost mystical, analogies between the "strange loops" from molecular biology and mathematical logic (analogies which I feel are of homologous nature, since biology comes from mathematics). In particular, Hofstadter saw the big picture of recursive systems and claimed that "By appropriate means, complex recursive systems could be strong enough to break free of any given pattern" and thus would be a foundation for the development of conscience and intelligence.

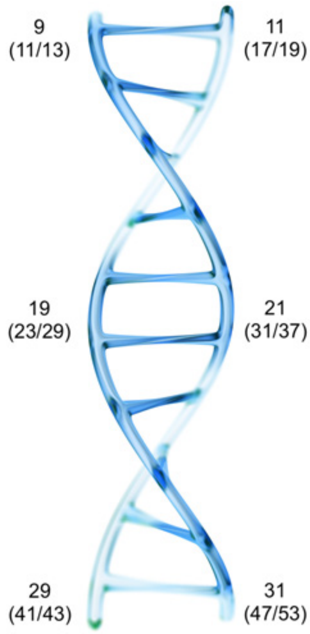


Fig. 5. The two helix strands with the nodes 9, 21, 29... and 11, 19, 31… as well as their respective prime number pairs, the double lotties. The mathematical structure of this double helix is sketched in the appendix. Photo from iStock.com/laremenko

As we have seen, the sequence 11, 21, 19 and 9 on the shorter of the Ishango bones can also be understood as a ladder with the two side rails 9, 19, 29, 39... and 11, 21, 31, 41… Upon reading Hofstadter's analogies from molecular biology and mathematics, the idea came to me that these sequences could resemble a double helix, as represented in Fig. 5. With the number 9, the first node in the double helix, the first ordinal number 154 of the primes 11 and 13 can be determined:

$\frac{31x9-19}{20}$ x P = 154 = 11 x (13 + 1)

In fact, the remainder of the ordinal numbers can be calculated with this method. I have sketched the prime number double helix in this text's appendix to highlight how each number is related to each other. For the first two ordinal numbers, 154 and 340, the formulas give the correct results. For the following two, we will need a correction term K, which we can calculate using the previous nodes on one of the strands of the helix, confirming the assumption that we have a helix structure:

$K_1 = 31^2 + (9^2 - 11^2) / 0 + 13$
$K_2 = 55^2 + (11^2 - 9^2) / 13 + 28.7$

Here we interestingly meet the two squares 9^2 and 11^2 once again. By symmetry, both of these formulas have a certain plausibility, since symmetry hardly appears by chance. Ergo, we have an exact model for all prime numbers up to 37. How do we calculate the other correction terms? For example:

$K_4 = K_1 \times \varphi$

Where φ is the golden ratio. This can also be expressed by trigonometric functions and the numerical values on the previous helix nodes:

$\varphi = \cos (31 + 143)^2 / \sin 30$

The trigonometric functions are useful for us, since the positive and negative values respectively give us the required positive and negative values for K. These are however the first fragments and speculations on our way to a comprehensive formula to calculate all the values of K. Thus the last door to the kingdom of the primes remains locked. Who will open it?

7. Gavrinis - Mathematics in Stone

Let us consider the third column on the shorter of the bones of Ishango with the sequence 11, 21, 19, 9 once again, which seems just as secretive as the "Section d'Or". We have already learned a possible way to interpret them: the four numbers form the first two pairs of the prime number double helix (Fig. 5). Here are two more possible ways:

$19/9 = 2.1\overline{11}$
$21/11 = 1.9\overline{09}$

9/19 = 0.4736...
11/21 = 0.5238...

Without being able to verify the first interpretation, there presumably are no further pairs of numbers which complement each other in such a beautiful and enigmatic way. What is interesting are their reciprocals 9/19 and 11/12, with approximate values of 0.4736 and 0.5238. These numbers also have a surprising geometric connection: at latitude 47°36', the sun's azimuth during sunrise on the summer solstice is 52°38'. The summer and winter solstices were of significant meaning in megalithic culture. Even Gavrinis, one of the greatest megalithic buildings in Europe, is oriented towards the sunrise on the day of winter solstice. And Gavrinis itself lies at latitude 47°34', which approximately corresponds to the value of 9/19.

