General Vector Algebra for Relativistic Spacetime and Its Application to Electromagnetism

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Abstract: Physical laws in spacetime are commonly represented by tensors. However, it is possible to give a vector form to them, where the vectors follow quite simple rules of sum, product and commutation. We present here a self-contained vector algebra for relativistic spacetime, which is a simplification and generalization of geometric (Clifford) and spacetime algebras, and apply it to electromagnetic theory. We emphasize that the vector algebra is convenient to use and flexible enough to accommodate Euclidean geometry and the symmetries of relativistic spacetime, providing an excellent mathematical framework for physics in spacetime.

Keywords: spacetime, vector algebra, electromagnetic field, Maxwell equations, commutation relations

1. Introduction

The equations of electromagnetic field are traditionally formulated in three-dimensional (3-d for short) Euclidean space and the real line. For this formulation, some elementary mathematical objects, such as 3-d vectors, real numbers and differential operators, are involved, and a number of operation methods, such as additions and multiplications (or products), for each kind of the objects and between them, are defined independently. [1, 2] Actually, our understanding of physics is restricted by this mathematical framework. For example, electric and magnetic fields are taken as two related but different things, and the fields around charges are described by four differential equations, namely Maxwell equations.

With the development of the special and general theory of relativity, it is realized that the fundamental laws of physics should be formulated in four-dimensional (4-d) spacetime, in which time is taken as the fourth dimension, in addition to the three spatial dimensions of space. Spacetime formulations always provide us deep insights into the underlying physics, and physical equations also find their most elegant expressions in spacetime. [3] For instance, the energy and momentum of a moving particle constitute its spacetime momentum. More importantly, electric and magnetic fields are combined into electromagnetic field as a whole, and the four Maxwell equations are unified into a single equation in spacetime.

However, constructing a mathematical tool fit for relativistic spacetime is not an easy task, [4, 5] because the tool should be as simple as possible in mathematics, and has to be compatible with Lorentz symmetry as well. [3, 6] Tensor is the most common tool used nowadays. While, since it does not consist of any information about physical space and time, many excessive operations are needed to fulfill Lorentz symmetry. [3, 7] For example, a displacement in spacetime is denoted by a 4-d tensor of the first rank, and two types of components, covariant and contravariant components, have to be introduced artificially to give its magnitude—spacetime interval. Furthermore, there are

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four types of components for a tensor of the second rank, and to figure out the connection between them one has to be very familiar with the artificial sign-changing rules.

A promising candidate is geometric algebra, in which points, (directed) line segments, (directed) plane segments and volumes are indicated by scalars (grade-0), vectors (grade-1), bi-vectors (grade-2) and tri-vectors (grade-3), respectively, providing us more geometric interpretations than tensors. [4, 5, 8-10] A geometric algebra that is applied to relativistic spacetime is called spacetime algebra. It treats time and space basis vectors as "real" and "imaginary" units respectively, so the product of two time basis vectors is +1, and that of two identical space basis vectors is -1. [11, 12] With this algebra, the spacetime expressions of physical laws are much more natural and elegant than ever.

It should be noted that, both geometric and spacetime algebras use the grade of vector to demonstrate the dimension of geometric object or that of physical quantity, and keep two grade-dependent operations, inner and outer (wedge) products, as important calculation methods. [8-12] From the practical point of view, the grade-dependent products are not easy to use, because one has to know the grade of each factor before operation and apply different rules for each condition. From the geometric point of view, the wide use of grade-dependent products makes the algebras stressing on the dimensional properties of mathematical objects, rather than the geometric relations between them.

In this paper, we introduce commutative and anticommutative products on the base of geometric product, instead of inner and outer products, by which all operation rules are independent of the grade of vector. We apply this grade-independent vector algebra to spacetime electromagnetism, and present some preliminary results.

2. General vectors and operation rules

It is assumed that general vectors follow three fundamental rules of operation:

(i) The sum of a row of vectors results in a vector, and the result is independent on their sequence.

(ii) The product of a row of vectors results in a vector, and product operation distributes over sum operation.

(iii) The product of two vectors are anticommutative if they are orthogonal, and commutative if they are parallel.

Rule (i) and (ii) define the fundamental operations of sum and product, and rule (iii) give the geometric interpretations of commutativity and anticommutativity in a product. In a row of vectors to be added, the operation order is not specified in rule (i), so sum operation is associative. For the same reason, product operation is also associative.

