

The Navier–Stokes problem

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A proposed solution to the millennium problem on the existence and smoothness of the Navier–Stokes equations.

1. Introduction

The Navier–Stokes equations are thought to govern the motion of a fluid in \mathbb{R}^d where $d \in \mathbb{N}$, see [1,3,7]. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^d$ be the velocity and let $p = p(\mathbf{x}, t) \in \mathbb{R}$ be the pressure, each dependent on position $\mathbf{x} \in \mathbb{R}^d$ and time $t \geq 0$. I take the externally applied force to be identically zero. The fluid is assumed to be incompressible with constant viscosity $\nu \geq 0$ and to fill all of \mathbb{R}^d . The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \quad (3)$$

where $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}) \in \mathbb{R}^d$. In these equations

$$\nabla = \left(\frac{\partial}{\partial \mathbf{x}_1}, \frac{\partial}{\partial \mathbf{x}_2}, \dots, \frac{\partial}{\partial \mathbf{x}_d} \right) \quad (4)$$

is the gradient operator and

$$\nabla^2 = \sum_{i=1}^d \frac{\partial^2}{\partial \mathbf{x}_i^2} \quad (5)$$

is the Laplacian operator. When $\nu = 0$, equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}_0(\mathbf{x} + Le_i) = \mathbf{u}_0(\mathbf{x}) \quad (6)$$

for $1 \leq i \leq d$ where e_i is the i^{th} unit vector in \mathbb{R}^d and $L > 0$ is a constant [7]. The initial condition \mathbf{u}_0 is a given C^∞ divergence-free vector field on \mathbb{R}^d . A solution of (1), (2), (3) is then accepted to be physically reasonable [3] if

$$\mathbf{u}(\mathbf{x} + Le_i, t) = \mathbf{u}(\mathbf{x}, t), \quad p(\mathbf{x} + Le_i, t) = p(\mathbf{x}, t) \quad (7)$$

on $\mathbb{R}^d \times [0, \infty)$ for $1 \leq i \leq d$ and

$$\mathbf{u}, p \in C^\infty(\mathbb{R}^d \times [0, \infty)). \quad (8)$$

2. Solution to the Navier–Stokes problem

I provide a proof of the following theorem [2,3,6,7].

Theorem. Take $\nu > 0$. Let \mathbf{u}_0 be any smooth, divergence-free vector field satisfying (6). Then there exist smooth functions \mathbf{u}, p on $\mathbb{R}^d \times [0, \infty)$ that satisfy (1), (2), (3), (7), (8).

Proof. It is sufficient to rule out the possibility that there is a smooth, divergence-free \mathbf{u}_0 for which (1), (2), (3) have a solution with a finite blowup time [3].

Let the Fourier series of \mathbf{u}, p be

$$\tilde{\mathbf{u}} = \sum_{\mathbf{L}=-\infty}^{\infty} \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}, \quad (9)$$

$$\tilde{p} = \sum_{\mathbf{L}=-\infty}^{\infty} p_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} \quad (10)$$

respectively. Here $\mathbf{u}_{\mathbf{L}} = \mathbf{u}_{\mathbf{L}}(t) \in \mathbb{C}^d$, $p_{\mathbf{L}} = p_{\mathbf{L}}(t) \in \mathbb{C}$, $i = \sqrt{-1}$, $k = 2\pi/L$, and $\sum_{\mathbf{L}=-\infty}^{\infty}$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^d$. The initial condition \mathbf{u}_0 is a Fourier series [2] of which is convergent for all $\mathbf{x} \in \mathbb{R}^d$. Substituting $\mathbf{u} = \tilde{\mathbf{u}}, p = \tilde{p}$ into (1) gives

$$\begin{aligned} & \sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} e^{ik\mathbf{L}\cdot\mathbf{x}} + \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}} \cdot ik\mathbf{M}) \mathbf{u}_{\mathbf{M}} e^{ik(\mathbf{L}+\mathbf{M})\cdot\mathbf{x}} \\ &= - \sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=-\infty}^{\infty} ik\mathbf{L} p_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}. \end{aligned} \quad (11)$$

