Once More on Potential vs. Actual Infinity

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Abstract

The *technique* of classical mathematics involves only potential infinity, i.e. infinity is understood only as a limit. However, *the basis* of classical mathematics does involve actual infinity: the infinite ring of integers Z is the starting point for constructing infinite sets with different cardinalities, and it is not even posed a problem whether Z can be treated as a limit of finite sets. On the other hand, finite mathematics starts from the ring $R_p = (0, 1, ...p-1)$ (where all operations are modulo p) and the theory contains only finite sets. We prove that Z can be treated as a limit of R_p when $p \to \infty$ and explain that, as a consequence, finite mathematics is more fundamental than classical one.

[1]

1 Introduction

According to Wikipedia: "In the philosophy of mathematics, the abstraction of actual infinity involves the acceptance (if the axiom of infinity is included) of infinite entities, such as the set of all natural numbers or an infinite sequence of rational numbers, as given, actual, completed objects. This is contrasted with potential infinity, in which a non-terminating process (such as "add 1 to the previous number") produces a sequence with no last element, and each individual result is finite and is achieved in a finite number of steps.".

The *technique* of classical mathematics involves only potential infinity, i.e. infinity is understood only as a limit. However, *the basis* of classical mathematics does involve actual infinity: the infinite ring of integers Z is the starting point for constructing infinite sets with different cardinalities, and it is not even posed a problem whether Z can be treated as a limit of finite sets. On the other hand, by definition, finite mathematics is a branch of mathematics which contains theories involving only finite sets. In particular, those theories cannot involve even the set of all natural

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numbers because this set is infinite. Known examples are theories of finite fields and finite rings described in a vast literature.

Finite mathematics starts from the ring $R_p = (0, 1, ...p - 1)$ where all operations are modulo p. In the literature the notation Z/p for R_p is often used. We believe that this notation is not quite consistent because it might give a wrong impression that finite mathematics starts from the infinite set Z and that Z is more fundamental than R_p. However, as proved in Sec. 4, the situation is the opposite: although R_p has less elements than Z, R_p is more fundamental than Z. Then, as explained in Sec. 5, as a consequence, finite mathematics is more fundamental than classical one, and, in particular, theories with actual infinity can be only special degenerated cases of theories based on finite mathematics. We believe that to better understand the above problems it is important first to discuss in Sec. 2 philosophical aspects of such a simple problem as operations with natural numbers.

2 Remarks on arithmetic

In the 20s of the 20th century the Viennese circle of philosophers under the leadership of Schlick developed an approach called logical positivism which contains verification principle: *A proposition is only cognitively meaningful if it can be definitively and conclusively determined to be either true or false* (see e.g. Refs. [1]). On the other hand, as noted by Grayling [2], *"The general laws of science are not, even in principle, verifiable, if verifying means furnishing conclusive proof of their truth. They can be strongly supported by repeated experiments and accumulated evidence but they cannot be verified completely"*. Popper proposed the concept of falsificationism [3]: *If no cases where a claim is false can be found, then the hypothesis is accepted as provisionally true.*

According to the principles of quantum theory, there should be no statements accepted without proof and based on belief in their correctness (i.e. axioms). The theory should contain only those statements that can be verified, at least in principle, where by "verified" physicists mean experiments involving only a finite number of steps. So the philosophy of quantum theory is similar to verificationism, not falsificationism. Note that Popper was a strong opponent of quantum theory and supported Einstein in his dispute with Bohr.

The verification principle does not work in standard classical mathematics. For example, it cannot be determined whether the statement that $a + b = b + a$ for all natural numbers a and b is true or false. According to falsificationism, this statement is provisionally true until one has found some numbers a and b for which $a + b \neq b + a$. There exist different theories of arithmetic (e.g. Peano arithmetic or Robinson arithmetic) aiming to solve foundational problems of standard arithmetic. However, those theories are incomplete and are not used in applications.

From the point of view of verificationism and principles of quantum theory, classical mathematics is not well defined not only because it contains an infinite number of numbers. For example, let us pose a problem whether 10+20 equals 30.

