# The Navier–Stokes problem

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A proposed solution to the millennium problem on the existence and smoothness of the Navier–Stokes equations.

## 1. Introduction

The Navier–Stokes equations are thought to govern the motion of a fluid in $\mathbb{R}^d$ where $d \in \mathbb{N}$, see [1,3]. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^d$ be the velocity and let $p = p(\mathbf{x}, t) \in \mathbb{R}$ be the pressure, each dependent on position $\mathbf{x} \in \mathbb{R}^d$ and time $t \geqslant 0$. We take the externally applied force to be identically zero. The fluid is assumed to be incompressible with constant viscosity $\nu \geqslant 0$ and to fill all of $\mathbb{R}^d$. The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \tag{3}$$

where $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}) \in \mathbb{R}^d$. In these equations

$$\nabla = (\frac{\partial}{\partial \mathbf{x}_1}, \frac{\partial}{\partial \mathbf{x}_2}, \dots, \frac{\partial}{\partial \mathbf{x}_d}) \tag{4}$$

is the gradient operator and

$$\nabla^2 = \sum_{i=1}^{d} \frac{\partial^2}{\partial {\mathbf{x}_i}^2} \tag{5}$$

is the Laplacian operator. When $\nu = 0$, equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}_0(\mathbf{x} + e_i) = \mathbf{u}_0(\mathbf{x}) \tag{6}$$

for $1 \leqslant i \leqslant d$ where $e_i$ is the $i^{\text{th}}$ unit vector in $\mathbb{R}^d$. The initial condition $\mathbf{u}_0$ is a given $C^\infty$ divergence-free vector field on $\mathbb{R}^d$. A solution of (1), (2), (3) is then accepted to be physically reasonable [3] if

$$\mathbf{u}(\mathbf{x} + e_i, t) = \mathbf{u}(\mathbf{x}, t), \quad p(\mathbf{x} + e_i, t) = p(\mathbf{x}, t) \tag{7}$$

on $\mathbb{R}^d \times [0, \infty)$ for $1 \leqslant i \leqslant d$ and

$$\mathbf{u}, p \in C^\infty(\mathbb{R}^d \times [0, \infty)). \tag{8}$$

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## 2. Solution to the Navier–Stokes problem

I provide a proof of the following theorem [2,3,6].
**Theorem**. Take $\nu > 0$. Let $\mathbf{u}_0$ be any smooth, divergence-free vector field satisfying (6). Then there exist smooth functions $\mathbf{u}, p$ on $\mathbb{R}^d \times [0, \infty)$ that satisfy (1), (2), (3), (7), (8).
**Proof**. It is sufficient to rule out the possibility that there is a smooth, divergence-free $\mathbf{u}_0$ for which (1), (2), (3) have a solution with a finite blowup time [3].
Let the Fourier series of $\mathbf{u}, p$ be

$$\tilde{\mathbf{u}} = \sum_{\mathbf{L}=-\infty}^{\infty} \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}, \tag{9}$$

$$\tilde{p} = \sum_{\mathbf{L}=-\infty}^{\infty} p_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{10}$$

respectively. Here $\mathbf{u}_{\mathbf{L}} = \mathbf{u}_{\mathbf{L}}(t) \in \mathbb{C}^d$, $p_{\mathbf{L}} = p_{\mathbf{L}}(t) \in \mathbb{C}$, $\mathrm{i} = \sqrt{-1}$, $k = 2\pi$, and $\sum_{\mathbf{L}=-\infty}^{\infty}$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^d$. The initial condition $\mathbf{u}_0$ is a Fourier series [2] of which is convergent for all $\mathbf{x} \in \mathbb{R}^d$. Substituting $\mathbf{u} = \tilde{\mathbf{u}}$, $p = \tilde{p}$ into (1) gives

$$\sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} e^{ik\mathbf{L}\cdot\mathbf{x}} + \sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}} \cdot ik\mathbf{M})\mathbf{u}_{\mathbf{M}} e^{ik(\mathbf{L}+\mathbf{M})\cdot\mathbf{x}}$$
$$= -\sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=-\infty}^{\infty} ik\mathbf{L} p_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}. \tag{11}$$

