Grimm's Conjecture

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Abstract: The collection of the consecutive composite integers is the composite connected, and no pair of its distinct integers may be generated by a single prime number. Consequently, it is possible to select at least one collection of distinct prime divisors by extracting only one prime from every single integer of the collection, and the Grimm's Conjecture holds.

Key words: Fundamental theorem of arithmetic, Bertrand's postulate.

Problem: *Every set of the consecutive composed integers has a collection of the distinct prime divisors selected by taking only one prime from every single integer of the set.*

Introduction

We define the set $\mathcal{S}$ to be a collection of the consecutive composed integers $x_1, x_2, x_3, \cdots, x_n$, perhaps placed between two consecutive primes α and β. All primes are outside of the set $\mathcal{S}$, and consequently the set $\mathcal{S}$ is the composite connected. The size of the set $\mathcal{S}$ cannot exceed α. Else, the size of the set would be greater then 2α, by Bertrand's postulate the set would contain at least one prime and would not be composite connected.

The collection of all prime divisors of an integer x in $\mathcal{S}$ is the prime divisors set $\mathrm{D}(x)$. According to the fundamental theorem of arithmetic each integer x is generated by its prime divisor set P, $x = \Pi p$, and will be identified by the pair (x, P). The set $\mathrm{D}(x, y) = \{d\colon d|x,\ d|y\}$ is the collection of all prime divisors common to both x and y integers. Further, $x = \mathrm{D}_x \Pi p'$ and $y = \mathrm{D}_y \Pi q'$, where D_x and D_y are products of the primes from $\mathrm{D}(x, y)$ particular to the integers x and y, and $\Pi p'$ and $\Pi q'$ the products of their additional composing primes.

Example: The following table shows the prime factorization and some collections of distinct prime factors of the composed integers between primes $\alpha = 89$ and $\beta = 97$. The set $\mathcal{S}$ has 7 and the collection $2, 3, 7, 5, 13, 19, 31, 43, 47$ of its prime divisors has 9 distinct members.

$\|i\rangle$	**89**	90	91	92	93	94	95	96	**97**
$\Pi\xi$	**89**	$2 \cdot 3^2 \cdot 5$	$7 \cdot 13$	$2^2 \cdot 43$	$3 \cdot 31$	$2 \cdot 47$	$5 \cdot 19$	$2^3 \cdot 3$	**97**
P_1		3	13	43	3	2	5	3	
P_2		2	7	43	31	47	19	3	
P_3		2	13	43	3	47	5	3	

Clearly, it is always possible to select a collection of 7 prime divisors, the primes $2, 3, 5$ are divisors of a few distinct integers from the set $\mathcal{S}$. However, the Grimm's conjecture requires the set P of 7 distinct prime divisors selected from the 7 distinct integers of the set $\mathcal{S}$. Notice that a such selection is not unique.

Description

Relations between two distinct integers from the set $\mathcal{S}$ are characterized by the set intersection of their prime divisors sets and by their greatest common divisor. For integers x and y in $\mathcal{S}$

1. $\mathrm{D}(x,y) = \emptyset$,
2. $\mathrm{D}(x,y) = \mathrm{D}(x) = \mathrm{D}(y)$,
3. $\mathrm{D}(x,y) = \mathrm{D}(x) \subset \mathrm{D}(y)$,
4. $\mathrm{D}(x,y) = \mathrm{D}(y) \cap \mathrm{D}(x) \neq \emptyset \,\&\, \mathrm{D}(x)$.

The greatest common divisor and $\mathrm{D}(x,y)$ are related. For, if $x = \Pi p^m$ and $y = \Pi p^n,\ n > m$ then

$$g = \Pi p^k,\ k = \max\{m,n\}\ \therefore\ p^k|x \,\&\, p^k\,|y \quad \Rightarrow \quad x = g\mathrm{G}_x,\ y = g\mathrm{G}_y,$$

where G_x and G_y are the g cofactor integers in the x and y. Two important cases are when integer x is the greatest common divisor of y, and when both x and y are generated by a single prime.

Corollary 1. *There is no distinct integers x and y in $\mathcal{S}$ such that either x divides y or that both x and y are generated by a single prime, or o pair of distinct integers from the set $\mathcal{S}$ is generated by a single prime.*

■ If $x|y$ the integer x is the greatest common divisor of x and y and its cofactor yx^{-1} in the y is an integer greater or equal to 2. By the Bertrand's postulate there is a prime between x and y and the set $\mathcal{S}$ is not the composition connected, contradiction.