The product of any two vectors can be expressed as a sum of "symmetric" and "antisymmetric" parts in the following way

$$\hat{a}\,\hat{b} = \tfrac{1}{2}\left(\hat{a}\,\hat{b} + \hat{b}\,\hat{a}\right) + \tfrac{1}{2}\left(\hat{a}\,\hat{b} - \hat{b}\,\hat{a}\right). \tag{1}$$

We can denote the symmetric part by

$$\hat{a} \cdot \hat{b} = \tfrac{1}{2}\left(\hat{a}\,\hat{b} + \hat{b}\,\hat{a}\right), \tag{2}$$

and the antisymmetric part by

$$\hat{a} \times \hat{b} = \tfrac{1}{2}\left(\hat{a}\,\hat{b} - \hat{b}\,\hat{a}\right), \tag{3}$$

then the product can be written as

$$\hat{a}\,\hat{b} = \hat{a}\cdot\hat{b} + \hat{a}\times\hat{b}. \tag{4}$$

The notation “·” in Equation (2) is a “mixed” operation of sum and product, and this mixed operation is distributive (over sum operation):

$$\begin{aligned}\hat{a}\cdot\left(\hat{b}+\hat{c}\right) &= \frac{1}{2}\hat{a}\left(\hat{b}+\hat{c}\right)+\frac{1}{2}\left(\hat{b}+\hat{c}\right)\hat{a}\\ &= \frac{1}{2}\left(\hat{a}\,\hat{b}+\hat{b}\,\hat{a}\right)+\frac{1}{2}\left(\hat{a}\,\hat{c}+\hat{c}\,\hat{a}\right)\\ &= \hat{a}\cdot\hat{b}+\hat{a}\cdot\hat{c}.\end{aligned} \tag{5}$$

Similarly, the mixed operation “×” in Equation (3) is also distributive:

$$\left(\hat{b}+\hat{c}\right)\times\hat{a} = \hat{b}\times\hat{a}+\hat{c}\times\hat{a}. \tag{6}$$

If distributivity is considered as an intrinsic property of “product”, the mixed operations, “·” and “×”, can be considered as some kinds of product. As the product “·” for any two vectors is commutative, and the product “×” for any two vectors is anticommutative:

$$\begin{cases}\hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{a}\\ \hat{c}\times\hat{d} = -\hat{d}\times\hat{c}\end{cases}, \tag{7}$$

we can call them commutative and anticommutative products, respectively.

It can be proved that both commutative and anticommutative products are not associative, so they are binary operations, which means that one has to give an operation order in which commutative or anticommutative product is performed for there or more vectors. Commutative and anticommutative products are not as fundamental as the operations of sum and product that defined in rule (i) and (ii), so they are just useful expressions of the fundamental operations.

Now, think about the product of a vector $\hat{a}$ with itself, which is also a vector according to rule (ii). If it is true that any vector $\hat{b}$ can be resolved into a vector $\hat{b}_{\parallel}$ parallel to $\hat{a}$ and a vector $\hat{b}_{\perp}$ orthogonal to $\hat{a}$:

$$\hat{b} = \hat{b}_{\parallel} + \hat{b}_{\perp}, \tag{8}$$

where

$$\begin{cases}\hat{b}_{\parallel}\,\hat{a} = \hat{a}\,\hat{b}_{\parallel}\\ \hat{b}_{\perp}\hat{a} = -\hat{a}\hat{b}_{\perp}\end{cases}. \tag{9}$$

Then, the product $\hat{a}\hat{a}$ is commutative with any vector $\hat{b}$:

$$\begin{aligned}(\hat{a}\hat{a})\hat{b} &= \hat{a}\hat{a}\hat{b}_{\parallel} + \hat{a}\hat{a}\hat{b}_{\perp}\\ &= \hat{a}\,\hat{b}_{\parallel}\hat{a} - \hat{a}\,\hat{b}_{\perp}\hat{a}\\ &= \hat{b}_{\parallel}\hat{a}\hat{a} + \hat{b}_{\perp}\hat{a}\hat{a}\\ &= \hat{b}(\hat{a}\hat{a}).\end{aligned} \tag{10}$$

We can call the product $\hat{a}\hat{a}$ a commutative vector, or a scalar, and denote it by a character without hat:

$$d = \hat{a}\hat{a}. \tag{11}$$

Therefore, a scalar is nothing else but a special vector that is commutative with any vector:

$$d\,\hat{b} = \hat{b}\,d, \tag{12}$$

and all operations defined above are applicable to scalars.