Gavrinis is a small island in the Gulf of Morbihan in Brittany, France. The megalithic monument of the same name consists of a 13 metre long passage which is built from 52 large stones and is covered by a round, 50 metre wide and 8 metre tall slab. The wave-formed ornamentation of the stone is of heavenly beauty and was unique in appearance for the megalithic era. The passage leads to a small chamber built from six stones, whose capstone weighs 17 tonnes. The site is around 6000 years old, and the water level was five metres lower than today, so the site itself was still on the mainland. Although Gavrinis is also denoted as Gavrinis Tomb Passage, there are no signs of any burials. It is discussed whether the site was used as a rendezvous or a cultural place of assembly. Was this megalithic monument incidentally built at latitude 47°34' or does this latitude and its respective azimuth during summer solstice point towards the pair of fractions 9/19 and 11/21, and through this the building blocks of the prime number double helix? Does this monument contain further indications of the prime numbers? As mentioned above, the first ordinal number 154 is calculated by multiplying 13 with the prime constant P, which can also be written as

$$13 \times 4 \times 2\pi \frac{\sqrt{8}}{6} = 52 \times 2\pi \frac{\sqrt{8}}{6}$$

Gavrinis is built from 52 stones and is covered by a circular block (2π). Six stones build the inner chamber and on the 8th stone in the passage, the square root of 8 is represented in an almost metaphysically beautiful manner. What is particularly interesting to see are the various stone axes, striking markings on some of the stones, which stand out considerably from the spherical wave-like patterns. On the 21st stone slab (counted anticlockwise), there are 18 such axes. Up to this stone, there are 20 stones in total, 6 of which confine the chamber, and two of these six build the passage's end. This gives us

360 = 18 axes x 20
108 = 18 axes x 6
36 = 18 axes x 2

and thereby the numbers on the “Section d’Or” on the bones of Ishango. Moreover, the four stone axes on the 10th stone slab lead us to the prime numbers, for 4 x 10 is the difference of the two square numbers 9^2 and 11^2. The Breton author and druid Gwenc’hlan Le Scouëzec, born on the 11th November 1929, wrote numerous mathematical speculations about Gavrinis in his book “Bretagne Mégalithique”, however, they had nothing to do with the prime numbers. On the same day as Le Scouëzec, Hans Magnus Enzensberger was born. Enzensberger wrote the very fine text “Zugbrücke ausser Betrieb. Die Mathematik im Jenseits der Kultur”.

The numbers 13 x 4 x 2π or 52 x 2π are also found in Mayan culture. The “Calendar round” in the Maya calendar is 18,980 days long. It is the smallest common multiple of the solar calendar, with 52 years consisting of 365 days each (52 x 2π), and the ritual calendar with 73 years, each lasting 260 days. The ritual calendar has 13 months, each with 20 days; 260/20 is again the number 13, which itself belongs in the prime number double helix. What is fascinating is the Danca del Volador, a ceremonial mesoamerican dance in which four men on ropes are tied to a long pole and gently fall to the ground, circling the pole exactly 13 times. 13 x 4 x 2π could not be more poetically expressed. Ishango, Gavrinis, and Maya: through a variety of ancient cultures, we find more cues to the secret of the primes.

8. The Dodecahedral Universe

Riemann only focus marginally on the Zeta function and the primes. In 1854, he presented the idea of a four-dimensional space and a new conception of force which was based on the curvature of space in his habilitation talk; insights which were the foundations for Einstein’s general relativity theory. Four-dimensional spaces lie beyond human perception. However, we can attempt to understand it with an analogy: The surface of the sphere is a two-dimensional surface of a three-dimensional sphere. Correspondingly, the hypersphere or 3-sphere is the three-dimensional “surface” of a four-dimensional sphere. The surface of the sphere in three dimensions has the equation $r^2 = x^2 + y^2 + z^2$, with r denoting the radius; for the hypersphere in four-dimensional space, we will need a further coordinate: $r^2 = x^2 + y^2 + z^2 + t^2$. If we noted down the amount of natural-numbered solutions and plot this

