Higher accuracy order in differentiation-by-integration

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Abstract

In this text explicit forms of several higher precision order kernel functions (to be used in the differentiation-by-integration procedure) are given for several derivative orders. Also, a system of linear equations is formulated which allows to construct kernels with arbitrary precision for arbitrary derivative order. A computer study is realized and it is shown that numerical differentiation based on higher precision order kernels performs much better (w.r.t. errors) than the same procedure based on usual Legendre-polynomial kernels. Presented results may have implications for numerical implementations of the differentiation-by-integration method.

1 Introduction

Derivative, if exists, can be expressed in an alternative way

$$f'(x)|_{x=x_0} = \lim_{h \to 0} \mathcal{D}_{n=1}^h (f)|_{x_0} \equiv \lim_{h \to 0} -\frac{1}{h} \int_{-1}^{1} k(t) f(x_0 + ht) \, dt, \qquad (1)$$

where k is an appropriate kernel function and n indicates the derivative order. The method is know as "*differentiation by integration*" (DbI) and an example was first published by Cioranescu [1]. Later, Lanczos [2] studied a version of (1) with kernel

$$k(t) = -\frac{3}{2}t \qquad (2)$$

and his name became associated with this type of differentiation[1]. DbI has some interesting features because it generalizes the ordinary derivative in several aspects. It may be used in situations where the latter does not exist, or, when generalized to higher orders, it can be (in some cases) seen as a way to define a fractional-order derivative [3]. The DbI also attracts attention for its potential applications in numerical analysis [4, 5, 6, 7, 8].

In published texts [9, 10, 11, 12, 13, 14, 15] two dominant ideas can be seen:

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[1]"*Generalized Lanczos derivative*" is to be seen as an alternative label to "DbI".

- The least-squares property: the derivative as defined by the Lanczos' formula (1)(2) behaves as the slope parameter of a line which is in the least-squares manner adjusted to the function f on the interval $[x_0 - h, x_0 + h]$. The idea was further generalized to higher derivatives in [10] where the kernel functions are based on the Legendre polynomials. This property is to be seen as very closely linked to the Savitzky–Golay filter [16] and related way of numerically computing a derivative from discrete data sets.

- Orthogonality approach: if one expands a function f into Taylor series

$$\int_{-1}^{1} k(t) f(x_0 + ht)\, dt = \sum_{n=0}^{\infty} \frac{1}{n!} h^n f^{(n)}(x_0) \int_{-1}^{1} k(t)\, t^n dt, \qquad (3)$$

one can search for a function k orthogonal to desired powers of t^n so as to make vanish all terms except those which we are interested in. For instance, to extract Dth derivative one can use a polynomial orthogonal to all t^n for $n < D$ (e.g. the Legendre polynomial L_D). If one then multiplies the resulting expression by an appropriate power of h, the limit $h \to 0$ makes disappear all higher terms $n > D$ and isolates the one with the Dth derivative. This approach can provide large generalizations (as presented in [13]) but remains limited to analytic functions f.

A universal procedure for constructing a kernel function for the first derivative was presented in [5], a full generalization to all derivative orders was given in [17]. A function k is a valid kernel function for computing Dth derivative if and only if

$$k = \frac{d^D}{dt^D} \omega(t), \qquad (4)$$

where ω is a weight function satisfying

$$\int_{-1}^{1} \omega(t)\, dt = 1, \qquad (5)$$

$$\frac{d^m}{dt^m} \omega(t)\,|_{t=-1} = \frac{d^m}{dt^m} \omega(t)\,|_{t=+1} = 0 \text{ for all } m < D, \qquad (6)$$

assuming that all appearing integrals and derivatives exist. The Dth derivative is then computed as

$$\frac{d^D}{dt^D} f(x)\,|_{x=x_0} = \lim_{h \to 0} \mathcal{D}_D^h(f)\,|_{x_0} \equiv \lim_{h \to 0} \left(\frac{-1}{h} \right)^D \int_{-1}^{1} k(t) f(x_0 + ht)\, dt. \qquad (7)$$

In the standard finite-difference approach is a large emphasis (in relation to numerical applications) given to the accuracy order in the discretization-parameter (h) expansion. It is know that the higher-order approaches perform

much better than the lower-order ones[2]. Systematic study of these effects is somewhat missing with respect to the DbI methods, yet higher accuracy orders lead (as demonstrated later) to largely improved overall numerical precision. In existing literature, formulas with higher accuracy order can be found (e.g. formula 4.1 in [13]), however explicit kernel function forms which could be directly used as a recipe for an immediate implementation are missing. One also lacks a numerical study of performance of such higher-order methods. The aim of this text is to

- provide explicit expressions for kernel functions for first few orders in precision and derivative (Sec. 2 and Appendix). Further, we want to

- give an explicit formulation of a linear system which, when solved, leads to kernel functions for any derivative order and any order in precision (Sec. 2). We also want to

- perform a numerical study of the higher accuracy order approach (Sec. 3).

