Factoring ANY Second Order Linear Ordinary Differential Equation

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Any Second Order Linear Ordinary Differential Equation may be factored via two linear differential operators.
The initial theorem demonstrates that for homogeneous 2nd order LODEs.
Theorems that follow improve by supplying the solution of such factorization, extending the theorem to inhomogeneous LODEs, and further providing a generalization which includes the constant coefficients and Cauchy-Euler LODEs; culminating in a single-parameter LODE solution formula with suggested usage examples.

Theorem I.1: Any Second Order Homogeneous Linear Ordinary Differential Equation may be factored via two linear differential operators.

Proof:

From the reduction of order formula:

$$y_1'' + Py_1' + Qy_1 = 0 \Rightarrow y_2'' + Py_2' + Qy_2 = 0 \quad , \quad \left(y_2 = y_1 \int y_1^{-2} e^{-\int P dx} dx \right)$$

Now, under the transformation: $y_1 = e^{\int s dx} \Leftrightarrow s = (\log y_1)'$:

$$y_2 = e^{\int s dx} \int e^{-2\int s dx} e^{\int P dx} dx = e^{\int s dx} \int e^{-\int (2s+P) dx} dx$$

So, let: $g = -s$

$$\Rightarrow y_2 = e^{-\int g dx} \int e^{-\int (-2g+P) dx} dx$$

Define: $h \equiv P - g \Rightarrow P = h + g$

$$\Rightarrow y_2 = e^{-\int g dx} \int e^{-\int (-2g+h+g) dx} dx = e^{-\int g dx} \int e^{\int (g-h) dx} dx$$
$$= e^{-\int g dx} \int e^{\int g dx} \left(e^{-\int h dx} \right) dx$$
$$\Rightarrow y_2 e^{\int g dx} = \int e^{\int g dx} \left(e^{-\int h dx} \right) dx$$
$$\Rightarrow \left(y_2 e^{\int g dx} \right)' = e^{\int g dx} \left(e^{-\int h dx} \right)$$
$$\Rightarrow e^{-\int g dx} \left(y_2 e^{\int g dx} \right)' = e^{-\int h dx}$$
$$\Rightarrow y_2' + g y_2 = e^{-\int h dx}$$
$$\Rightarrow (D+g) y_2 = e^{-\int h dx}$$
$$\Rightarrow (D+h)(D+g) y_2 = (D+h) e^{-\int h dx} = -h e^{-\int h dx} + h e^{-\int h dx} = 0$$
$$= (D+h)(y_2' + g y_2)$$
$$= D y_2' + D(g y_2) + h y_2' + h g y_2$$
$$= y_2'' + g' y_2 + g y_2' + h y_2' + h g y_2$$
$$= y_2'' + (g+h) y_2' + (g' + hg) y_2$$
$$= y_2'' + P y_2' + (-s' - (P-g)s) y_2$$
$$= y_2'' + P y_2' + (-s' - (P+s)s) y_2$$
$$= y_2'' + P y_2' + (-s' - s^2 - Ps) y_2$$

But:

$$0 = y_1'' + Py_1' + Qy_1 = (sy_1)' + P(sy_1) + Qy_1$$
$$= s' y_1 + s^2 y_1 + P s y_1 + Q y_1$$
$$= (s' + s^2 + Ps + Q) y_1$$
$$\Rightarrow Q = -s' - s^2 - Ps$$
$$\Rightarrow 0 = y_2'' + Py_2' + Qy_2 = (D+h)(D+g) y_2 \quad , \quad \left(P = h + g \ , \ Q = g' + hg \right)$$

alternatively written:

$$\Rightarrow 0 = y_2'' + Py_2' + Qy_2 = [D + (P+s)](D-s) y_2 \quad , \quad (Q = -s' - s^2 - Ps)$$

or:

$$\Rightarrow 0 = y_2'' + Py_2' + Qy_2 = \left(P + \left[D + \frac{y_1'}{y_1} \right] \right) \left[D - \frac{y_1'}{y_1} \right] y_2 \quad , \quad \left(Q = -\frac{y_1''}{y_1} - P \frac{y_1'}{y_1} \right)$$

□

Theorem I.2: A solution of a Second Order Homogeneous Linear Ordinary Differential Equation factored as:

$$(D+h)(D+g) y = 0$$

may be written:

$$y = k e^{-\int g dx} \int e^{\int (g-h) dx} dx$$

Proof:

$(D+h)(D+g) y = 0$

Let: $(D+g) y = U$

$$\Rightarrow 0 = (D+h) U = U' + hU = \left(U e^{\int h dx} \right)' e^{-\int h dx} \Rightarrow U = k e^{-\int h dx}$$
$$\Rightarrow k e^{-\int h dx} = (D+g) y = y' + gy = \left(y e^{\int g dx} \right)' e^{-\int g dx}$$
$$\Rightarrow y = k e^{-\int g dx} \int e^{\int (g-h) dx} dx$$

□

Corollary I.3: The total solution of a Second Order Homogeneous Linear Ordinary Differential Equation may be written:

$$y = c_1 e^{-\int g dx} + c_2 e^{-\int g dx} \int e^{\int (2g-P)dx} dx$$

Proof:

$$y_1 = c_1 e^{-\int g dx} \Rightarrow y_1' = -g y_1 \Rightarrow y_1'' = -g' y_1 + g^2 y_1$$
$$\Rightarrow y_1'' + (h+g)y_1' = -g' y_1 + g^2 y_1 - (h+g)gy$$
$$\Rightarrow y_1'' + (h+g)y_1' = -g' y_1 + g^2 y_1 - hgy_1 - g^2 y_1 = -(g' + hg)y$$
$$\Rightarrow y'' + Py' + Qy = 0 \quad , \quad \big(P = h+g \quad , \quad Q = g' + hg\big)$$

And, by the reduction of order formula:

$$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int P dx} dx = c_1 e^{-\int g dx} \int \left(e^{-\int g dx}\right)^{-2} e^{-\int P dx} dx = c_1 e^{-\int g dx} \int e^{\int (2g-P)dx} dx$$

But this is just the factored solution, so:

$$y = c_1 e^{-\int g dx} + c_2 e^{-\int g dx} \int e^{\int (2g-P)dx} dx$$

□

Corollary I.4: A Second Order Linear Ordinary Differential Equation may be factored via two linear differential operators.

$$(D+h)(D+g)y = W \Rightarrow y = y_{h_1} \int y_{h_1}^{-2} e^{-\int P dx} \left(\int W y_{h_1} e^{\int P dx} dx \right) dx$$

(where: $y_h'' + Py_h' + Qy_h = 0$)
($P = g + h$, $Q = g' + gh$)

Proof:

$$y_1 = c_1 e^{-\int g dx} \Rightarrow y_1' = -g y_1 \Rightarrow y_1'' = -g' y_1 + g^2 y_1$$

Using: $y_{h_1} = e^{-\int g dx}$ and $P = g + h$ and reversing the order of steps in theorem I.1:

$$(D+h)(D+g)y = W$$
$$\Rightarrow (D+h)U = W \text{ and } (D+g)y = U$$
$$\Rightarrow U = e^{-\int h dx} \int W e^{\int h dx} dx \text{ and } y = e^{-\int g dx} \int U e^{\int g dx} dx$$
$$\Rightarrow y = e^{-\int g dx} \int \left(e^{-\int h dx} \int W e^{\int h dx} dx \right) e^{\int g dx} dx = y_{h_1} \int y_{h_1}^{-2} e^{-\int P dx} \left(\int W y_{h_1} e^{\int P dx} dx \right) dx \ ,$$
$$(Q = g' + gh)$$

□

Note: $y = e^{-\int g dx} \int \left(e^{-\int h dx} \int W e^{\int h dx} dx \right) e^{\int g dx} dx$

is the 2nd order inhomogeneous LODE formula [6].

Corollary I.5: For differentiable functions P, Q :

$$y'' + Py' + Qy = 0 \Rightarrow \exists u :$$
$$y = c_1 e^{-\frac{1}{2} \int \left[P + \left(u' \pm \sqrt{(u'+P)^2 - 4Q} \right) \right] dx} + c_2 e^{-\frac{1}{2} \int \left[P + \left(u' \pm \sqrt{(u'+P)^2 - 4Q} \right) \right] dx} \int e^{\int \left(u' \pm \sqrt{(u'+P)^2 - 4Q} \right) dx} dx$$

Proof:

From Theorems I.1 , I.2 & I.3:

$$Q = g' + gh = g' + g(P-g) = g' - g^2 + Pg$$
$$g = e^u \Rightarrow Q = u'(e^u) - (e^u)^2 + P(e^u) \Rightarrow (e^u)^2 - (u'+P)(e^u) + Q = 0$$
$$\Rightarrow g = (e^u) = \tfrac{1}{2} \left[(u'+P) \pm \sqrt{(u'+P)^2 - 4Q} \right]$$
$$= \tfrac{1}{2} \left[P + \left(u' \pm \sqrt{(u'+P)^2 - 4Q} \right) \right]$$
$$\Rightarrow h = P - g = \tfrac{1}{2} \left[(-u'+P) \mp \sqrt{(u'+P)^2 - 4Q} \right]$$
$$= \tfrac{1}{2} \left[P - \left(u' \pm \sqrt{(u'+P)^2 - 4Q} \right) \right]$$
$$\Rightarrow y'' + Py' + Qy = (D+h)(D+g)y = 0 \Rightarrow y = k e^{-\int g dx} \int e^{\int (g-h)dx} dx \ ,$$
$$\big(P = g + h \quad , \quad Q = g' + hg\big)$$
$$\Rightarrow \exists u : y = k e^{-\frac{1}{2} \int \left[P + \left(u' \pm \sqrt{(u'+P)^2 - 4Q} \right) \right] dx} \int e^{\int \left(u' \pm \sqrt{(u'+P)^2 - 4Q} \right) dx} dx$$
$$\Rightarrow \exists u : y = c_1 e^{-\frac{1}{2} \int \left[P + \left(u' \pm \sqrt{(u'+P)^2 - 4Q} \right) \right] dx} +$$
$$+ c_2 e^{-\frac{1}{2} \int \left[P + \left(u' \pm \sqrt{(u'+P)^2 - 4Q} \right) \right] dx} \int e^{\int \left(u' \pm \sqrt{(u'+P)^2 - 4Q} \right) dx} dx$$

□

Examples:

$$u = \ln(aR + b) : u' = \frac{aR'}{aR + b}$$
$$\Rightarrow s = -g = -\tfrac{1}{2} \left[P + \left(u' \pm \sqrt{(u'+P)^2 - 4Q} \right) \right] \quad , \quad g = e^u \Rightarrow s' = -g' = -u'e^u = -u'g = u's$$
$$\Rightarrow \left[s + \tfrac{1}{2}(u'+P)^2 \right]^2 = \tfrac{1}{4}\left((u'+P)^2 - 4Q \right) = s^2 + sP + \tfrac{1}{4}(u'+P)^2 \Leftrightarrow s^2 + s(u'+P) = -Q$$
$$\Leftrightarrow s' + s^2 + sP = -Q \Leftrightarrow g' - g^2 + gP = Q = u'e^u - e^u e^u + e^u P = aR' - (aR+b)^2 + P(aR+b)$$
$$\Leftrightarrow \exists R : Q = aR' - (aR+b)^2 + P(aR+b)$$

(Note: most of the 2nd order HLODE solutions in [4] are of this form)

$R = 1 : \exists a, b : Q = -(a+b)^2 + P(a+b) \Rightarrow -k^2 + kP$

This is a generalization of the 2nd order Euler Constant Coefficients solution[3][5]:

$$k = \frac{-P \pm \sqrt{P^2 - 4Q}}{2}$$

$$R = \frac{1}{x} \; : \exists a, b \, : \, Q = -\frac{a}{x^2} - \left(\frac{a}{x} + b\right)^2 + P\left(\frac{a}{x} + b\right) = -\frac{a(a+1)}{x^2} - \frac{2ab}{x} + \frac{a}{x}P + bP - b^2$$

This is a generalization of the 2nd order Cauchy-Euler solution[3][5]:

$$\left(P = \frac{A}{x}, Q = \frac{B}{x^2}\right)$$

$$\exists a, b \, : \, \frac{B}{x^2} = -\frac{a(a+1-A)}{x^2} + \frac{b(-2a+A)}{x} - b^2$$

$$(b = 0) \, :$$

$$\exists a \, : \, \frac{B}{x^2} = -\frac{a(a+1-A)}{x^2}$$

$$\Rightarrow B = -a^2 + a(1-A) \Rightarrow a = \frac{(1-A) \pm \sqrt{(1-A)^2 - 4B}}{2}$$

And the special case Bessel: $\left(P = \frac{1}{x}, Q = \frac{B}{x^2} + C\right)$

$$\exists a, b \, : -\frac{B}{x^2} - C = -\frac{a(a+1-1)}{x^2} + \frac{b(-2a+1)}{x} - b^2$$

$$\left(a = \tfrac{1}{2}\right) : \quad \exists b \, : \, a = \pm B, b = \pm\sqrt{C}$$

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