Then one should describe an experiment which gives the answer to this problem. Any computing device can operate only with a finite amount of resources and can perform calculations only modulo some number p. Say $p = 40$, then the experiment will confirm that 10+20=30 while if $p = 25$ then one will get that 10+20=5. *So the statements that 10+20=30 and even that* $2 \cdot 2 = 4$ *are ambiguous because they do not contain information on how they should be verified.* On the other hands, the statements

$$10 + 20 = 30\,(mod\,40),\;\; 10 + 20 = 5\,(mod\,25),$$

$$2 \cdot 2 = 4\,(mod\,5),\;\; 2 \cdot 2 = 2\,(mod\,2)$$

are well defined because they do contain such an information. So, from the point of view of verificationism and principles of quantum theory, only operations modulo a number are well defined.

We believe the following observation is very important: although classical mathematics (including its constructive version) is a part of our everyday life, people typically do not realize that *classical mathematics is implicitly based on the assumption that one can have any desired amount of resources.* In other words, standard operations with natural numbers are implicitly treated as limits of operations modulo p when $p \to \infty$. Usually in mathematics, legitimacy of every limit is thoroughly investigated, but in the simplest case of standard operations with natural numbers it is not even mentioned that those operations can be treated as limits of operations modulo p. In real life such limits even might not exist if, for example, the Universe contains a finite number of elementary particles.

Classical mathematics proceeds from standard arithmetic which does not contain operations modulo a number while finite mathematics necessarily involves such operations. In the present paper we explain that, regardless of philosophical preferences, finite mathematics is more fundamental than classical one.

3 Comparison of different theories

A belief of the overwhelming majority of scientists is that classical mathematics (involving the notions of infinitely small/large and continuity) is fundamental while finite mathematics is something inferior what is used only in special applications. This belief is based on the fact that the history of mankind undoubtedly shows that classical mathematics has demonstrated its power in many areas of science.

The notions of infinitely small/large, continuity etc. were proposed by Newton and Leibniz more than 300 years ago. At that time people did not know about existence of atoms and elementary particles and believed that any body can be divided by an arbitrarily large number of arbitrarily small parts. However, now it is obvious that standard division has only a limited applicability because when we reach the level of atoms and elementary particles the division operation loses its meaning. In nature there are no infinitely small objects and no continuity because on the very fundamental level nature is discrete. So, as far as application of mathematics to physics is concerned, classical mathematics is only an approximation which in many cases works with very high accuracy but the ultimate quantum theory cannot be based on classical mathematics.

In view of those remarks, the fact that even standard quantum theory is based on continuous mathematics is not natural but it is probably a consequence of historical reasons. The founders of quantum theory were highly educated scientists but they used only classical mathematics, and even now discrete and finite mathematics is not a part of standard mathematical education at physics departments.

Theories dealing with foundation of mathematics and/or physics are called fundamental, and in the present paper we consider only such theories. In abstract mathematics there is no notion that one branch of mathematics is more fundamental than the other. For example, classical and finite mathematics are treated as fully independent theories dealing with different problems. However, in physics this notion is well known. Probably the most known example is that nonrelativistic theory (NT) can be obtained from relativistic theory (RT) in the formal limit $c \to \infty$ where c is the speed of light. RT can reproduce any result of NT with any desired accuracy if c is chosen to be sufficiently large. On the contrary, when the limit is already taken then one cannot return back from NT to RT, and NT can reproduce results of RT only in relatively small amount of cases when speeds are much less than c. Therefore RT is more fundamental than NT, and NT is a special degenerated case of RT. Other known examples are that classical theory is a special degenerated case of quantum one in the formal limit $\hbar \to 0$ where $\hbar$ is the Planck constant, and Poincare invariant theory is a special degenerated case of de Sitter invariant theories in the formal limit $R \to \infty$ where R is the parameter defining contraction from the de Sitter Lie algebras to the Poincare Lie algebra. In view of these examples, we propose the following

Definition: *Let theory A contain a finite parameter and theory B be obtained from theory A in the formal limit when the parameter goes to zero or infinity. Suppose that with any desired accuracy theory A can reproduce any result of theory B by choosing a value of the parameter. On the contrary, when the limit is already taken then one cannot return back to theory A and theory B cannot reproduce all results of theory A. Then theory A is more fundamental than theory B and theory B is a special degenerated case of theory A.*

A question arises whether **Definition** can be used for proving that finite mathematics is more fundamental than classical one. As noted above, in abstract mathematics there is no notion that one theory is more fundamental than the other. However, if in applications finite mathematics is more pertinent than classical one then, as a consequence

Main Statement: Even classical mathematics itself is a special degenerated case of finite mathematics in the formal limit when the characteristic of the field or ring in the latter goes to infinity.