We will briefly discuss the presented results in Sec. 4 and close the text by providing summary and conclusion (Sec. 5).

2   Higher order formulas

Different strategies can be adopted when constructing higher precision order kernels. Orthogonality approach is certainly an option: by an appropriate choice of a kernel k one can set to zero all h^n terms in (3) up to a desired value of $n < N$, with exception of the term containing the derivative one is interested in. Yet, we prefer to chose a different strategy which stems from basic principles (4), (5) and (6). We adopt the most natural choice and search for kernels in form of least-degree polynomials. This minimalist approach is allowed by a "brute force" computation which provides us with full control over coefficients and leads to explicit relations between them.

A general polynomial respecting (6) is written as

$$\omega\left(x\right) = \mathcal{N}\left(1-x\right)^{D}\left(1+x\right)^{D}\sum_{n=0}^{N}a_n x^n, \tag{8}$$

$$= \mathcal{N}\left[\sum_{j=0}^{2D}\frac{1}{2}\left\{1+(-1)^{j}\right\}C_{D,j/2}\left(-1\right)^{j/2}x^{j}\right]\sum_{n=0}^{N}a_n x^n,$$

where $\mathcal{N}$ is normalization, D is derivative order and $C_{a,b}$ represents binomial coefficient $\binom{a}{b}$. Formally appearing non-integer powers in the last line are canceled

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[2]A simple forward difference is over-performed by the central difference, which is on its turn over-performed by, for example, the five-point rule.

by the $1 + (-1)^j$ term. The two sums can be multiplied

$$\omega(x) = \mathcal{N} \sum_{n=0}^{N+2D} \Omega_n x^n, \tag{9}$$

where

$$\Omega_n = \sum_{k=n-2D}^{n} \frac{1}{2} H_k H_{N-k} \left\{ 1 + (-1)^{n-k} \right\} C_{D,(n-k)/2} (-1)^{(n-k)/2} a_k, \tag{10}$$

with H indicating a step function

$$H_i = \begin{cases} 0 & \text{if } i < 0 \\ 1 & \text{else} \end{cases}.$$

The condition (5) implies the normalization

$$\mathcal{N} = \left(\sum_{n=0}^{\lfloor (N+2D)/2 \rfloor} \frac{2\Omega_{2n}}{2n+1} \right)^{-1}, \tag{11}$$

where symbol $\lfloor \ldots \rfloor$ stands for the floor function. For the needs of formula (4) one has to compute higher-order derivatives

$$\omega^{(m)} = \mathcal{N} \sum_{n=0}^{N+2D-m} \frac{(n+m)!}{n!} \Omega_{n+m} x^n \Rightarrow \omega^{(D)} = \mathcal{N} \sum_{n=0}^{N+D} \frac{(n+D)!}{n!} \Omega_{n+D} x^n. \tag{12}$$

Expanding a function f into Taylor series and plugging (12) into (7) using (4) one gets[3]

$$f^{(D)} = \left(-\frac{1}{h} \right)^D \int_{-1}^{1} \mathcal{N} \sum_{n=0}^{N+D} \alpha_n t^n \sum_{m=0}^{\infty} \frac{1}{m!} h^m f^{(m)} t^m dt,$$

$$= \mathcal{N} (-1)^D \sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)} \left\{ \sum_{n=0}^{N+D} \frac{\alpha_n}{n+m+1} \left[1 + (-1)^{m+n} \right] \right\} h^{m-D},$$

$$\text{with } \alpha_n \equiv \frac{(n+D)!}{n!} \Omega_{n+D},$$

where one controls the accuracy order by adjusting the coefficients in front of the powers of h. The series formally contains also negative powers (for $m < D$), formula (7) however implies that their coefficients are identically zero. Furthermore, formula (7) also implies[4] that the overall coefficient in front of the constant term (h^0) is $f^{(D)}$. The interesting terms are so those with $m > D$.