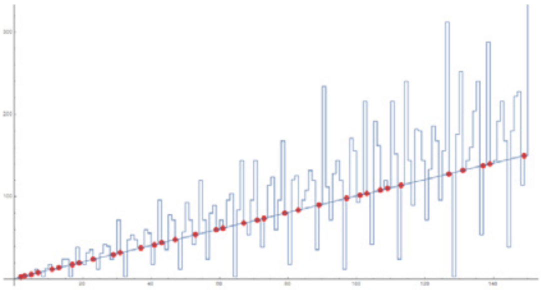


Fig. 6. The number of integer solutions of the hypersphere equation all lie for r^2 = prime on a straight line. Diagram by Karl-Heinz Kuhl in “Primzahlen – Altbekanntes und Neues”, December 2018

as a function value of r^2, then all these values of r^2 = prime (and only for these) lie on a line. This was very nicely depicted in the physicist Karl-Heinz Kuhl’s inspiring book on prime numbers (Fig. 6).

It is a strange coincidence of mathematical history that the prime numbers not only manifest on a line in the Riemannian Zeta function, but also in the four-dimensional space which was also introduced by Riemann. The hypersphere line illustrates the fundamental meaning of the fourth power or dimension for mathematics and even our universe. The speed of light appears in Einstein’s equivalence of mass and energy in the fourth dimension: $E^2 = m^2c^4 + p^2c^2$ (p denoting momentum, the more well-known $E = mc^2$ only holds for stationary masses).

What would a four-dimensional universe look like? The hypersphere, as hypothesised by Riemann and Einstein, is only one of many possible models. We differentiate models between euclidean, spherical and hyperbolic geometries. Due to the analysis of cosmic background radiation, the French scientist Jean-Pierre Luminet and his colleagues posted the hypothesis in the scientific journal “Nature” that the universe is the three dimensional analogue of a dodecahedral surface with slightly spherical-curved edges, also known as the “Poincaré dodecahedron space”. This space would also be the three-dimensional “surface” of the four-dimensional universe and, as with the hypersphere, it would be indeed finite but without limits (see arxiv.org and Die Form des Weltraums, Bild der Wissenschaft 9/2011). In 1919, the physicist Theodor Kaluza conjectured a universe consisting of four space dimensions and a time dimension. Einstein wrote to Kaluza that he had “great respect for the beauty and the boldness of your thought”. Just six years later, the physicist Wolfgang Pauli introduced the fourth dimension into the quantum world. He recognized that one would need four degrees of freedom, or

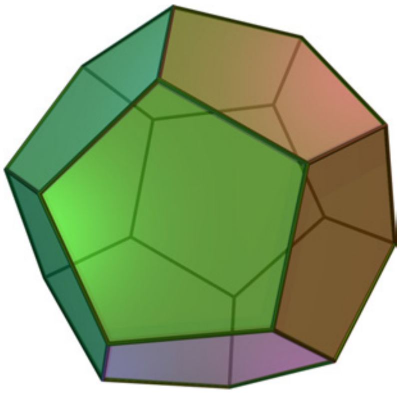


Fig. 7. Regular dodecahedron as a universal principle of order. Photo from Wikipedia

quantum numbers, to fully represent the bound electrons inside the hydrogen atom. Even modern theories which describe the fundamental forces in the universe work with four dimensions (such as the models of Lisa Randall and Raman Sundrum).

Returning to the Ishango-Bones: The central column of the shorter of the bones with the "Section d'Or" can also be read as follows:

108 = 9 x 12
360 = 90 x 4
12 = 5 + 7

Or in words: we can inscribe the 108 degree angles in a circle and we get a pentagon. Then we arrange 12 pentagons together to form a dodecahedron. In the pentagon and the dodecahedron, we notice numerous golden ratios and golden angles. The "Section d'Or" numbers are reunited once again. Does the shorter Ishango bone point us beyond the primes to the dodecahedron form of the universe? Are there perhaps further hints of the nature of the dodecahedron in our world? Let us go to the atoms, from the macrocosmos to the microcosmos, and we consider again Balmer's magic formula (in the generalised Rydberg form):