Equating like powers of the exponentials in (11) yields

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})\mathbf{u}_{\mathbf{M}} = -\nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} - ik\mathbf{L} p_{\mathbf{L}} \tag{12}$$

on using the Cauchy product type formula [4]

$$\sum_{l=-\infty}^{\infty} a_l x^l \sum_{m=-\infty}^{\infty} b_m x^m = \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_{l-m} b_m x^l. \tag{13}$$

Substituting $\mathbf{u} = \tilde{\mathbf{u}}$ into (2) gives

$$\sum_{\mathbf{L}=-\infty}^{\infty} ik\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} = 0. \tag{14}$$

Equating like powers of the exponentials in (14) yields

$$\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} = 0. \tag{15}$$

Applying $\mathbf{L}\cdot$ to (12) and noting (15) leads to

$$p_{\mathbf{L}} = -\sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \tag{16}$$

where $p_{\mathbf{0}}$ is arbitrary and $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$ is the unit vector in the direction of $\mathbf{L}$. Then substituting (16) into (12) gives

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} = -\sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \mathrm{i}k\mathbf{M})\mathbf{u}_{\mathbf{M}} - \nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} + \sum_{\mathbf{M}=-\infty}^{\infty} \mathrm{i}k\mathbf{L}(\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \tag{17}$$

where $\mathbf{u}_{\mathbf{0}} = \mathbf{u}_{\mathbf{0}}(0)$. The equations for $\mathbf{u}_{\mathbf{L}}$ are to be solved for all $\mathbf{L} \in \mathbb{Z}^d$.
Let

$$\mathbf{u}_{\mathbf{L}} = \mathbf{a}_{\mathbf{L}} + \mathrm{i}\mathbf{b}_{\mathbf{L}}, \tag{18}$$

$$p_{\mathbf{L}} = c_{\mathbf{L}} + \mathrm{i}d_{\mathbf{L}} \tag{19}$$

where $\mathbf{a}_{\mathbf{L}} \in \mathbb{R}^d$, $\mathbf{b}_{\mathbf{L}} \in \mathbb{R}^d$, $c_{\mathbf{L}} \in \mathbb{R}$, and $d_{\mathbf{L}} \in \mathbb{R}$. Substituting (18), (19) into (12) gives

$$\begin{aligned} &\frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} + \mathrm{i}\frac{\partial \mathbf{b}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} ((\mathbf{a}_{\mathbf{L}-\mathbf{M}} + \mathrm{i}\mathbf{b}_{\mathbf{L}-\mathbf{M}}) \cdot \mathrm{i}k\mathbf{M})(\mathbf{a}_{\mathbf{M}} + \mathrm{i}\mathbf{b}_{\mathbf{M}}) \\ &= -\nu k^2 |\mathbf{L}|^2 (\mathbf{a}_{\mathbf{L}} + \mathrm{i}\mathbf{b}_{\mathbf{L}}) - \mathrm{i}k\mathbf{L}(c_{\mathbf{L}} + \mathrm{i}d_{\mathbf{L}}). \end{aligned} \tag{20}$$

Equating real and imaginary parts in (20) gives

$$\frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} (-(\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}}) = -\nu k^2 |\mathbf{L}|^2 \mathbf{a}_{\mathbf{L}} + k\mathbf{L}d_{\mathbf{L}}, \tag{21}$$

$$\frac{\partial \mathbf{b}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} ((\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}}) = -\nu k^2 |\mathbf{L}|^2 \mathbf{b}_{\mathbf{L}} - k\mathbf{L}c_{\mathbf{L}}. \tag{22}$$

Substituting (18) into (15) gives

$$\mathbf{L} \cdot (\mathbf{a}_{\mathbf{L}} + \mathrm{i}\mathbf{b}_{\mathbf{L}}) = 0. \tag{23}$$