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[3] Argument of f is omitted, one implicitly assumes it takes value x_0 (point of differentiation).

[4] Both of these statements can be verified in an explicit computation (but do not need to be).

An important observation is that the odd powers of h are controlled only by odd-indexed coefficients $a_1, a_3, a_5, \ldots$. It can be verified in an explicit way, that the index of Ω has the same parity as the power of h (symbol $\sim$ means conservation of index and power-related properties)

$$h^{m-D} \sum_{n=0}^{N+D} \frac{\alpha_n \left[1 + (-1)^{m+n}\right]}{n+m+1} \sim \sum_{n=0}^{N+D} h^{m+n-(D+n)} \Omega_{n+D} \left[1 + (-1)^{m+n}\right]$$

$$= \sum_{n=0}^{N+D} h^{m+n-Z_n} \Omega_{Z_n} \left[1 + (-1)^{m+n}\right]$$

$$\sim \sum_{n=0}^{N+D} h^{2i_n - Z_n} \Omega_{Z_n} = \sum_{n=0}^{N+D} h^{Z_n + 2i_n} \Omega_{Z_n},$$

where the factor $1 + (-1)^{m+n}$ allows only for even-valued sums of the two numbers $m + n = 2i$. The expression for Ω_Z is, on its turn, given only by terms whose coefficients a_i have index of the same parity as Z:

$$\Omega_{2i+1} \sim \sum_{k=n-2D}^{n} \left\{1 + (-1)^{2i+1-k}\right\} (-1)^{(2i+1-k)/2} a_k, \quad 0 < k < N$$

$$\sim \sum_{k=n-2D}^{n} \left\{1 + (-1)^{1+k}\right\} (-1)^{(2i+1-k)/2} a_k.$$

If k is even, then $1 + k$ is odd and $1 + (-1)^{1+k}$ is zero for all related a_k terms. Therefore Ω_{2i+1} depends only on the odd-index a_i coefficients. This being established, in what follows we will study only even-index based weight functions[5], i.e. even function of x on the interval $[-1, 1]$. Even-indexed coefficients, controlling the even powers of h, cannot be set all to zero because one needs to fulfill the normalization condition (5).

We are now in the situation, where we need to adjust the even-indexed coefficients $a_0, a_2, a_4, \ldots$ appearing in

$$\mathcal{N} \frac{(-1)^D}{m!} f^{(m)} \left\{ \sum_{n=0}^{N+D} \frac{\alpha_n (a_k)}{n+m+1} \left[1 + (-1)^{m+n}\right] \right\} h^{m-D}$$

$$= \mathcal{D}_{n=D}^h (f) = f^{(D)} + c_2 h^2 + c_4 h^4 + c_6 h^6 + \ldots$$

so as to make chosen c_i coefficients vanish. Let us denote the highest even-power h term which is meant to be set to zero (together with all lower terms) by h^{2i}. For achieving that we need to match i (even-indexed) coefficients and formulate i equations. In a compact form the jth equation can be written ($1 \le j \le i$)

$$\mathrm{E}_j : \quad \sum_{n=0}^{2i+D} \frac{\alpha_n (\Delta [a_k])}{n+2j+D+1} \left[1 + (-1)^{2j+D+n}\right] = 0, \quad (13)$$

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[5]Aiming highest accuracy order, we have no motivation to keep or study odd-h-power terms. We directly set them all to zero.

where the dependence of α_n on coefficients is

$$
\Delta\,[a_k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 1 \le k \le 2i \text{ and } k \text{ is odd} \\ a_k & \text{if } 1 \le k \le 2i \text{ and } k \text{ is even} \\ 0 & \text{else} \end{cases}
$$

and equality

$$
N = 2i
$$

is assumed in all expressions. Here, without loss of generality, we set the first coefficient to one $a_0 = 1$. Initially a general number, N (as an unspecified number of polynomial terms) can be cut to $2i$ (it comprises all coefficients, odd-indexed included) and is so determined only by the accuracy order. Once the system is solved and a_i coefficients are found, one computes the normalization $\mathcal{N}$ from (11) and constructs the weight function (8). The kernel function is given as its Dth derivative (4).