Equating like powers of the exponentials in (11) yields

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M}) \mathbf{u}_{\mathbf{M}} = -\nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} - ik\mathbf{L} p_{\mathbf{L}} \quad (12)$$

on using the Cauchy product type formula [4]

$$\sum_{l=-\infty}^{\infty} a_l x^l \sum_{m=-\infty}^{\infty} b_m x^m = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{l-m} b_m x^l. \quad (13)$$

Substituting $\mathbf{u} = \tilde{\mathbf{u}}$ into (2) gives

$$\sum_{\mathbf{L}=-\infty}^{\infty} ik\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} = 0. \quad (14)$$

Equating like powers of the exponentials in (14) yields

$$\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} = 0. \quad (15)$$

Applying $\mathbf{L} \cdot$ to (12) and noting (15) leads to

$$p_{\mathbf{L}} = - \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \quad (16)$$

where p_0 is arbitrary and $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$ is the unit vector in the direction of \mathbf{L} . Then substituting (16) into (12) gives

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} = - \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})\mathbf{u}_{\mathbf{M}} - \nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} + \sum_{\mathbf{M}=-\infty}^{\infty} ik\mathbf{L}(\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \quad (17)$$

where $\mathbf{u}_0 = \mathbf{u}_0(0)$. Without loss of generality [2], I take $\mathbf{u}_0 = \mathbf{0}$. The equations for $\mathbf{u}_{\mathbf{L}}$ are to be solved for all $\mathbf{L} \in \mathbb{Z}^d$.

Let

$$\mathbf{u}_{\mathbf{L}} = \mathbf{a}_{\mathbf{L}} + i\mathbf{b}_{\mathbf{L}}, \quad (18)$$

$$p_{\mathbf{L}} = c_{\mathbf{L}} + id_{\mathbf{L}} \quad (19)$$

where $\mathbf{a}_{\mathbf{L}} = \mathbf{a}_{\mathbf{L}}(t) \in \mathbb{R}^d$, $\mathbf{b}_{\mathbf{L}} = \mathbf{b}_{\mathbf{L}}(t) \in \mathbb{R}^d$, $c_{\mathbf{L}} = c_{\mathbf{L}}(t) \in \mathbb{R}$, and $d_{\mathbf{L}} = d_{\mathbf{L}}(t) \in \mathbb{R}$. Substituting (18), (19) into (12) gives

$$\begin{aligned} & \frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} + i \frac{\partial \mathbf{b}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} ((\mathbf{a}_{\mathbf{L}-\mathbf{M}} + i\mathbf{b}_{\mathbf{L}-\mathbf{M}}) \cdot ik\mathbf{M})(\mathbf{a}_{\mathbf{M}} + i\mathbf{b}_{\mathbf{M}}) \\ & = -\nu k^2 |\mathbf{L}|^2 (\mathbf{a}_{\mathbf{L}} + i\mathbf{b}_{\mathbf{L}}) - ik\mathbf{L}(c_{\mathbf{L}} + id_{\mathbf{L}}). \end{aligned} \quad (20)$$

Equating real and imaginary parts in (20) gives

$$\frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} (-\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}} = -\nu k^2 |\mathbf{L}|^2 \mathbf{a}_{\mathbf{L}} + k\mathbf{L}d_{\mathbf{L}}, \quad (21)$$

$$\frac{\partial \mathbf{b}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} ((\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}}) = -\nu k^2 |\mathbf{L}|^2 \mathbf{b}_{\mathbf{L}} - k\mathbf{L}c_{\mathbf{L}}. \quad (22)$$

Substituting (18) into (15) gives

$$\mathbf{L} \cdot (\mathbf{a}_{\mathbf{L}} + i\mathbf{b}_{\mathbf{L}}) = 0. \quad (23)$$