In our publications (see e.g. Refs. [4, 5]) we discussed an approach called Finite Quantum Theory (FQT) where quantum theory is based not on classical but on finite mathematics. Physical states in FQT are elements of a linear space over a finite field or ring, and operators of physical quantities are linear operators in this space. It has been proved in Refs. [6, 7] that FQT is more fundamental than standard quantum theory and the latter is a special degenerated case of the former in the formal limit when the characteristic of the field or ring in FQT goes to infinity.

In other words, the problem what mathematics is the most fundamental is the problem of physics, not mathematics. However, the first stage in proving **Main Statement** is pure mathematical: one has to prove

Statement 1: The ring Z is the limit of the ring R_p when $p \to \infty$ since the result of any finite combination of additions, subtractions and multiplications in Z can be reproduced in R_p if p is chosen to be sufficiently large. On the contrary, when the limit is already taken then one cannot return back from Z to R_p, and in Z it is not possible to reproduce all results in R_p because in Z there are no operations modulo a number.

Then, according to **Definition**, *the ring R_p is more fundamental than Z, and Z is a special degenerated case of R_p.*

4 Proof of Statement 1

Since operations in R_p are modulo p, one can represent R_p as a set $\{0, \pm 1, \pm 2, ..., \pm(p-1)/2)\}$ if p is odd and as a set $\{0, \pm 1, \pm 2, ..., \pm(p/2-1), p/2\}$ if p is even. Let f be a function from R_p to Z such that $f(a)$ has the same notation in Z as a in R_p. If elements of Z are depicted as integer points on the x axis of the xy plane then, if p is odd, the elements of R_p can be depicted as points of the circumference in Fig. 1. and analogously if p is even. This picture is natural since R_p has a property that if

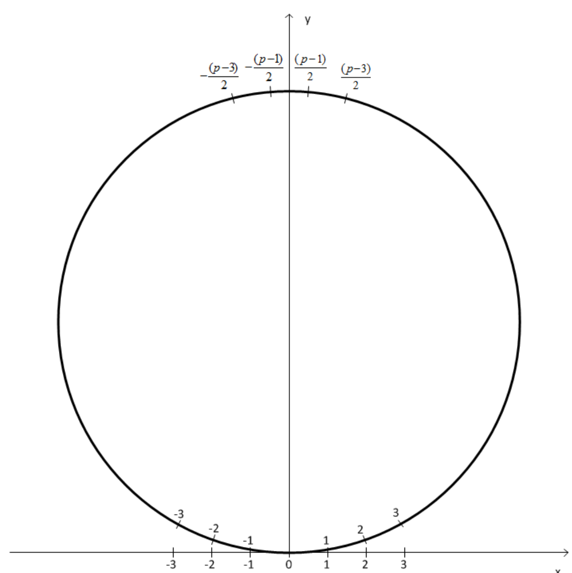

Figure 1: Relation between R_p and Z

we take any element $a \in R_p$ and sequentially add 1 then after p steps we will exhaust the whole set R_p by analogy with the property that if we move along a circumference in the same direction then sooner or later we will arrive at the initial point.

We define the function $h(p)$ such that $h(p) = (p-1)/2$ if p is odd and $h(p) = p/2 - 1$ if p is even. Let n be a natural number and $U(n)$ be a set of elements $a \in R_p$ such that $|f(a)|^n \leq h(p)$. Then $\forall m \leq n$ the result of any m operations of addition, subtraction or multiplication of elements $a \in U(n)$ is the same as for the corresponding elements $f(a)$ in Z, i.e. in this case operations modulo p are not explicitly manifested.

