Using Decimals to Prove $\zeta(n \geq 2)$ is Irrational

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Abstract

With a strange and ironic twist an open number theory problem, show $\zeta(n)$ is irrational for natural numbers greater than or equal to 2, is solved with the easiest of number theory concepts: the rules of representing fractions with decimals.

Introduction

If you are like me, someone who likes irrationality proofs, you probably delighted in experiencing how repeated divisions gives

$$\frac{1}{7} = .\overline{142857},$$

a repeating decimal. Also

$$\frac{1}{6} = .1\overline{6}$$

forms a mixed decimal. Hardy [2] reviews why numbers relatively prime to the base used give repeating decimals, like $1/7$, and fractions with denominators that share some but not all prime factors with the base used form mixed decimals. We see the latter with $1/6$. This shares a factor of 2 in its denominator with a 2 factor in the base used 10. Just to exhaust the possibilities, a fraction like $1/4$ shares all prime factors of its denominator with the base 10. It has a finite decimal representation, but there's more to the story.

We could write $1/4 = .25$ with $.24\overline{9}$ or $.(24)(99)$ in base 100 where the parentheses indicate a single symbol designating the number inside. With a little thought all fractions less than 1 can be written with a single decimal in some base and also as a single decimal with a trail of $(b-1)$, where b is the base. A natural name for this is the $.\overline{9}$ phenomenon.

This phenomenon introduces a annoying ambiguity when using decimal numbers. You can't really say that there is a unique decimal representation for single digits in a base. You can say there is a unique decimal representation for fractions that are repeating and mixed because their denominators are relatively prime to the base or, like $1/6$ base 10, share some but not all prime factors with the base. These are uniquely expressed with a decimal system. The other type of number, the irrationals, are also uniquely expressed or represented by a decimal base – they never repeat and require an infinite number of decimals.

We claim we can use this here-to-fore annoyance of decimal system to good avail. We can use it to give a proof of the irrationality of

$$z_n = \zeta(2) - 1 = \sum_{k=1}^{\infty} \frac{1}{k^n} - 1.$$

We will also use the notation

$$s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

For a proofs that z_2 and z_3 are irrational see [1]. These are currently the only two cases that are known to be irrational – at least without debate. We use a few non-debatable results from the debated proof given by [3] for the general result in the title of this article.

Assumption

We assume from a previous article the following.

Definition 1.

$$D_{j^n} = \{0, 1/j^n, \ldots, (j^n - 1)/j^n\} = \{0, .1, \ldots, .(j^n - 1)\} \; base \; j^n$$

Definition 2.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

Corollary 1.

$$s_k^n \notin \Xi_k^n$$

The corollary says that partial sums are not expressible with one finite decimal when the bases are denominators of the terms used in the partial. We claim this immediately implies the irrationality of z_n.

Reasoning

If z_n was rational then for some k

$$z_n \in \Xi_k^n.$$

This forces z_n to be a repeating, mixed, or finite decimal in every base. But it can only be a single decimal with the $.\overline{9}$ phenomenon in one base. But that base will have to be given by the denominators of s_k^n and this is not possible. The base perpetually changes and exceeds any given base, as the corollary indicates. Hence it can't be given by any single decimal using the $.\overline{9}$ phenomenon. It must not be rational.

Compare z_n's situation with that of any single decimal expressed with the $.\overline{9}$ phenomenon and you will see it!

References

[1] P. Eymard and J.-P. Lafon, *The Number π*, American Mathematical Society, Providence, RI, 2004.

[2] G. H. Hardy, E. M. Wright, R. Heath-Brown, J. Silverman, and A. Wiles, *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, London, 2008.

[3] T. W. Jones, A Simple Proof of the Irrationality of $\zeta(n \geq 2)$, available at `http://vixra.org/abs/1801.0140`.