Equating real and imaginary parts in (23) gives

$$\mathbf{L} \cdot \mathbf{a}_{\mathbf{L}} = 0, \quad (24)$$

$$\mathbf{L} \cdot \mathbf{b}_{\mathbf{L}} = 0. \quad (25)$$

From (21) and in light of (24) it is possible to write

$$\frac{\partial \mathbf{a}_L}{\partial t} \cdot \hat{\mathbf{a}}_L + \sum_{M=-\infty}^{\infty} (-\mathbf{a}_{L-M} \cdot k\mathbf{M})\mathbf{b}_M - (\mathbf{b}_{L-M} \cdot k\mathbf{M})\mathbf{a}_M \cdot \hat{\mathbf{a}}_L = -\nu k^2 |\mathbf{L}|^2 \mathbf{a}_L \cdot \hat{\mathbf{a}}_L \quad (26)$$

where $\hat{\mathbf{a}}_L = \mathbf{a}_L/|\mathbf{a}_L|$ is the unit vector in the direction of \mathbf{a}_L . Then (26) implies

$$\frac{\partial |\mathbf{a}_L|}{\partial t} + \sum_{M=-\infty}^{\infty} (-\mathbf{a}_{L-M} \cdot k\mathbf{M})\mathbf{b}_M - (\mathbf{b}_{L-M} \cdot k\mathbf{M})\mathbf{a}_M \cdot \hat{\mathbf{a}}_L = -\nu k^2 |\mathbf{L}|^2 |\mathbf{a}_L|. \quad (27)$$

From (27) it is possible to write

$$\frac{\partial |\mathbf{a}_L|}{\partial t} \leq \sum_{M=-\infty}^{\infty} (|\mathbf{a}_{L-M}|k|\mathbf{M}||\mathbf{b}_M| + |\mathbf{b}_{L-M}|k|\mathbf{M}||\mathbf{a}_M|) - \nu k^2 |\mathbf{L}|^2 |\mathbf{a}_L| \quad (28)$$

on using the Cauchy–Schwarz inequality [5]

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|. \quad (29)$$

It then follows from (28) that

$$\begin{aligned} \sum_{L=-\infty}^{\infty} \frac{\partial |\mathbf{a}_L|}{\partial t} e^{k|\mathbf{L}||\mathbf{x}|} &\leq \sum_{L=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} |\mathbf{a}_{L-M}|k|\mathbf{M}||\mathbf{b}_M| e^{k|\mathbf{L}||\mathbf{x}|} \\ &+ \sum_{L=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} |\mathbf{b}_{L-M}|k|\mathbf{M}||\mathbf{a}_M| e^{k|\mathbf{L}||\mathbf{x}|} - \sum_{L=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{a}_L| e^{k|\mathbf{L}||\mathbf{x}|} \end{aligned} \quad (30)$$

implying that

$$\begin{aligned} \sum_{L=-\infty}^{\infty} \frac{\partial |\mathbf{a}_L|}{\partial t} e^{k|\mathbf{L}||\mathbf{x}|} &\leq \sum_{L=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} |\mathbf{a}_L|k|\mathbf{M}||\mathbf{b}_M| e^{k|\mathbf{L}+\mathbf{M}||\mathbf{x}|} \\ &+ \sum_{L=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} |\mathbf{b}_L|k|\mathbf{M}||\mathbf{a}_M| e^{k|\mathbf{L}+\mathbf{M}||\mathbf{x}|} - \sum_{L=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{a}_L| e^{k|\mathbf{L}||\mathbf{x}|} \end{aligned} \quad (31)$$

in light of (13), which yields

$$\begin{aligned} \sum_{L=-\infty}^{\infty} \frac{\partial |\mathbf{a}_L|}{\partial t} e^{k|\mathbf{L}||\mathbf{x}|} &\leq \sum_{L=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} |\mathbf{a}_L|k|\mathbf{M}||\mathbf{b}_M| e^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|} \\ &+ \sum_{L=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} |\mathbf{b}_L|k|\mathbf{M}||\mathbf{a}_M| e^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|} - \sum_{L=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{a}_L| e^{k|\mathbf{L}||\mathbf{x}|} \end{aligned} \quad (32)$$

on using the triangle inequality [5]