Equating real and imaginary parts in (23) gives

$$\mathbf{L} \cdot \mathbf{a}_{\mathbf{L}} = 0, \tag{24}$$

$$\mathbf{L} \cdot \mathbf{b}_{\mathbf{L}} = 0. \tag{25}$$

From (21) and in light of (24) it is possible to write

$$\frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} \cdot \hat{\mathbf{a}}_{\mathbf{L}} + \sum_{\mathbf{M}=-\infty}^{\infty} (-(\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}}) \cdot \hat{\mathbf{a}}_{\mathbf{L}} = -\nu k^2 |\mathbf{L}|^2 \mathbf{a}_{\mathbf{L}} \cdot \hat{\mathbf{a}}_{\mathbf{L}} \tag{26}$$

where $\hat{\mathbf{a}}_{\mathbf{L}} = \mathbf{a}_{\mathbf{L}}/|\mathbf{a}_{\mathbf{L}}|$ is the unit vector in the direction of $\mathbf{a}_{\mathbf{L}}$. Then (26) implies

$$\frac{\partial|\mathbf{a}_{\mathbf{L}}|}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty}(-(\mathbf{a}_{\mathbf{L}-\mathbf{M}}\cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}}\cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}})\cdot\hat{\mathbf{a}}_{\mathbf{L}} = -\nu k^2|\mathbf{L}|^2|\mathbf{a}_{\mathbf{L}}|. \tag{27}$$

From (27) it is possible to write

$$\frac{\partial|\mathbf{a}_{\mathbf{L}}|}{\partial t} \leqslant \sum_{\mathbf{M}=-\infty}^{\infty}(|\mathbf{a}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}| + |\mathbf{b}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}|) + \nu k^2|\mathbf{L}|^2|\mathbf{a}_{\mathbf{L}}| \tag{28}$$

on using the Cauchy–Schwarz inequality [5]

$$|\mathbf{a}\cdot\mathbf{b}| \leqslant |\mathbf{a}||\mathbf{b}|. \tag{29}$$

It then follows from (28) that

$$\sum_{\mathbf{L}=-\infty}^{\infty}\frac{\partial|\mathbf{a}_{\mathbf{L}}|}{\partial t}\mathrm{e}^{k|\mathbf{L}||\mathbf{x}|} \leqslant \sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{a}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}|\mathrm{e}^{k|\mathbf{L}||\mathbf{x}|}$$
$$+\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{b}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}|\mathrm{e}^{k|\mathbf{L}||\mathbf{x}|} + \sum_{\mathbf{L}=-\infty}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{a}_{\mathbf{L}}|\mathrm{e}^{k|\mathbf{L}||\mathbf{x}|} \tag{30}$$

implying that

$$\sum_{\mathbf{L}=-\infty}^{\infty}\frac{\partial|\mathbf{a}_{\mathbf{L}}|}{\partial t}\mathrm{e}^{k|\mathbf{L}||\mathbf{x}|} \leqslant \sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{a}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}|\mathrm{e}^{k|\mathbf{L}+\mathbf{M}||\mathbf{x}|}$$
$$+\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{b}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}|\mathrm{e}^{k|\mathbf{L}+\mathbf{M}||\mathbf{x}|} + \sum_{\mathbf{L}=-\infty}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{a}_{\mathbf{L}}|\mathrm{e}^{k|\mathbf{L}||\mathbf{x}|} \tag{31}$$

in light of (13), which yields

$$\sum_{\mathbf{L}=-\infty}^{\infty}\frac{\partial|\mathbf{a}_{\mathbf{L}}|}{\partial t}\mathrm{e}^{k|\mathbf{L}||\mathbf{x}|} \leqslant \sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{a}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}|\mathrm{e}^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|}$$
$$+\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{b}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}|\mathrm{e}^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|} + \sum_{\mathbf{L}=-\infty}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{a}_{\mathbf{L}}|\mathrm{e}^{k|\mathbf{L}||\mathbf{x}|} \tag{32}$$

on using the triangle inequality [5]

$$|\mathbf{a}+\mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|. \tag{33}$$