If both x and y are generated by a single prime p then $x = p^m$ and $y = p^n,\ n > m$. Hence $y = xp^{n-m}$ and x is the greatest common divisor of x and y with cofactor p^{n-m} in y greater or equal to 2. Again, by the Bertrand's postulate must be a prime between x and y, contradiction. Hence, there are no two distinct integers in the set $\mathcal{S}$ generated by the single prime. □

Corollary 2. *Each pair (x,y) of the distinct composed integers from the set $\mathcal{S}$ lets a pair $|p,q\rangle$. of distinct prime divisors $p|x$ and $q|y$.*

■ We will consider one by one the set intersection cases of the prime divisor sets of a pair (x,y) of distinct composed integers $x \sim (p,\mathrm{P})$ and $y \sim (q,\mathrm{Q})$ from the set $\mathcal{S}$. We may assume $y > x$ when it is necessary.

1. When $\mathrm{D}(x,y) = \emptyset$ the integers x and y are composed on two distinct sets of the primes. It is sufficient to take a prime divisor p of x and a prime divisor q of y to make the pair $|p,q\rangle$ of the distinct primes.

2. When $\mathrm{D}(x) = \mathrm{D}(y)$ the set $\mathrm{D}(x,y)$ cannot be a single prime set, see the Corollary 1. Consequently, the set $\mathrm{D}(x,y)$ must have at least two distinct prime divisors and the vector $|p,q\rangle$ of the distinct prime divisors of x and y cannot be empty.

3. When $\mathrm{D}(x) \subset \mathrm{D}(y)$ all primes in the set $\mathrm{D}(x,y)$ are the divisors of the x, and there is at least one prime q divisor of y not in the $\mathrm{D}(x,y)$. The vector $|p,q\rangle$, $p \in \mathrm{D}(x,y)$, is a pair of distinct prime divisors of x and y respectively.

4. Finally, when $\mathrm{D}(x,y)$ is nonempty intersection $\mathrm{D}(y) \cap \mathrm{D}(x)$ not identical to $\mathrm{D}(x)$ there are at least two distinct primes $p \in \mathrm{D}(x) \setminus \mathrm{D}(y)$ and $q \in \mathrm{D}(y) \setminus \mathrm{D}(x)$ such that $x = p\Pi d_x$ and $y = q\Pi d_y$, where $d_x, d_y \in \mathrm{D}(x,y)$. Definitely, at least $|p,q\rangle$ is the vector of two distinct prime divisors of the pair (x,y). □

Remark: Further, we are selecting a collection $\mathrm{P} = \{p_1, p_2, p_3, \cdots, p_n\}$ of the prime divisors from a set of the consecutive composed integers $\mathcal{S} = \{x_1, x_2, x_3, \cdots, x_n\}$, and require that the following, either affirmative or negative, statements of the Grimm's Conjecture hold

$$\forall \mathcal{S},\ \exists \mathrm{P} = |p_1, p_2, \cdots, p_i, \cdots, p_n\rangle,\ p_i|x_i\ \therefore \forall i \neq j \Rightarrow p_i \neq p_j,$$
$$\exists \mathcal{S},\ \forall \mathrm{P} = |p_1, p_2, \cdots, p_i, \cdots, p_n\rangle,\ p_i|x_i\ \therefore \exists i \neq j \Rightarrow p_i = p_j.$$

The explicit verbal statement of the affirmative Grimm's Conjecture, "*Every set of the consecutive composed integers has a collection of the prime divisors selected from its distinct integers such that two prime divisors are identical only when the composed integers are identical,*" is equivalent to the statement, "*Every set of the consecutive composed integers must have at least one collection of the distinct prime divisors, only one divisor is selected from each integer.*"

The most direct statement of the negative of the Grimm's Conjecture, "*Any collection selecting one prime from every integer of a set of the consecutive composed integers must have some identical primes,*" is equivalent to the explicit verbal negative statement: "*There is at least one set of the consecutive composed integers such that every collection of the prime divisors selected one per integer of its distinct composed integers must have at least one pair of the identical prime divisors.*" Shortly, "*There is a set of the consecutive composed integers such that in each collection of the prime divisors singly selected from its distinct composed integers at least two are identical.*"

Conclusion

Here is the final statement on the Grimm's Conjecture.

Corollary 3 *Every set of the consecutive composed integers must have at least one collection of the distinct prime divisors, only one divisor is selected from each integer.*

■ We assume that Grimm's Conjecture is not true. This means that there is a set of consecutive composed integers such that in each collection of the prime divisors singly selected from its distinct composed integers at least two are identical. Consequently, there are two distinct composed integers (X, P) and (Y, Q) such that the prime divisors $p\,|\mathrm{X}$ and $q\,|\mathrm{Y}$ are identical. Thus $p = \xi = q$. The prime ξ is the only prime divisors of X and Y. For, if not we would find a prime η distinct from $p = q$ and the prime divisors collection with two identical prime divisors would not be ever selected, Thus must be $p = q = \xi$ and both X and Y are generated by the same prime ξ. Consequently, by Corollary 1, the set of the consecutive composed integers is not the composition connected, a contradiction. Hence, the affirmative statement of the Grimm's Conjecture is true. □

References

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