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|. \quad (33)$$

From (22) and in light of (25) it is possible to write

$$\frac{\partial \mathbf{b}_L}{\partial t} \cdot \hat{\mathbf{b}}_L + \sum_{M=-\infty}^{\infty} ((\mathbf{a}_{L-M} \cdot k\mathbf{M})\mathbf{a}_M - (\mathbf{b}_{L-M} \cdot k\mathbf{M})\mathbf{b}_M) \cdot \hat{\mathbf{b}}_L = -\nu k^2 |\mathbf{L}|^2 \mathbf{b}_L \cdot \hat{\mathbf{b}}_L \quad (34)$$

where $\hat{\mathbf{b}}_L = \mathbf{b}_L / |\mathbf{b}_L|$ is the unit vector in the direction of \mathbf{b}_L . Then (34) implies

$$\frac{\partial |\mathbf{b}_L|}{\partial t} + \sum_{M=-\infty}^{\infty} ((\mathbf{a}_{L-M} \cdot k\mathbf{M})\mathbf{a}_M - (\mathbf{b}_{L-M} \cdot k\mathbf{M})\mathbf{b}_M) \cdot \hat{\mathbf{b}}_L = -\nu k^2 |\mathbf{L}|^2 |\mathbf{b}_L|. \quad (35)$$

From (35) it is possible to write

$$\frac{\partial |\mathbf{b}_L|}{\partial t} \leq \sum_{M=-\infty}^{\infty} (|\mathbf{a}_{L-M}|k|\mathbf{M}||\mathbf{a}_M| + |\mathbf{b}_{L-M}|k|\mathbf{M}||\mathbf{b}_M|) - \nu k^2 |\mathbf{L}|^2 |\mathbf{b}_L| \quad (36)$$

on using the Cauchy–Schwarz inequality. It then follows from (36) that

$$\begin{aligned} \sum_{L=-\infty}^{\infty} \frac{\partial |\mathbf{b}_L|}{\partial t} e^{k|\mathbf{L}||\mathbf{x}|} &\leq \sum_{L=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} |\mathbf{a}_{L-M}|k|\mathbf{M}||\mathbf{a}_M| e^{k|\mathbf{L}||\mathbf{x}|} \\ &+ \sum_{L=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} |\mathbf{b}_{L-M}|k|\mathbf{M}||\mathbf{b}_M| e^{k|\mathbf{L}||\mathbf{x}|} - \sum_{L=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{b}_L| e^{k|\mathbf{L}||\mathbf{x}|} \end{aligned} \quad (37)$$

implying that

$$\begin{aligned} \sum_{L=-\infty}^{\infty} \frac{\partial |\mathbf{b}_L|}{\partial t} e^{k|\mathbf{L}||\mathbf{x}|} &\leq \sum_{L=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} |\mathbf{a}_L|k|\mathbf{M}||\mathbf{a}_M| e^{k(|\mathbf{L}+|\mathbf{M}||\mathbf{x}|)} \\ &+ \sum_{L=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} |\mathbf{b}_L|k|\mathbf{M}||\mathbf{b}_M| e^{k(|\mathbf{L}+|\mathbf{M}||\mathbf{x}|)} - \sum_{L=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{b}_L| e^{k|\mathbf{L}||\mathbf{x}|} \end{aligned} \quad (38)$$

in light of (13), which yields

$$\begin{aligned} \sum_{L=-\infty}^{\infty} \frac{\partial |\mathbf{b}_L|}{\partial t} e^{k|\mathbf{L}||\mathbf{x}|} &\leq \sum_{L=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} |\mathbf{a}_L|k|\mathbf{M}||\mathbf{a}_M| e^{k(|\mathbf{L}+|\mathbf{M}||\mathbf{x}|)} \\ &+ \sum_{L=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} |\mathbf{b}_L|k|\mathbf{M}||\mathbf{b}_M| e^{k(|\mathbf{L}+|\mathbf{M}||\mathbf{x}|)} - \sum_{L=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{b}_L| e^{k|\mathbf{L}||\mathbf{x}|} \end{aligned} \quad (39)$$

on using the triangle inequality.

