Letter N°3 : Elementary Formula

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ABSTRACT. We recall a elementary formula involving Pi.
Keywords. number Pi , hypergeometric function .

I. Introduction .

Recall that:

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.14159265 \ldots \tag{1}$$

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \ , \ |x| < 1 \tag{2}$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \ , \ |x| < 1 \tag{3}$$

$$\ln(x) = 2 \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{x-1}{x+1} \right)^{2n+1} \ , \ x > 0 \tag{4}$$

Remark: $F(a, b; c; x)$ is the Gauss hypergeometric function , $\tan^{-1}(x)$ is the inverse tangent function and $\ln(x)$ is the logarithm function.

II. Elementary Formula .

Entry 1. If a, b, c, s are defined by

$$a = \frac{1}{20} \sqrt{10 + 2\sqrt{5}} \left(\sqrt{5 + 4\sqrt{5}} \sqrt{5} - 5 \right) \tag{5}$$

$$b = 3 + 2\sqrt{5} + 2\sqrt{5 + 4\sqrt{5}} \tag{6}$$

$$c = \frac{7 + 3\sqrt{5} + \sqrt{70 + 34\sqrt{5}}}{4} \tag{7}$$

$$s = \frac{1}{4} \left(\sqrt{5} - 1 \right) \left(-1 + \sqrt{5 + 4\sqrt{5}} \right) \tag{8}$$

then

$$s\,F\left(1,\ \frac{1}{5};\ \frac{6}{5};\ -s^5\right) =$$

$$\frac{\sqrt{10-2\sqrt{5}}}{20}\,\pi + \left(\frac{\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}}}{10}\right)\tan^{-1}(a) + \frac{\ln(b)}{20} + \frac{\ln(c)}{4\sqrt{5}}$$

Entry 2. We have

$$F\left(1,\ \frac{1}{5};\ \frac{6}{5};\ -s^5\right) = \frac{1}{1+s^5}\,F\left(1,\ 1;\ \frac{6}{5};\ \frac{s^5}{1+s^5}\right) \tag{10}$$

$$F\left(1,\ \frac{1}{5};\ \frac{6}{5};\ -s^5\right) = \frac{1}{\sqrt[5]{1+s^5}}\,F\left(\frac{1}{5},\ \frac{1}{5};\ \frac{6}{5};\ \frac{s^5}{1+s^5}\right) \tag{11}$$

$$s\,F\left(1,\ \frac{1}{5};\ \frac{6}{5};\ -s^5\right) = \sum_{n=0}^{\infty}\frac{(-1)^n s^{5n+1}}{5n+1} =$$

$$s\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1} F\left(\frac{1}{5},\ -n;\ \frac{6}{5};\ s^5\right) = \frac{s}{\sqrt[5]{1-s^5}}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1} F\left(\frac{1}{5},\ n+\frac{6}{5};\ \frac{6}{5};\ \frac{s^5}{s^5-1}\right) = \tag{12}$$

$$\frac{s}{2}\sum_{n=0}^{\infty}\left(\frac{1-s^5}{2}\right)^{n} F\left(-n,\ 1;\ \frac{6}{5};\ \frac{s^5}{s^5-1}\right) = s\sum_{n=0}^{\infty}\left(\frac{1-s^5}{2}\right)^{n+1} F\left(1,\ n+\frac{6}{5};\ \frac{6}{5};\ s^5\right)$$

Entry 3. We have

$$a^8 - 7a^6 + 4a^4 - 3a^2 + 1 = 0 \tag{13}$$

$$b^4 - 12b^3 - 26b^2 - 268b + 1 = 0 \tag{14}$$

$$c^4 - 7c^3 + 4c^2 - 3c + 1 = 0 \tag{15}$$

$$s^4 - s^3 + s^2 + 4s - 4 = 0 \tag{16}$$

III. References .

[1] Arndt, J., and Haenel, C.: π unleashed. Springer-Verlag , 2001.

[2] Berndt, B.: Ramanujan’s Notebooks. Part I , Springer - Verlag , 1985.

[3] Blatner, D.: The Joy of π . Walker Publ., 1999.