We will note $\omega_D^{[2i]}$ and $k_D^{[2i]}$ those functions, which provide Dth derivative with order of precision $\mathcal{O}\left(h^{2i+2}\right)$

$$
\mathcal{D}_D^h\left(f, k_D^{[2i]}\right) = f^{(D)} + \mathcal{O}\left(h^{2i+2}\right).
$$

This notations reflects the fact that the highest even-power term which is annihilated is h^{2i}. Since the next one is also zero (because has an odd power), the first nonzero term corresponds to the power $2i+2$. Solving system (13) one gets as example results

$$
\omega_1^{[2]}(x) = \frac{45}{32}\,(1-x)\,(1+x)\left(1 - \frac{7}{3}x^2\right),
$$

$$
k_1^{[2]}(x) = \frac{1}{8}\left(105x^3 - 75x\right),
$$

$$
\omega_1^{[4]}(x) = \frac{525}{256}\,(1-x)\,(1+x)\left(1 - 6x^2 + \frac{33}{5}x^4\right),
$$

$$
k_1^{[4]}(x) = -\frac{1}{128}\left(10395x^5 - 13230x^3 + 3675x\right),
$$

$$
\omega_2^{[2]}(x) = \frac{105}{64}\,(1-x)^2\,(1+x)^2\left(1 - 3x^2\right),
$$

$$
k_2^{[2]}(x) = -\frac{1}{32}\left(4725x^4 - 4410x^2 + 525\right),
$$

$$
\omega_2^{[4]}(x) = \frac{4725}{2048}\,(1-x)^2\,(1+x)^2\left(1 - \frac{22}{3}x^2 + \frac{143}{15}x^4\right),
$$

$$
k_2^{[4]}(x) = \frac{1}{256}\left(315315x^6 - 467775x^4 + 178605x^2 - 11025\right),
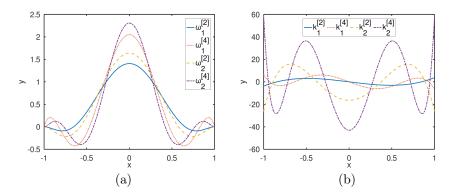$$

Figure 1: Selected weight functions (a) and corresponding kernel functions (b).

where cases with $2i = 0$ are omitted because they correspond to the known Legendre-polynomial based kernels[6] (e.g formula (2)). The example functions are depicted in Fig. (1). All cases with $1 \leq D \leq 4$ and $2 \leq 2i \leq 10$ are listed in Appendix.

3   Numerical investigations

The topic under consideration is, from the perspective of numerical study[7], rather large and multidimensional. One could by interested in the dependence of results on the

- size of the discretization parameter h, or on the

- precision order of the kernel function, or on the

- derivative order of the kernel function, or on the

- choice of the test functions.

To reduce the complexity, we opt for our purposes only for the $\mathcal{O}\left(h^6\right)$ precision kernels $k_D^{2i=4}$. Next, the optimal size of the discretization parameter is analyzed. For that we select 3 test functions with fixed point of differentiation x_0

$$\sin\left(x\right) \text{ with } x_0 = 1; \quad \exp\left(x\right) \text{ with } x_0 = \pi \text{ and } \ln\left(x\right) \text{ with } x_0 = \frac{1}{2}.$$

Using these settings we scan 8 orders of magnitude in discretization parameter $10^{-8} \leq h \leq 10^{-1}$, changing the step in geometrical progression with factor 10.

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[6]Our approach represents a generalization which includes previously known results as special cases.

[7]Numerical analysis was done with the GNU Octave [18] software 5.1.0 configured for "x86_64-suse-linux-gnu". The *quadl* function was used for numerical integration. Author also greatly profited from the WxMaxima tools [19].