Let

$$\psi = \sum_{L=-\infty}^{\infty} |\mathbf{a}_L| e^{k|\mathbf{L}||\mathbf{x}|}, \quad (40)$$

$$\phi = \sum_{\mathbf{L}=-\infty}^{\infty} |\mathbf{b}_{\mathbf{L}}| e^{k|\mathbf{L}|X} \quad (41)$$

where $X = |\mathbf{x}|$ and note that $|\tilde{\mathbf{u}}| \leq Q$ where $Q = \psi + \phi$. Then (32) can be written as

$$\frac{\partial \psi}{\partial t} \leq \psi \frac{\partial \phi}{\partial X} + \phi \frac{\partial \psi}{\partial X} - v \frac{\partial^2 \psi}{\partial X^2} \quad (42)$$

and (39) can be written as

$$\frac{\partial \phi}{\partial t} \leq \psi \frac{\partial \psi}{\partial X} + \phi \frac{\partial \phi}{\partial X} - v \frac{\partial^2 \phi}{\partial X^2}. \quad (43)$$

Adding (42) and (43) yields

$$\frac{\partial Q}{\partial t} \leq Q \frac{\partial Q}{\partial X} - v \frac{\partial^2 Q}{\partial X^2}. \quad (44)$$

Equation (44) can be written as

$$\frac{\partial Q}{\partial t} - Q \frac{\partial Q}{\partial X} + v \frac{\partial^2 Q}{\partial X^2} = H \quad (45)$$

where $H = H(X, t) \leq 0$. Here $Q|_{t=0}$ converges for all $X \geq 0$ since $\tilde{\mathbf{u}}|_{t=0}$ converges for all $\mathbf{x} \in \mathbb{R}^d$. Note also that

$$\frac{\partial^s Q}{\partial X^s} \geq 0 \text{ for } s \geq 0. \quad (46)$$

At points where Q is a maximum,

$$\frac{\partial Q}{\partial t} \geq 0. \quad (47)$$

The extreme case is then $Q = \Omega$ where

$$\frac{\partial \Omega}{\partial t} = \Omega \frac{\partial \Omega}{\partial X} - v \frac{\partial^2 \Omega}{\partial X^2}. \quad (48)$$

Let

$$\Omega = \lambda \frac{\partial A}{\partial X} / A = \lambda \frac{\partial}{\partial X} \log_e A \quad (49)$$

where λ is a constant. Substituting (49) into (48) gives

$$\lambda \frac{\partial}{\partial X} \left(\frac{\partial A}{\partial t} / A \right) = \lambda^2 \frac{1}{2} \frac{\partial}{\partial X} \left(\left(\frac{\partial A}{\partial X} / A \right)^2 \right) - \lambda v \frac{\partial}{\partial X} \left(\left(\frac{\partial^2 A}{\partial X^2} A - \left(\frac{\partial A}{\partial X} \right)^2 \right) / A^2 \right). \quad (50)$$

Then with $\lambda = -2\nu$, equation (50) gives

$$\frac{\partial}{\partial X} \left(\frac{\partial A}{\partial t} / A \right) = -\nu \frac{\partial}{\partial X} \left(\frac{\partial^2 A}{\partial X^2} / A \right) \quad (51)$$

which leads to

$$\frac{\partial A}{\partial t} = -\nu \frac{\partial^2 A}{\partial X^2} + hA \quad (52)$$

where $h = h(t)$ is arbitrary.