Indeed, the vectors defined by the three fundamental rules are general enough to accommodate a wide variety of mathematical objects. The objects that have directions, such as basis vectors of 4-d spacetime, are indicated by non-commutative vectors. The objects that are directionless, such as real numbers and functions, are indicated by commutative vectors (scalars). One can distinguish commutative product and dot product by that, commutative product can be operated on a basis vector and a real number, but dot product can not. [2] Commutative product is also different from inner product, because the inner product of a grade-1 vector and a grade-2 vector (or bi-vector) is anticommutative, [5, 11] but the commutative product is always commutative, with independence on the factors.

Unlike conventional algebra, an operator of differentiation with respect to a coordinate, say x, is considered here as a scalar. Its application to a function $f(x)$ (f for short) is given by their product:

$$\frac{\partial}{\partial x}f = f\frac{\partial}{\partial x} = \frac{\partial f}{\partial x}. \tag{13}$$

And its application to the product of two functions $f(x)$ and $g(x)$ is expressed by

$$\frac{\partial}{\partial x}fg = f\frac{\partial}{\partial x}g = fg\frac{\partial}{\partial x} = \frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}. \tag{14}$$

Thus we have a so abstract and general definition of vector that, all mathematical objects we are going to deal with have been considered as vectors. They follow the same, and only three fundamental rules, which are the rules of sum, product and commutation. The geometric relation we care most about is closely related to the commutation rule.

3. Spacetime vectors and invariants

A vector algebra for spacetime is built up from combinations of one time basis vector $\hat{\gamma}_\tau$ and three space basis vectors, $\{\hat{\gamma}_x, \hat{\gamma}_y, \hat{\gamma}_z\}$, which are orthonormal:

$$\begin{cases} \hat{\gamma}_i\,\hat{\gamma}_j = -\hat{\gamma}_j\,\hat{\gamma}_i \; (i \neq j) \\ \hat{\gamma}_i\,\hat{\gamma}_i = 1 \end{cases}, \tag{15}$$

If a physical quantity is the result of a measurement (or an observation), it is typically indicated by a real number or a series of real numbers. The measurement is surely performed at some instant, and would last for a while, so each number in the result is definitely related to a time basis vector. Thus, the position of a moving particle at some instant in spacetime is indicated by

$$\hat{\gamma} = \tau\,\hat{\gamma}_\tau + x\,\hat{\gamma}_x\hat{\gamma}_\tau + y\,\hat{\gamma}_y\hat{\gamma}_\tau + z\,\hat{\gamma}_z\hat{\gamma}_\tau, \tag{16}$$

and a small displacement by

$$d\hat{\gamma} = d\tau\,\hat{\gamma}_\tau + dx\,\hat{\gamma}_x\hat{\gamma}_\tau + dy\,\hat{\gamma}_y\hat{\gamma}_\tau + dz\,\hat{\gamma}_z\hat{\gamma}_\tau. \tag{17}$$

For convenience, we rewrite the product of basis vectors as one vector with a sequence notation:

$$\hat{\gamma}_i\,\hat{\gamma}_j = \hat{\gamma}_{ij}. \tag{18}$$

Then, the small displacement vector can be expressed as

$$d\hat{\gamma} = d\tau\,\hat{\gamma}_\tau + dx\,\hat{\gamma}_{x\tau} + dy\,\hat{\gamma}_{y\tau} + dz\,\hat{\gamma}_{z\tau}, \tag{19}$$

and its magnitude, spacetime interval, is

$$d\gamma = \sqrt{d\hat{\gamma}d\hat{\gamma}} = \sqrt{d\tau^2 - dx^2 - dy^2 - dz^2}, \tag{20}$$

where a system of units with the speed of light being set as

$$c=1 \tag{21}$$

is used.

One may notice that a spacetime interval is a number without any basis vector. Since a spacetime basis is always associated with an inertial frame of reference, the absence of basis vectors means that a spacetime interval is independent on the choice of reference frame, and is a relativistic invariant.