$\lambda = A \times m^2 / (m^2 - n^2)$

This is the formula behind the wavelengths of the hydrogen atom, from which all other atoms arise from, and also the formula which built the foundation of Niels Bohr's atomic model, who wrote himself: "As soon as I saw Balmer's formula the whole thing was immediately clear to me." Balmer's work is interpolated by Helmut Reis in his paper "100 Jahre Balmerformel". Reis himself recognised and described the relationship between Balmer's formula and the dodecahedron in his anniversary publication, and how it was related to the world of crystallography (the pentagon here is however not regular, as one of the sides is slightly longer than the rest). Upon further inspection of the length ratios of the so called "Pyrite Dodecahedron" with the crystallographic notation {hk0}, Reis found the same values which the expression $m^2 / (m^2 - n^2)$ had produced. There are no further examples for the incidence of these relationships in crystallography. Furthermore, if we assume radius 1 for the circumscribed sphere around the crystalline dodecahedron, the radius of the inscribed sphere is precisely root of $h^2 / (h^2 - k^2)$, and connection between the dodecahedron and Balmer's formula becomes obvious. Is the dodecahedron the basis of both the universe and the hydrogen atom? Platon already believed that the Gods created the world in the form of a dodecahedron, as Hans Wussing writes in "6000 Jahre Mathematik".

Let us journey further through the land of the golden ratios. The "Section d'Or" firstly lead us to the helix structure of the primes, and then to the dodecahedral nature of the universe and the hydrogen atom. In the next chapter, we will discover the golden ratio of DNA's double helix and its surprising connection to the dodecahedron.

9. The Cosmic Order

"The Book of Nature is written in the language of mathematics," wrote Galilei. "All things are numbers" was the motto of the Pythagoreans, who believed that immersing oneself in the kingdom of numbers would allow access to transcendence to understand the nature of the world. A further step was taken by the cosmologist Max Tegmark with his "Mathematical Universe Hypothesis": The universe is not only described by mathematics, it itself is mathematics. It is thereby very wonderful but not too surprising that the primes are arranged in a spiral. The spiral is a universal principle of order, from the DNA double helix to the nautilus shell and even to the galaxies of the universe. The nautilus shell is in the form of a logarithmic spiral, also denoted by Jakob Bernoulli as Spira mirabilis (miraculous spiral). A special case of the logarithmic spiral is the golden spiral, which is the expansion of the Fibonacci numbers. The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13… is formed when we begin with two ones and recursively create each new term by summing the two previous terms. The ratio of two consecutive Fibonacci numbers converges to the Golden Ratio. Fibonacci sequences and the Golden Ratio are, together with spheres and spirals as well as recursion, self-similarity and symmetry, the basic mathematical principles of nature.

The mathematical discipline of group theory deals with symmetries. Even here, the dodecahedron takes on a special role: its rotational group, A_5, is the smallest non-abelian simple group and is also the smallest non-solvable group and has order 60. Mathematical symmetries are the key to the understanding of our world. They are “the poetical essence of the Universe” as Edgar Allan Poe wonderfully formulated in “Eureka”. We can also think about whether the standard model of particle physics can be represented with the mathematical symmetries of the octonions. The Canadian physicist, mathematician and passionate accordionist Cohl Furey is convinced that our world is octonionic, as her portrayal in the Quanta Magazine shows.

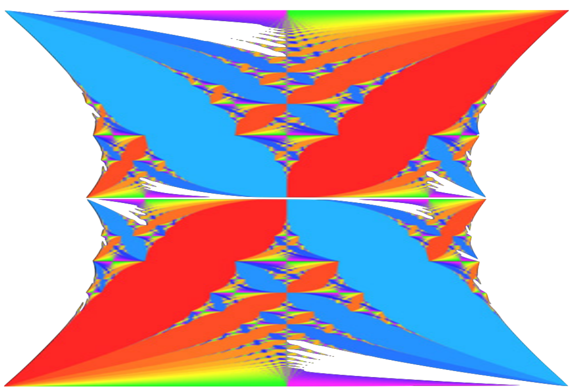
It was the brilliant biochemist Rosalind Franklin who first visualised the DNA helix structure with the usage of x-ray crystallography, and thus set the foundation for Watson and Crick’s model. Surprisingly, we meet the golden ratio once again: the base pairs are at a 36-degree angle to one another and thus divide the circle from above in exactly 10 triangles. These triangles have angles of 36°, 72° and 72°, the lengths of the sides are of the golden ratio, making it the obvious reason why there are exactly ten and not another amount of base pairs per revolution. We thus find the number 10 in the DNA double helix, the basis of all life. And ten is, as we have seen, the foundation of the prime number double helix with the two starting numbers 9 and 11 (10 +/– 1) as well as the steps of 10 in the numerator and denominator (see appendix). Interestingly, the two universal spacial structures of DNA and the dodecahedron are closely related to each other: the DNA has base pairs and the dodecahedron opposite pentagon faces, both of which are 36° to each other. Two opposite-facing pentagons, like the ten DNA base pairs, make a decagon made of ten golden triangles. The golden ratios in the decagon are also a possible reason for the ten fingers we have and thus for the decimal system's development.