From (22) and in light of (25) it is possible to write

$$\frac{\partial\mathbf{b}_{\mathbf{L}}}{\partial t}\cdot\hat{\mathbf{b}}_{\mathbf{L}} + \sum_{\mathbf{M}=-\infty}^{\infty}((\mathbf{a}_{\mathbf{L}-\mathbf{M}}\cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}}\cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}})\cdot\hat{\mathbf{b}}_{\mathbf{L}} = -\nu k^2|\mathbf{L}|^2\mathbf{b}_{\mathbf{L}}\cdot\hat{\mathbf{b}}_{\mathbf{L}} \tag{34}$$

where $\hat{\mathbf{b}}_{\mathbf{L}} = \mathbf{b}_{\mathbf{L}}/|\mathbf{b}_{\mathbf{L}}|$ is the unit vector in the direction of $\mathbf{b}_{\mathbf{L}}$. Then (34) implies

$$\frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty}((\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}}) \cdot \hat{\mathbf{b}}_{\mathbf{L}} = -\nu k^2|\mathbf{L}|^2|\mathbf{b}_{\mathbf{L}}|. \tag{35}$$

From (35) it is possible to write

$$\frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t} \leqslant \sum_{\mathbf{M}=-\infty}^{\infty}(|\mathbf{a}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}| + |\mathbf{b}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}|) + \nu k^2|\mathbf{L}|^2|\mathbf{b}_{\mathbf{L}}| \tag{36}$$

on using the Cauchy–Schwarz inequality. It then follows from (36) that

$$\sum_{\mathbf{L}=-\infty}^{\infty}\frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t}e^{k|\mathbf{L}||\mathbf{x}|} \leqslant \sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{a}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}|e^{k|\mathbf{L}||\mathbf{x}|} + \sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{b}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}|e^{k|\mathbf{L}||\mathbf{x}|} + \sum_{\mathbf{L}=-\infty}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{b}_{\mathbf{L}}|e^{k|\mathbf{L}||\mathbf{x}|} \tag{37}$$

implying that

$$\sum_{\mathbf{L}=-\infty}^{\infty}\frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t}e^{k|\mathbf{L}||\mathbf{x}|} \leqslant \sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{a}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}|e^{k|\mathbf{L}+\mathbf{M}||\mathbf{x}|} + \sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{b}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}|e^{k|\mathbf{L}+\mathbf{M}||\mathbf{x}|} + \sum_{\mathbf{L}=-\infty}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{b}_{\mathbf{L}}|e^{k|\mathbf{L}||\mathbf{x}|} \tag{38}$$

in light of (13), which yields

$$\sum_{\mathbf{L}=-\infty}^{\infty}\frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t}e^{k|\mathbf{L}||\mathbf{x}|} \leqslant \sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{a}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}|e^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|} + \sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{b}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}|e^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|} + \sum_{\mathbf{L}=-\infty}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{b}_{\mathbf{L}}|e^{k|\mathbf{L}||\mathbf{x}|} \tag{39}$$

on using the triangle inequality.
Let

$$\psi = \sum_{\mathbf{L}=-\infty}^{\infty}|\mathbf{a}_{\mathbf{L}}|e^{k|\mathbf{L}|X}, \tag{40}$$

$$\phi = \sum_{\mathbf{L}=-\infty}^{\infty}|\mathbf{b}_{\mathbf{L}}|e^{k|\mathbf{L}|X} \tag{41}$$

where $X = |\mathbf{x}|$ and note that

$$|\tilde{\mathbf{u}}| \leqslant Q \tag{42}$$

where $Q = \psi + \phi$. Then (32) can be written as

$$\frac{\partial \psi}{\partial t} \leqslant \psi \frac{\partial \phi}{\partial X} + \phi \frac{\partial \psi}{\partial X} + \nu \frac{\partial^2 \psi}{\partial X^2} \tag{43}$$

and (39) can be written as

$$\frac{\partial \phi}{\partial t} \leqslant \psi \frac{\partial \psi}{\partial X} + \phi \frac{\partial \phi}{\partial X} + \nu \frac{\partial^2 \phi}{\partial X^2}. \tag{44}$$