Let

$$A = \sum_{\mathbf{L}=-\infty}^{\infty} A_{\mathbf{L}} e^{k|\mathbf{L}|X} \quad (53)$$

where $A_{\mathbf{L}} = A_{\mathbf{L}}(t)$. Substituting (53) into (52) gives

$$\sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial A_{\mathbf{L}}}{\partial t} e^{k|\mathbf{L}|X} = -\nu \sum_{\mathbf{L}=-\infty}^{\infty} k^2 |\mathbf{L}|^2 A_{\mathbf{L}} e^{k|\mathbf{L}|X} + h \sum_{\mathbf{L}=-\infty}^{\infty} A_{\mathbf{L}} e^{k|\mathbf{L}|X}. \quad (54)$$

Equating like powers of the exponentials in (54) leads to

$$\frac{\partial A_{\mathbf{L}}}{\partial t} = -\nu k^2 |\mathbf{L}|^2 A_{\mathbf{L}} + A_{\mathbf{L}} h. \quad (55)$$

Equation (55) is easily solved to find

$$A_{\mathbf{L}} = A_{\mathbf{L}}(0) e^{-\nu k^2 |\mathbf{L}|^2 t + \int_0^t h(\tau) d\tau}. \quad (56)$$

It then follows that

$$\Omega = \frac{\partial}{\partial X} \log_e \left(\left(\sum_{\mathbf{L}=-\infty}^{\infty} A_{\mathbf{L}}(0) e^{-\nu k^2 |\mathbf{L}|^2 t} e^{k|\mathbf{L}|X} \right)^{-2\nu} \right). \quad (57)$$

Now with

$$\Omega = \sum_{\mathbf{L}=-\infty}^{\infty} \Omega_{\mathbf{L}} e^{k|\mathbf{L}|X}, \quad \Omega_0 = 0 \quad (58)$$

where $\Omega_{\mathbf{L}} = \Omega_{\mathbf{L}}(t) \geq 0$ it follows that

$$\begin{aligned} A|_{t=0} &= e^{\int^X \frac{\Omega}{\lambda} dX} \Big|_{t=0} \\ &= e^{\frac{1}{\lambda} \sum_{\mathbf{L} \neq 0} \frac{\Omega_{\mathbf{L}}(0) e^{k|\mathbf{L}|X}}{k|\mathbf{L}|}} \\ &= 1 + \frac{1}{\lambda} \sum_{\mathbf{L} \neq 0} \frac{\Omega_{\mathbf{L}}(0) e^{k|\mathbf{L}|X}}{k|\mathbf{L}|} + \frac{1}{2} \left(\frac{1}{\lambda} \sum_{\mathbf{L} \neq 0} \frac{\Omega_{\mathbf{L}}(0) e^{k|\mathbf{L}|X}}{k|\mathbf{L}|} \right)^2 + \dots \end{aligned} \quad (59)$$

For consistency, matching (53) with (59) yields

$$A_{\mathbf{0}}(0) = 1, \quad A_{\mathbf{L}}(0) = \frac{\Omega_{\mathbf{L}}(0)}{\lambda k |\mathbf{L}|} + O\left(\frac{1}{\lambda^2 k^2}\right) \text{ for } \mathbf{L} \neq \mathbf{0}. \quad (60)$$

Then (57) becomes

$$\Omega = \frac{\partial}{\partial X} \log_e(A^\lambda) \quad (61)$$

where

$$A = 1 + \sum_{\mathbf{L} \neq \mathbf{0}} \left(\frac{\Omega_{\mathbf{L}}(0)}{\lambda k |\mathbf{L}|} + O\left(\frac{1}{\lambda^2 k^2}\right) \right) e^{-\nu k^2 |\mathbf{L}|^2 t} e^{k |\mathbf{L}| X} = e^{\int^X \frac{\Omega}{\lambda} dX}. \quad (62)$$

