$A^x + B^y = C^z$
- Part 1: My theorem

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Abstract

Adding to the known partial results, two famous Math problems : Beal conjecture and the Fermat-Catalan conjecture are proved by one theorem - QS theorem that we propose in this article, and also means that the elementary proof of FLt has been found.

1 The theorem

theorem 1. *(General - theorem: denoted by QG theorem)*
For all positive integers n and x_i, all integers $A_i \neq \pm 1$ and $(A_1, A_2, ..., A_n) = 1$
$A_1^{x_1} + A_2^{x_2} + ... + A_{n-1}^{x_{n-1}} + A_n^{x_n} = 0 \Rightarrow \dfrac{1}{x_1} + \dfrac{1}{x_2} + ... + \dfrac{1}{x_{n-1}} + \dfrac{1}{x_n} + \dfrac{n}{LCM(x_1, x_2, ..., x_{n-1}, x_n)} > 1.$

Notes:
Sign(+) before $A_i$can be replaced with sigh (-) in any place.
LCM $(x_1, x_2, ..., x_{n-1}, x_n)$: least common multiples of $x_1, x_2, ..., x_{n-1}, x_n$.
Remark:
If we let $B_1^{x_1} + B_2^{x_2} + ... + B_{n-1}^{x_{n-1}} = B$
Multiply both sides by $B^{x_1 x_2 ... x_{n-1}}$
we have $B_1^{x_1} B^{x_1 x_2 ... x_{n-1}} + B_2^{x_2} B^{x_1 x_2 ... x_{n-1}} + ... + B_{n-1}^{x_{n-1}} B^{x_1 x_2 ... x_{n-1}} = B^{x_1 x_2 ... x_{n-1} + 1}$

$(B_1 B^{x_2 ... x_{n-1}})^{x_1} + (B_2 B^{x_1 ... x_{n-1}})^{x_2} + ... + (B_{n-1} B^{x_1 x_2 ... x_{n-2}})^{x-1} = B^{x_1 x_2 ... x_{n-1} + 1}$
clearly, if all A_i has a common factor, the equation above have solution in integer.

For n = 3, we have $A_1^{x_1} + A_2^{x_2} + A_3^{x_3} = 0$
This is the same as equation $A^x + B^y = C^z$, and we get the theorem for specific case $n = 3$ as below:

theorem 2. *(Specific - theorem: denoted by QS-theorem)*
For positive integers x, y, z, and $A, B, C \neq \pm 1$, coprime integers:
The equation $A^x + B^y = C^z \Rightarrow \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} + \dfrac{3}{LCM(x, y, z)} > 1$
In other worlds, the equation $A^x + B^y = C^z$has no solution $(A, B, C \neq \pm 1$, coprime $)$in integer if

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{3}{LCM(x, y, z)} \leqslant 1$$

Notes:
LCM (x,y,z): least common multiples of x,y and z.

As above, if A, B and C have a common factor, we always find a solution as below:
Let $a_0^x + b_0^x = c_0$ then $c_0^x a_0^x + c_0^x b_0^x = c_0^{x+1}$
so $A^x + B^x = C^{x+1}$ ($A = c_0 a_0, B = c_0 b_0, C = c_0$)
Or
Let $a_0^x + b_0^y = c_0$ then $c_0^{xy} a_0^x + c_0^{xy} b_0^y = c_0^{xy+1}$
so $A^x + B^y = C^{xy+1}$, ($A = c_0^y a_0, B = c_0^x b_0, C = c_0$

2 Consequence:

For Beal's conjecture [1]
There are no solution to the equation $A^x + B^y = C^z$ in positive integers A,B,C,x,y,z with A,B,C being pairwise coprime and all of x,y,z being greater than 2.

The cases:$A^3+B^3 = C^3$, $A^4+B^4 = C^4$, $A^3+B^3 = C^4$, $A^4+B^4 = C^3$, $A^3+B^3 = C^5$, $A^5+B^5 = C^5$(and all permutation of the ordered triple (x,y,z)) have been shown to have no solution in integer (A,B and C coprimes). By QS - theorem, Beal's conjecture is proven for other cases.That mean Beal's conjecture (including Fermat's Last theorem) is true for all positive integers $x, y, z \geqslant 3$

For Fermat- Catalan conjecture [2]
There are only finitely many triples of relatively integers A,B,C for which $A^x + B^y = C^z$ with x,y,z are positive integers satisfying $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1$

All cases of triple:$x, y, z = (2, 4, 5); (2, 3, 7); (2, 3, 8); (2, 3, 9); (2, 3, 10); (2, 5, 5); (4, 4, 4); (5, 5, 5); (3, 3, 4); (4, 4, 3); (3, 3, 5)$ (and all permutation of the ordered triple (x,y,z))satisfy :$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1 \leqslant \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{3}{LCM(x, y, z)}$ have been shown [2] to have only finitely many solutions in integer as follows:

The cases:$x, y, z = (2, 4, 5); (2, 3, 7); (2, 3, 8); (2, 3, 9)$ have some solutions in integer.

$2^5 + 7^2 = 3^4, 7^3 + 13^2 = 2^9, 2^7 + 17^3 = 71^2$,
$3^5 + 11^4 = 122^2, 17^7 + 76271^3 = 21063928^2 1414^3 + 2213459^2 = 65^7$,
$9262^3 + 15312283^2 = 113^7, 43^8 + 96222^3 = 30042907^2, 33^8 + 1549034^2 = 15613^3$

And for $1^x + B^y = C^z$, the $1^x + 2^3 = 3^2$ is the only solution where one of a,b and c is 1 [3]. The cases:$(2, 3, 10); (2, 5, 5); (4, 4, 4); (5, 5, 5); (3, 3, 4); (4, 4, 3); (3, 3, 5)$ have no solution in integer.

By QS -theorem, other cases $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{3}{LCM(x, y, z)} \leqslant 1$ have no solution in integer, so that if x,y,z satisfy $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1$, then the equation $A^x + B^y = C^z$ has only finitely many solutions in integer.

References

[1] Beal conjecture - Wikipedia

[2] Fermat Catalan conjecture - Wikipedia

[3] Catalan's conjecture - Wikipedia

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