The spacetime interval of a moving particle is also the time that the particle (or the clock fixed on the particle) experienced, so it is also called proper time. Thus, the spacetime velocity of the particle is indicated by the direction of the small displacement vector:

$$\hat{\mu} = \frac{d\hat{\gamma}}{d\gamma} = \frac{d\tau}{d\gamma}\,\hat{\gamma}_\tau + \frac{dx}{d\gamma}\,\hat{\gamma}_{x\tau} + \frac{dy}{d\gamma}\,\hat{\gamma}_{y\tau} + \frac{dz}{d\gamma}\,\hat{\gamma}_{z\tau}. \tag{22}$$

And the spacetime momentum is

$$\hat{p} = m_0\hat{\mu}, \tag{23}$$

where m_0 is the rest mass of the particle. The time component of the momentum is the energy (or mass) m:

$$m = m_0\frac{d\tau}{d\gamma} = \frac{m_0}{\sqrt{1-\frac{dx^2}{d\tau}-\frac{dy^2}{d\tau}-\frac{dz^2}{d\tau}}} = \frac{m_0}{\sqrt{1-v^2}}, \tag{24}$$

and the magnitude of the momentum is the rest mass m_0:

$$p = \sqrt{\hat{p}\hat{p}} = m_0\sqrt{\hat{\mu}\hat{\mu}} = m_0 \tag{25}$$

which is also a relativistic invariant.

4. Spacetime equations of electromagnetic field

The inverse of a vector $\hat{a}$, if it exists, is denoted by $\hat{a}^{-1}$ and defined by the equation

$$\hat{a}\,\hat{a}^{-1} = \hat{a}^{-1}\hat{a} = 1. \tag{26}$$

Thus, we can write the vector differential operator as

$$\begin{aligned}\hat{\partial} &= \frac{\partial}{\partial\tau}\,\hat{\gamma}_\tau^{-1} + \frac{\partial}{\partial x}\,\hat{\gamma}_{x\tau}^{-1} + \frac{\partial}{\partial y}\,\hat{\gamma}_{y\tau}^{-1} + \frac{\partial}{\partial z}\,\hat{\gamma}_{z\tau}^{-1} \\ &= \partial_\tau\,\hat{\gamma}_\tau + \partial_x\,\hat{\gamma}_{\tau x} + \partial_y\,\hat{\gamma}_{\tau y} + \partial_z\,\hat{\gamma}_{\tau z}.\end{aligned} \tag{27}$$

Then the spacetime divergence and curl of a vector field $\hat{A}$ can be expressed as

$$\begin{cases} div\,\hat{A} = \hat{\partial}\cdot\hat{A} \\ curl\,\hat{A} = \hat{\partial}\times\hat{A} \end{cases}. \tag{28}$$

For an electromagnetic potential,

$$\hat{A} = \varphi\hat{\gamma}_\tau + A_x\,\hat{\gamma}_{x\tau} + A_y\,\hat{\gamma}_{y\tau} + A_z\,\hat{\gamma}_{z\tau}, \tag{29}$$

its curl gives the electromagnetic field:

$$\begin{aligned}\hat{\partial}\times\hat{A} &= \left(\partial_\tau\,\hat{\gamma}_\tau + \partial_x\,\hat{\gamma}_{\tau x} + \partial_y\,\hat{\gamma}_{\tau y} + \partial_z\,\hat{\gamma}_{\tau z}\right)\times\left(\varphi\hat{\gamma}_\tau + A_x\,\hat{\gamma}_{x\tau} + A_y\,\hat{\gamma}_{y\tau} + A_z\,\hat{\gamma}_{z\tau}\right) \\ &= -\partial_\tau\left(A_x\,\hat{\gamma}_x + A_y\,\hat{\gamma}_y + A_z\,\hat{\gamma}_z\right) - \left(\partial_x\,\hat{\gamma}_x + \partial_y\,\hat{\gamma}_y + \partial_z\,\hat{\gamma}_z\right)\varphi \\ &\quad +\left(\partial_y A_z\,\hat{\gamma}_{yz} + \partial_z A_y\,\hat{\gamma}_{zy} + \partial_z A_x\,\hat{\gamma}_{zx} + \partial_x A_z\,\hat{\gamma}_{xz} + \partial_x A_y\,\hat{\gamma}_{xy} + \partial_y A_x\,\hat{\gamma}_{yz}\right) \\ &= \left[-(\partial_\tau A_x + \partial_x\varphi)\hat{\gamma}_x - \left(\partial_\tau A_y + \partial_y\varphi\right)\hat{\gamma}_y - (\partial_\tau A_z + \partial_z\varphi)\hat{\gamma}_z\right] \\ &\quad +\left[\left(\partial_y A_z - \partial_z A_y\right)\hat{\gamma}_{yz} + (\partial_z A_x - \partial_x A_z)\,\hat{\gamma}_{zx} + \left(\partial_x A_y - \partial_y A_x\right)\hat{\gamma}_{xy}\right] \\ &= \left[E_x\hat{\gamma}_x + E_y\hat{\gamma}_y + E_z\hat{\gamma}_z\right] + \left[B_x\hat{\gamma}_{yz} + B_y\hat{\gamma}_{zx} + B_z\hat{\gamma}_{xy}\right] \\ &= \hat{E} + \hat{B} \\ &= \hat{F},\end{aligned} \tag{30}$$