Let $n = g(p)$ be a function of p and $G(p)$ be a function such that the set $U(g(p))$ contains the elements $\{0, \pm 1, \pm 2, ..., \pm G(p)\}$. In what follows $M > 0$ and $n_0 > 0$ are natural numbers. If there is a sequence of natural numbers (a_n) then standard definition that $(a_n) \to \infty$ is that $\forall M\ \exists n_0$ such that $a_n \geq M\ \forall n \geq n_0$. By analogy with this definition we will now prove

Proposition: There exist functions $g(p)$ *and* $G(p)$ *such that* $\forall M\ \exists p_0 > 0$ *such that* $g(p) \geq M$ *and* $G(p) \geq 2^M\ \forall p \geq p_0$.

Proof. $\forall p > 0$ there exists a unique natural n such that $2^{n^2} \leq h(p) < 2^{(n+1)^2}$. Define $g(p) = n$ and $G(p) = 2^n$. Then $\forall M\ \exists p_0$ such that $h(p_0) \geq 2^{M^2}$. Then $\forall p \geq p_0$ the conditions of *Statement 1* are satisfied. ■

As a consequence of *Proposition* and **Definition**, *Statement 1* is valid, i.e. the ring Z is the limit of the ring R_p when $p \to \infty$ and Z is a special degenerated case of R_p.

When p is very large then $U(g(p))$ is a relatively small part of R_p, and in general the results in Z and R_p are the same only in $U(g(p))$. This is analogous to the fact mentioned in Sec. 3 that the results of NT and RT are the same only in relatively small cases when velocities are much less than c. However, when the radius of the circumference in Fig. 1 becomes infinitely large then a relatively small vicinity of zero in R_p becomes the infinite set Z when $p \to \infty$. *This example demonstrates that once we involve actual infinity and replace* R_p *by* Z *then we automatically obtain a degenerated theory because in* Z *there are no operations modulo a number.*

5 Why finite mathematics is more fundamental than classical one

As noted in Sec. 3, finite mathematics is more fundamental than classical one if finite mathematics is more pertinent for applications than classical one. Since quantum theory is the most fundamental physical theory (i.e. all other physical theories are special cases of quantum one), the answer to this question depends on whether standard quantum theory based on classical mathematics is the most fundamental or is a special degenerated case of a more general quantum theory.

In classical mathematics, the ring Z is the starting point for introducing the notions of rational, real, complex numbers etc. Therefore those notions arise from a degenerated set. Then a question arises whether the fact that R_p is more fundamental than Z implies that finite mathematics is more fundamental than classical one, i.e. whether finite mathematics can reproduce all results obtained by applications of classical mathematics. For example, if p is prime then R_p becomes the Galois field F_p, and the results in F_p considerably differ from those in the set Q of rational numbers even when p is very large. In particular, $1/2$ in F_p is a very large number $(p+1)/2$.

As noted in Sec. 3, de Sitter invariant quantum theory is more fundamental than Poincare invariant quantum theory. In the former, quantum states are described by representations of the de Sitter algebras. According to principles of quantum theory, from the ten linearly independent operators defining such representations one should construct a maximal set S of mutually commuting operators defining independent physical quantities and construct a basis in the representation space such that the basis elements are eigenvectors of the operators from S. In Secs. 4.1 and 8.2 of Ref. [7] we have proved that

Statement 2: For the de Sitter algebras there exist sets S and representations such that basis vectors in the representation spaces are eigenvectors of the operators from S with eigenvalues belonging to Z. Such representations reproduce standard representations of the Poincare algebra in the formal limit $R \to \infty$. Therefore the remaining problem is whether or not quantum theory based on finite mathematics can be a generalization of standard quantum theory where states are described by elements of a separable complex Hilbert spaces H.

Let $(e_1, e_2, ...)$ be a basis of H normalized such that the norm of each e_j is an integer. The known fact in the theory of Hilbert spaces is that *with any desired accuracy each element of H can be approximated by a finite linear combination of the basis elements with rational coefficients* because the set of such linear combinations is dense in H.