Before we come to the butterflies, a little mystery of numbers or a short story of life. As we have seen, the mysterious number 19 is the starting point of the prime number double helix: 19 + 5 + 7 = 31. It also conveys the connection between the fourth dimension and the decimal world: $3^4 + 19 = 100$. Strangely enough, the fraction 1/19’s sum of the repeating digits is also 3^4, and reading the 18-digit period backwards (and carrying over each power of 10), we get the geometric series 1, 2, 4, 8, 16, …, which many growth processes are modelled after. The number 19 and its reciprocal value symbolise the double helix and cell growth, the two fundamental principles of life. Even the Menton Cycle takes almost exactly 19 years. This lunisolar cycle was already known in ancient times and is perhaps why the number 19 has a significant meaning in Arabian number mysticism. The profound article “Alles rechnet sich” by George Szpiro on the topic numerology in the also recommendable NZZ-folio about superstition is recommended for readers that find this topic too numerological.

Apart from his work “Gödel, Escher, Bach”, Hoftstadter was also credited as the discoverer of Hofstadter's butterfly, a wonderful self-similar fractal in the form of a butterfly (Fig. 8). The butterfly visualises how nature deals with the continuous energy levels of electrons in a crystal lattice as they meet with the discrete energy levels of electrons in a magnetic field. This solution is made clear when the significant parameter is expanded to a continued fraction. Hofstadter’s fractal butterfly thereby reflects the concurrence of the continuous and discrete worlds.

Physics is the story of the infinite divisibility of space and the finite divisibility of matter, as Harald Lesch vividly tells us in his post “Die grosse Krise der Physik”. The search for the theory of everything is the search for the common origin of continuous gravitation and discrete quantum mechanics. Albert Einstein sought it for his whole life, to no avail. Does this origin even exist? Perhaps the world is, in its bare essentials, of dualistic nature and its discrete and continuous parts are as convoluted as Benoît Mandelbrot’s fractals. Clues towards the fractal nature of the union of space and matter can be found in “Asymptotic Safety”, a theory of quantum gravitation. Nowadays, the theory of everything that we have been looking for might just be happily fluttering around our ears, like a butterfly of the genus Hofstadter.

Fig. 8. Infinitely many infinities. The fractal butterfly by Douglas Hofstadter, 1975. Photo from Wikipedia/Mytomi



10. Hen kai pan

The bones of Ishango are an example of the classic Out-of-place artifact, for the abstract notion of a number presumably did not exist before the Neolithic age. But beware! We often tend to systematically underestimate our predecessors, like the perhaps most famous example of an out-of-place artifact, the Antikythera mechanism, shows. This ingenious clock comes from a time where one assumed, for a long time, that it was before the time of technical development. With this very clock, the aforementioned 19-year-old Meton cycle can also be represented. Did the people of Ishango already have outstanding mathematical skills? Where did this knowledge come from and who was it passed on to? Africa is the cradle of humanity. It is not hard to guess where the cradle of mathematics is, especially if one knows Sona geometry or the differentiated mathematical structures of Central African music (see the article “Musik in Zentralafrika” in “Spektrum der Wissenschaft Spezial 2/2006” on ethnomathematics).

Perhaps the author of the bones of Ishango was a great but lonely mathematician, as they have always existed, from Euler to Gauss to Ramanujan? Even Einstein had other-worldly visions, very well expressed in the cartoon that was published a day after his death in the Washington Post. His insights in special relativity came to him, far away from the academic world, during his work for the Swiss Patent Office as, as he once called himself, a “dignified Swiss ink shitter”. Over and over again, “travellers on the edge of the world” gain access to deeper truths from another dimension. In earlier times, these trance-like states could also be reached with the help of dance, music, and drugs, as we have seen from the Mayans.