where electric and magnetic fields can be expressed as

$$\begin{cases} \hat{E} = E_x\hat{\gamma}_x + E_y\hat{\gamma}_y + E_z\hat{\gamma}_z \\ \hat{B} = B_x\hat{\gamma}_{yz} + B_y\hat{\gamma}_{zx} + B_z\hat{\gamma}_{xy} \end{cases}. \tag{31}$$

According to Maxwell equations, the divergence of electromagnetic field is zero,

$$\begin{aligned}\hat{\partial}\cdot\hat{F} &= \left(\partial_\tau\,\hat{\gamma}_\tau + \partial_x\,\hat{\gamma}_{\tau x} + \partial_y\,\hat{\gamma}_{\tau y} + \partial_z\,\hat{\gamma}_{\tau z}\right)\cdot(\hat{E} + \hat{B}) \\ &= \partial_\tau\left(B_x\hat{\gamma}_{\tau yz} + B_y\hat{\gamma}_{\tau zx} + B_z\hat{\gamma}_{\tau xy}\right) + \left(\partial_x B_x + \partial_y B_y + \partial_z B_z\right)\hat{\gamma}_{\tau xyz} \\ &\quad +\left[\left(\partial_y E_z - \partial_z E_y\right)\hat{\gamma}_{\tau yz} + (\partial_z E_x - \partial_x E_z)\,\hat{\gamma}_{\tau zx} + \left(\partial_x E_y - \partial_y E_x\right)\hat{\gamma}_{\tau xy}\right] \\ &= 0\ ,\end{aligned} \tag{32}$$

and the curl of electromagnetic field is electric current density (over ε_0)

$$\begin{aligned}\hat{\partial}\times\hat{F} &= \left(\partial_\tau\,\hat{\gamma}_\tau+\partial_x\,\hat{\gamma}_{\tau x}+\partial_y\,\hat{\gamma}_{\tau y}+\partial_z\,\hat{\gamma}_{\tau z}\right)\times\left(\hat{E}+\hat{B}\right)\\ &= -\partial_\tau\,\hat{E}\,\hat{\gamma}_\tau+\left(\partial_x\,\hat{\gamma}_{\tau x}+\partial_y\,\hat{\gamma}_{\tau y}+\partial_z\,\hat{\gamma}_{\tau z}\right)\times\left(\hat{E}+\hat{B}\right)\\ &= -\partial_\tau\left(E_x\hat{\gamma}_{x\tau}+E_y\hat{\gamma}_{y\tau}+E_z\hat{\gamma}_{z\tau}\right)+\left(\partial_x\,E_x+\partial_y E_y+\partial_z\,E_z\right)\hat{\gamma}_\tau\\ &\quad+\left[\left(\partial_y B_z-\partial_z B_y\right)\hat{\gamma}_{x\tau}+\left(\partial_z B_x-\partial_x B_z\right)\hat{\gamma}_{y\tau}+\left(\partial_x B_y-\partial_y B_x\right)\hat{\gamma}_{z\tau}\right]\\ &= \frac{\hat{J}}{\varepsilon_0}\ ,\end{aligned}\tag{33}$$

where the electric current density

$$\hat{J}=\rho_0\hat{\mu}.\tag{34}$$

Combining equation (32) and (33), we have

$$\hat{\partial}\hat{F}=\frac{\hat{J}}{\varepsilon_0},\tag{35}$$

which is the spacetime expression of Maxwell equations.

Applying divergence on both sides of equation (33), the left side is

$$\begin{aligned}\hat{\partial}\cdot\left(\hat{\partial}\times\hat{F}\right) &= \frac{1}{2}\hat{\partial}\cdot\left(\hat{\partial}\hat{F}-\hat{F}\hat{\partial}\right)\\ &= \frac{1}{4}\left(\hat{\partial}\hat{\partial}\hat{F}+\hat{\partial}\hat{F}\hat{\partial}-\hat{\partial}\hat{F}\hat{\partial}-\hat{F}\hat{\partial}\hat{\partial}\right)\\ &= \frac{\left(\hat{\partial}\hat{\partial}\right)\hat{F}-\hat{F}\left(\hat{\partial}\hat{\partial}\right)}{4}\\ &= \frac{\partial^2\hat{F}-\hat{F}\,\partial^2}{4}\\ &= 0.\end{aligned}\tag{36}$$

where the D'Alembertian operator,

$$\partial^2=\hat{\partial}\hat{\partial}=\partial_\tau^2-\partial_x^2-\partial_y^2-\partial_z^2,\tag{37}$$

is commutative with any vector field. Then, the right side must be zero:

$$\hat{\partial}\cdot\hat{J}=0,\tag{38}$$

and this equation is recognized as the conservation law of electric charges.