The next observation is that spaces in quantum theory are projective, i.e. for any complex number $c \neq 0$ and any element $x \in H$, x and cx describe the same state. This follows from the physical fact that not the probability itself but only ratios of probabilities have a physical meaning. In view of this property, the linear combination approximating the element $x \in H$ can be multiplied by a common denominator of all the rational coefficients in this combination. As a result, we have

Statement 3: Each element of H can be approximated by a finite linear combination with the coefficients $a_j + ib_j$ where all the numbers a_j and b_j belong to Z.

In the literature it is also considered a version of quantum theory based not on real but on p-adic numbers (see e.g. the review paper [8] and references therein). Both, the sets of real and p-adic numbers are the completions of the set of rational numbers but with respect to different metrics. Therefore the set of rational numbers is dense in both, in the set of real numbers and in the set of p-adic numbers $\mathcal{Q}_p$. In the p-adic case, the Hilbert space analog of H is the space of complex-valued functions $L^2(\mathcal{Q}_p)$ and therefore there is an analog of *Statement 3*.

We conclude that Hilbert spaces in standard quantum theory contain a big redundancy of elements. Indeed, although formally the description of states in standard quantum theory involves rational and real numbers, such numbers play only an auxiliary role because with any desired accuracy each state can be described by using only integers. Therefore, as follows from **Definition** and *Statements 1-3*,

- Standard quantum theory based on classical mathematics is a special degenerated case of quantum theory based on finite mathematics.
- **Main Statement** is valid.

6 Discussion

As noted above, the problem what mathematics is more fundamental, finite mathematics or classical one, is the problem of physics, not mathematics. However, the first step in proving that finite mathematics is more fundamental is the proof that Z is the limit of R_p when $p \to \infty$. Classical mathematics starts from the infinite ring Z and, usually in mathematics, legitimacy of every limit is thoroughly investigated. However, it is not even posed a problem whether Z can be treated as a limit of finite sets.

As noted in Sec. 4, introducing actual infinity automatically implies transition to a degenerated theory because in this case operations modulo a number are lost. Therefore *even from the pure mathematical point of view* the notion of actual infinity cannot be fundamental, and theories involving actual infinities can be only approximations of more fundamental theories.

Legitimacy of the above limit is problematic because when R_p is replaced by Z, we get classical mathematics which has foundational problems. For example, Gödel's incompleteness theorems state that no system of axioms can ensure that all facts about natural numbers can be proven and the system of axioms in classical mathematics cannot demonstrate its own consistency. The efforts of many great mathematicians to resolve those problems have not been successful yet. The philosophy of Cantor, Fraenkel, Gödel, Hilbert, Kronecker, Russell, Zermelo and other great mathematicians was based on macroscopic experience in which the notions of infinitely small, infinitely large, continuity and standard division are natural. However, as noted above, those notions contradict the existence of elementary particles and are not natural in quantum theory. The illusion of continuity arises when one neglects the discrete structure of matter.

The above construction has a known historical analogy. For many years people believed that the Earth was flat and infinite, and only after a long period of time they realized that it was finite and curved. It is difficult to notice the curvature dealing only with distances much less than the radius of the curvature. Analogously one might think that the set of numbers describing nature in our Universe has a "curvature" defined by a very large number p but we do not notice it dealing only with numbers much less than p.

In Sec. 5 we have explained that quantum theory based on finite mathematics is more fundamental than standard quantum theory and therefore classical mathematics is a special degenerated case of finite one in the formal limit $p \to \infty$. The fact that at the present stage of the Universe p is a huge number explains why in many cases classical mathematics describes natural phenomena with a very high accuracy. At the same time, as shown in Refs. [5, 7], the explanation of several phenomena can be given only in the theory where p is finite.

One of the examples is that in our approach gravity is a manifestation of the fact that p is finite. In Ref. [7] we have derived the approximate expression for the gravitational constant which depends on p as $1/lnp$. By comparing this expression with the experimental value we get that lnp is of the order of 10^{80} or more, i.e. p is a huge number of the order of $exp(10^{80})$ or more. However, since lnp is "only" of the order of 10^{80} or more, the existence of p is observable while in the formal limit $p \to \infty$ gravity disappears.

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