Equation (62) can be written as

$$\begin{aligned} A &= 1 + \left\{ \frac{1}{2} \left(\frac{1}{\lambda k} \right)^2 \sum_{\mathbf{L} \neq \mathbf{0}} \sum_{\mathbf{M} \neq \mathbf{0}} \frac{\Omega_{\mathbf{L}}(0) \Omega_{\mathbf{M}}(0)}{|\mathbf{L}| |\mathbf{M}|} e^{k(|\mathbf{L}|+|\mathbf{M}|)X} e^{-\nu k^2 (|\mathbf{L}|+|\mathbf{M}|)^2 t} \right. \\ &\quad + \frac{1}{24} \left(\frac{1}{\lambda k} \right)^4 \sum_{\mathbf{L} \neq \mathbf{0}} \sum_{\mathbf{M} \neq \mathbf{0}} \sum_{\mathbf{N} \neq \mathbf{0}} \sum_{\mathbf{P} \neq \mathbf{0}} \frac{\Omega_{\mathbf{L}}(0) \Omega_{\mathbf{M}}(0) \Omega_{\mathbf{N}}(0) \Omega_{\mathbf{P}}(0)}{|\mathbf{L}| |\mathbf{M}| |\mathbf{N}| |\mathbf{P}|} \\ &\quad \times e^{k(|\mathbf{L}|+|\mathbf{M}|+|\mathbf{N}|+|\mathbf{P}|)X} e^{-\nu k^2 (|\mathbf{L}|+|\mathbf{M}|+|\mathbf{N}|+|\mathbf{P}|)^2 t} + \dots \left. \right\} + \left\{ \left(\frac{1}{\lambda k} \right) \sum_{\mathbf{L} \neq \mathbf{0}} \frac{\Omega_{\mathbf{L}}(0)}{|\mathbf{L}|} e^{k |\mathbf{L}| X} e^{-\nu k^2 |\mathbf{L}|^2 t} \right. \\ &\quad + \frac{1}{6} \left(\frac{1}{\lambda k} \right)^3 \sum_{\mathbf{L} \neq \mathbf{0}} \sum_{\mathbf{M} \neq \mathbf{0}} \sum_{\mathbf{N} \neq \mathbf{0}} \frac{\Omega_{\mathbf{L}}(0) \Omega_{\mathbf{M}}(0) \Omega_{\mathbf{N}}(0)}{|\mathbf{L}| |\mathbf{M}| |\mathbf{N}|} e^{k(|\mathbf{L}|+|\mathbf{M}|+|\mathbf{N}|)X} e^{-\nu k^2 (|\mathbf{L}|+|\mathbf{M}|+|\mathbf{N}|)^2 t} \\ &\quad \left. + \dots \right\}. \quad (63) \end{aligned}$$

In light of (63) and due to $A \in [0, 1]$ it is then clear that A increases with increasing $t \geq 0$. Note that it is sufficient to take X to be in a finite domain due to the spatially periodic boundary conditions. It then follows that Ω has no finite-time singularity and $|\tilde{\mathbf{u}}| \leq \Omega$. \therefore blowup is ruled out. \square

References

- [1] Batchelor G. 1967. *An introduction to fluid dynamics*. Cambridge U. Press, Cambridge.
- [2] Doering C. 2009. The 3D Navier–Stokes problem. *Annu. Rev. Fluid Mech.* **41**: 109–128.
- [3] Fefferman C. 2000. Existence and smoothness of the Navier–Stokes equation. *Clay Mathematics Institute*. Official problem description.
- [4] Hardy G. 1949. *Divergent series*. Oxford University Press.
- [5] Kreyszig E. 1989. *Introductory functional analysis with applications*. Wiley Classics Library.
- [6] Ladyzhenskaya O. 1969. *The mathematical theory of viscous incompressible flows*. Gordon and Breach, New York.
- [7] Tao T. 2013. Localisation and compactness properties of the Navier–Stokes global regularity problem. *Analysis and PDE*. **6**: 25–107.