With the Lorentz gauge

$$\hat{\partial}\cdot\hat{A}=0,\tag{39}$$

the spacetime Maxwell equation can also be expressed as

$$\partial^2\hat{A}=\frac{\hat{J}}{\varepsilon_0}.\tag{40}$$

The motion equation of a charge q in the electromagnetic field $\hat{F}$ is

$$\frac{d}{d\gamma}\hat{p}=q\hat{F}\times\hat{\mu},\tag{41}$$

where the right side is the general Lorentz force,

$$\hat{f}=q\hat{F}\times\hat{\mu}.\tag{42}$$

5. Discussion and conclusions

Although commutative and anticommutative products are not fundamental operations, they are widely used to indicate the geometric relations between various vectors, and thus have strong effects on the whole vector algebra. Firstly, they are so easy to operate that one need not to know the grades of the factors before operation. Secondly, since all kinds of vectors follow the same rules of operation, it is not necessary to classify them by grade. In other words, grade become meaningless for operation. Thirdly, the concept of vector is greatly generalized. Not only real numbers but also differential operators can be taken as general vectors, and all their operation rules are unified. Finally, the main theme of vector algebra is changed from the dimensional property of an object to its geometric relations with the others.

Actually, the concepts of “parallel” and “orthogonal” are also generalized here. A real number is commutative with a basis vector, so they are “parallel” to each other according to rule (iii). If the real number corresponds to a point, and the basis vector a directed line segment, what dose “parallel” mean for them? Suppose we had defined that, two objects are parallel if only one direction can be obtained from them. On this assumption, a directionless point and a directed line segment are parallel. However, the commutativity in product is a more compact and general way for the definition of parallel.

Conventionally, a physical quantity in spacetime, such as the position of a particle at some instant, is represented by a 4-d vector that consists of a time component and three space components. To fill the need of Lorentz symmetry (or relativistic invariance), the time and space basis vectors must follow different rules of operation, which inevitably increases the number of rules and breaks the beautiful symmetry of Euclidean geometry. It seems reasonable, because the conception of time is intuitively different from that of space.

From purely mathematical considerations, Lorentz symmetry and Euclidean geometry can be fulfilled simultaneously without changing any rule. If a 4-d vector is constructed with a time component and three space-time components, as shown in equation (16), it is naturally consistent with Lorentz symmetry, and the Euclidean geometry is preserved as well. Thus, every operation in this vector algebra has a clear geometric meaning as that in the conventional vector algebra, and the noncommutativity of any two vectors is finally attribute to the anticommutativity between the basis vectors, which bring us great convenience in operation and conception.

Beyond that, some underlying physics and beautiful symmetries emerge with this algebra. The weird expression of a displacement vector indicates that each component is associated with a measurement. The unit vectors for each component of a displacement vector can be replaced by gamma matrices, and vice versa. The vector differential operator is precisely the same as the Dirac operator, [11] which is usually used together with gamma matrices in the Dirac equation. The connections between various quantities in electromagnetic theory become so clear. The curl of electromagnetic potential is naturally electromagnetic field, the divergence of electromagnetic field is zero, and the curl of electromagnetic field is electric current density over ε_0 (Maxwell equation). The divergence of electromagnetic potential is zero according to Lorentz gauge, and the divergence of electric current density is zero, manifesting the conservation of electric charges. A remarkable notation is that, the conservation law of charges can be derived from Maxwell equation in such a simple way that not even a coordinate system is needed, as shown in equation (36).

In this paper we have presented a brief introduction to the generalized, grade-independent vector algebra, and have demonstrated its practicality in formulating the spacetime equations of electromagnetic field. This algebra not only generalizes the beautiful symmetries of Euclidean space into relativistic spacetime, but also produces tremendous insight about the electromagnetic theory in such a simple way. This general vector algebra is also applicable to other relativistic theories.

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