Adding (43) and (44) yields

$$\frac{\partial Q}{\partial t} \leqslant Q \frac{\partial Q}{\partial X} + \nu \frac{\partial^2 Q}{\partial X^2}. \tag{45}$$

Here $Q|_{t=0}$ converges for all $X \geqslant 0$ since $\tilde{\mathbf{u}}|_{t=0}$ converges for all $\mathbf{x} \in \mathbb{R}^d$. Note also that

$$\frac{\partial^s Q}{\partial X^s} \geqslant 0 \quad \text{for} \quad s \geqslant 0. \tag{46}$$

At points where $Q$ is a maximum,

$$\frac{\partial Q}{\partial t} \geqslant 0. \tag{47}$$

The extreme case is then $Q = \Omega$ where

$$\frac{\partial \Omega}{\partial t} = \Omega \frac{\partial \Omega}{\partial X} + \nu \frac{\partial^2 \Omega}{\partial X^2}. \tag{48}$$

Let

$$\Omega = \lambda \frac{\partial A}{\partial X} / A = \lambda \frac{\partial}{\partial X} \log_{\mathrm{e}} A \tag{49}$$

where $\lambda$ is a constant. Substituting (49) into (48) gives

$$\lambda \frac{\partial}{\partial X} (\frac{\partial A}{\partial t} / A) = \lambda^2 \frac{1}{2} \frac{\partial}{\partial X} ((\frac{\partial A}{\partial X} / A)^2) + \lambda \nu \frac{\partial}{\partial X} ((\frac{\partial^2 A}{\partial X^2} A - (\frac{\partial A}{\partial X})^2) / A^2). \tag{50}$$

Then with $\lambda = 2\nu$, equation (50) gives

$$\frac{\partial}{\partial X} (\frac{\partial A}{\partial t} / A) = \nu \frac{\partial}{\partial X} (\frac{\partial^2 A}{\partial X^2} / A) \tag{51}$$

which leads to

$$\frac{\partial A}{\partial t} = \nu \frac{\partial^2 A}{\partial X^2} + hA \tag{52}$$

where $h = h(t)$ is arbitrary.
Let

$$A = \sum_{l=0}^{\infty} A_l \mathrm{e}^{klX} \tag{53}$$

where $A_l = A_l(t)$. Substituting (53) into (52) gives

$$\sum_{l=0}^{\infty} \frac{\partial A_l}{\partial t} \mathrm{e}^{klX} = \sum_{l=0}^{\infty} \nu k^2 l^2 A_l \mathrm{e}^{klX} + \sum_{l=0}^{\infty} A_l h \mathrm{e}^{klX}. \tag{54}$$

Equating like powers of the exponentials in (54) yields

$$\frac{\partial A_l}{\partial t} = \nu k^2 l^2 A_l + A_l h. \tag{55}$$

Equation (55) is easily solved to find

$$A_l = c_l \mathrm{e}^{\nu k^2 l^2 t + \int h\, dt} \tag{56}$$

where $c_l$ are constants. It then follows that

$$\Omega = 2\nu \frac{\partial}{\partial X} \log_{\mathrm{e}}\Big(\sum_{l=0}^{\infty} c_l \mathrm{e}^{\nu k^2 l^2 t} \mathrm{e}^{klX}\Big). \tag{57}$$

It is sufficient to take $X$ to be in a finite domain due to the spatially periodic boundary conditions. Then $\Omega$ has no finite-time singularity and

$$|\tilde{\mathbf{u}}| \leqslant \Omega. \tag{58}$$

$\therefore$ blowup is ruled out. □

## References


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