Derek Tucker DcannonT@gmail.com October 2019

Prime Intra Squares Conjecture (PISC): a computational approach. Using the conventional notation of *p()* denoting the least factor of the argument (used in the first proof), and subscript p_x to denote the x^{th} prime number.

We identify three levels of concreteness, *theoretical*, *possible*, and *relevant;* where theoretical results are not bound to realistic contraints, possible results are bound to one but not other constraining variables, and relavent results are constrained by existing relavent variables.

Lemma 1. At any given point of the number line, there exists exactly one *greatest* prime contributing as the least factor in composite numbers, it is the greatest prime with square less than that point.

Proof. $\forall (n, x) \in \mathbb{N}: 1 < x < p_n,\ p(p_n(p_n - x)) < p_n.$ ■.

In words, this says that for any arbitrary prime number p_n, the least factor in a number defined by the product of p_n with a number strictly between one and p_n is a prime less than p_n. Which is not contraversial.

Lemma 2. Prime factors in the primorial multiple $kp_n\#$, i.e. $k\,(2*3*5\ldots p_n)$ produce prime gaps from $kp_n\# - p_{n+1}$ to $kp_n\# - 2$, and $kp_n\# + 2$ to $kp_n\# + p_{n+1}$. For example, with n equal to three, and k equal to one, we have from 23 to 28 and 32 to 37 consecutive composites around 2*3*5 = 30. The general proof is elementary and left to the reader.

Lemma 3. All original composites greater than three appear exclusively in the inner two positions of the infinite *4-tuple* (0, *p*, *5p*, *6p*). For example, five's first two cycles are (0, 5, 25, 30) and (30, 35, 55, 60). Seven's first two cycles are (0, 7, 35, 42) and (42, 49, 77, 84). All composites with least factor greater than three appear in this fashion.

Proof. All other multiples have least factor of two or three, the elements in this cycle are *(0,* ***1x, 5x*** *6x)*, *(6x,* ***7x, 11x,*** *12x)* strictly avoid all and only multiples of two and three. ■.

We define the *theoretical* prime gap limit to be $2p_n$ which describes the diameter of a circle centered on a number of the form $k6p_n$. For example, there theoretically could occur a gap from 25 to 35 when n equals three and k equals one because there is a perimeter extending ± 5 known to be composite. The theoretical range for these gaps are primes no smaller than $p_n^2 - p_n$.

Now, for such gaps to occur requires all primes smaller than p_n to also be factors of $k6p_n$. But this leaves the positions $k6p_n \pm 1$ as prime candidates without factors, producing a twin prime, not a maximum prime gap. We see this occurs, for example, at 29 and 31. This means that for a gap the size of $2p_n$ to be possible, there must be at least two other factors in orbit available to contribute composite numbers in the $k6p_n \pm 1$ positions.

We define the *possible* prime gap limit to be $2p_{n-1}$, which describes the diameter of a circle centered on a number of the form $kp_{n-2}\#$. For example, when n equals five, and k equals four, the gap size 14 $(2p_{5-1} = 2p_4 = 2*7)$ occurs between 113 and 127 around $k = 4\,(p_{5-2}\# = p_3\# = 2*3*5 = 30) = 120$. The range for these gaps are primes no smaller than $p_n^2 - p_n$.

Now the size of $p_{n-2}\#$ grows faster than p_n^2 with increasing n. When n is six, we have 210 and 169 respectively, and when n is seven, we have 2310 and 289 respectively. This inequality is known to only increase further with increased n (Wikipedia, 2019) (Panaitopol, 2000).

We conclude that the *relevant* prime gap limit for n six or greater is always less than $2p_{n-1}$, for prime gaps beginning no earlier than $p_n^2 - p_n$. To define the theorem, we use s to denote the relevant primes squared, and attend carefully to the subscripts.

$$(1)\ \boldsymbol{Gap\ Cap\ Theorem}:$$

$$\forall n: (p_s^2 - p_s) < p_n < (p_{s+1}^2 - p_{s+1}),$$

$$(p_{n+1} - p_n) \le 2p_{s-1}.$$

Now, for the difference between consecutive square numbers, we have

$$(2)\ n^2 - (n+1)^2 = 2n + 1$$

. This is always less than the difference between consecutive prime squares.

Proof. $\forall n > 1,\ p_{n+1}^2 - p_n^2 > (p_n + 1)^2 - p_n^2 . \because p_{n+1} > p_n + 1. ■.$

Given that when the n is p_n, we now know that the prime gap upper bound is $2p_{n-1}$, clearly less than $2p_n+1$, we can confidently conclude that there is always a prime between perfect squares. If n is not prime, it is greater than the prime, hence the difference between squares is even greater than the prime gap limit, which remains $2p_{s-1}$ until $p_{s+1}^2 - p_{s+1}$.

*Table 1. Comparison of Gap Cap Theorem Upper Bounds with observed Maximal Gaps as displayed on Wikipedia.org/wiki/Prime_gap. Here starting with the second gap, *indicates the non-prime number one was used for the calculation. The Gap Cap is never violated, and cannot be violated as the empirical gaps become a mere fraction of the Gap Cap due to the exponentially growing primorial against the square of primes, a phenomenon known as Bonse inequality (Panaitopol, 2000) (Wikipedia, 2019). This fact secures the validity of the Legendre conjecture proof,*

p_s	p_s^2	$2p_{s-1}$	Max Gap? #	Size	P_n
2	4	2*	Yes 2	2	3
3	9	4	Yes 3	4	7
5	25	6	Yes 4	6	23
7	49	10	No	8	89
11	121	14	Yes 6	14	113
23	529	38	Yes 7	18	523

A prime number exists between all perfect squares (Legendre's Conjecture):

Proof. By (1) the maximum gap between primes changes around prime squares, and equals, where p_s is the local prime squared, $2p_{s-1}$. By (2), the difference between arbitrary squares is $2n + 1$. Since $n \geq p_s$ we see that the difference between arbitrary squares is less than the maximum gap between prime numbers, i.e. $2p_{s-1} < 2p_s + 1$. ■.

References

Panaitopol, L. (2000, 11). *An inequality involving Prime Numbers*. Retrieved from pefmath2etf.rs: http://pefmath2.etf.rs/files/120/901.pdf

Wikipedia. (2019, October 20). *Bonse Inequality*. Retrieved from Wikipedia: https://en.wikipedia.org/wiki/Bonse's_inequality