Last but not least: We cannot and do not want to exclude the case that friendly aliens had left the bones of Ishango behind or had built Gavrinis. Scientists are busy intensively searching for extraterrestrial life and intelligence. Recommended is the book “Da draussen – Leben auf unserem Planeten und anderswo” by English astrophysicist Ben Moore. The SETI Projects’ search for artificial radio waves is still unsuccessful to this day. Perhaps higher-developed civilisations do not use electromagnetic waves for interstellar communication, but instead neutrinos. Despite primitive tries, we have not come to the technical possibilities to receive such neutrino messages.

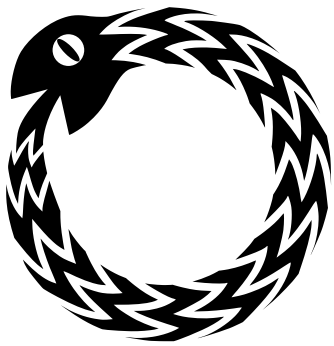


Fig. 9. The ouroboros symbolises cosmic unity, the equivalence of micro and macrocosmos. Photo from Wikipedia/Zanaq

The universe, the hydrogen atom, DNA and dodecahedrons: several things connected in a mysterious manner. It is the impartible unit of existence, or as the Greeks named it, hen kai pan. The polymath Athanasius Kircher wrote in 1654 that “the world is bound with secret knots”. This principle is easy to recognise but is difficult to apprehend in great detail. It is the perpetual passion of mankind to get the bottom of these connections. We, however, are at the end of our mathematical walk. After Marquis de Laplace claimed at the beginning of the 19th century that the universe is entirely deterministic, Werner Heisenberg and Kurt Gödel ended the Laplacian dream with their uncertainty principle (1927) and incompleteness theorem (1931), respectively. The prime number double helix is a beautiful diagram and perhaps also the foundation of the world between determinism and uncertainty: the ordinal numbers of the double helix can be calculated with a formula; the pairs of primes determined through the ordinal numbers, double lotties, can however not be determined analytically. Determinism and Uncertainty: the two dance, tightly entangled, through the night. It is the dance of Ishango.

above the fields –
detached from everything
a skylark sings

Matsuo Bashō

Literature Recommendations

Translation by Wayne Zeng

The Prime Number Double Helix

$$\frac{\overbrace{(19+5+7)}^{31}\cdot \mathbf{9}-19}{\mathbf{20}} = 13$$

$$13 \cdot \mathbf{P} = 11\,(13+1)$$

$$K_1 = \frac{31^2+(9^2-11^2)-0}{0+13}$$

$$\frac{\overbrace{(55+17+19)}^{91}\cdot \mathbf{19}-55+K_2}{\mathbf{30}} = 58.24$$

$$58.24 \cdot \mathbf{P} = 23\cdot(29+1)$$

$$\frac{\overbrace{(143+31+37)}^{211}\cdot \mathbf{29}-143+K_4}{\mathbf{40}} = 152.26$$

$$152.26 \cdot \mathbf{P} = 41\cdot(43+1)$$

$$\frac{\overbrace{(31+11+13)}^{55}\cdot \mathbf{11}-31}{\mathbf{20}} = 28.7$$

$$28.7 \cdot \mathbf{P} = 17\cdot(19+1)$$

$$K_2 = \frac{55^2+(11^2-9^2)-13}{13+28.7}$$

$$\frac{\overbrace{(91+23+29)}^{143}\cdot \mathbf{21}-91+K_1}{\mathbf{30}} = 99.43$$

$$99.43 \cdot \mathbf{P} = 31\cdot(37+1)$$

$$\frac{\overbrace{(211+41+43)}^{295}\cdot \mathbf{31}-211+K_3}{\mathbf{40}} = 214.22$$

$$214.22 \cdot \mathbf{P} = 47\cdot(53+1)$$

Ishango Bone I

11 13 17 19

11 21 19 9

Ishango Bone II

$$19+5+7=31$$

$$\mathbf{P} = \tfrac{4}{3}\cdot\pi\cdot\sqrt{8}$$

CB 9.9.19