D	Opt. for	$\sin(x)$, $x_0=1$			$\exp(x)$, $x_0=\pi$			$\ln(x)$, $x_0=\frac{1}{2}$		
		h^{opt}	$\left|\Delta_{abs}^{HO}\right|$	$\left|\Delta_{abs}^{LS}\right|$	h^{opt}	$\left|\Delta_{abs}^{HO}\right|$	$\left|\Delta_{abs}^{LS}\right|$	h^{opt}	$\left|\Delta_{abs}^{HO}\right|$	$\left|\Delta_{abs}^{LS}\right|$
1	HO	10^{-2}	$\mathbf{1.62\times10^{-14}}$	5.40×10^{-6}	10^{-2}	$\mathbf{6.64\times10^{-13}}$	2.31×10^{-4}	10^{-3}	$\mathbf{8.53\times10^{-14}}$	1.60×10^{-6}
	LS	10^{-5}	1.32×10^{-11}	$\mathbf{1.39\times10^{-11}}$	10^{-5}	7.73×10^{-10}	$\mathbf{4.26\times10^{-10}}$	10^{-5}	2.37×10^{-11}	$\mathbf{1.57\times10^{-10}}$
2	HO	10^{-1}	$\mathbf{7.82\times10^{-12}}$	6.00×10^{-4}	10^{-1}	$\mathbf{2.10\times10^{-10}}$	1.65×10^{-2}	10^{-2}	$\mathbf{2.60\times10^{-11}}$	6.86×10^{-4}
	LS	10^{-4}	5.74×10^{-7}	$\mathbf{1.41\times10^{-8}}$	10^{-4}	1.11×10^{-5}	$\mathbf{4.26\times10^{-7}}$	10^{-4}	5.13×10^{-7}	$\mathbf{4.23\times10^{-8}}$
3	HO	10^{-1}	$\mathbf{2.47\times10^{-11}}$	3.00×10^{-4}	10^{-1}	$\mathbf{4.26\times10^{-10}}$	1.29×10^{-2}	10^{-2}	$\mathbf{1.20\times10^{-8}}$	4.27×10^{-3}
	LS	10^{-3}	1.63×10^{-5}	$\mathbf{6.08\times10^{-7}}$	10^{-3}	3.58×10^{-5}	$\mathbf{6.82\times10^{-6}}$	10^{-3}	7.56×10^{-6}	$\mathbf{4.35\times10^{-5}}$
4	HO	10^{-1}	$\mathbf{4.08\times10^{-11}}$	3.82×10^{-4}	10^{-1}	$\mathbf{7.77\times10^{-8}}$	1.05×10^{-2}	10^{-2}	$\mathbf{1.39\times10^{-4}}$	3.49×10^{-2}
	LS	10^{-2}	1.67×10^{-4}	$\mathbf{1.87\times10^{-6}}$	10^{-2}	1.49×10^{-3}	$\mathbf{1.31\times10^{-5}}$	10^{-3}	1.32	$\mathbf{8.53\times10^{-3}}$

Table 1: Absolute error for 4 derivatives orders (D), two kernel-function types (higher order HO vs. least-squares LS) and three test functions with fixed test points. The discretization parameter h is tuned to an optimal value for one of the methods (in bold), the the result of other method is shown also.

For comparison purposes we add results from the least-squares (LS) approach, which is based on the Legendre polynomials.

As seen from the summary Table (1), the optimal h for higher precision order (HO) methods is $h \sim 10^{-1}, 10^{-2}$, for the LS methods it is systematically smaller[8] $h \sim 10^{-3}, 10^{-4}$. The important observation is that the HO kernels significantly over-perform the LS kernels (if each method uses its own optimal value of h). If the HO kernels are used with an h-size that is optimal for the LS methods, the HO results remain competitive.

Fixing the discretization parameter size to near-optimal values $h^{HO} = 10^{-2}$ and $h^{LS} = 10^{-3}$, we study the error behavior on a whole interval: $[-\pi, \pi]$ for $\sin(x)$, $[-2, 2]$ for $\exp(x)$ and $[0.1, 2]$ for $\ln(x)$. Results are shown in Fig. (2). One can draw from them a straightforward conclusion: when interested in absolute errors, the HO differentiation provides several orders of magnitude more precise results than the LS methods and, when it comes to implementing a numerical differentiation-by-integration method on a computer, an approach based on the HO kernels is clearly the preferred option.

4   Discussion

So far we did not address the question of numerical (i.e. round-off) errors. The precision order is to be related to the so-called discretization error which can be made arbitrary small[9] and can be handled with full mathematical rigorosity. However, on a real computer the use of a (very) high-accuracy approach would fail: as seen from Fig. (1) higher-order weight functions become more sharply peaked near zero which can, in some sense, be interpreted as effective shrinking of the interval $[x_0 - h, x_0 + h]$. If interval becomes small, numerical errors grow. In addition, higher-order weight functions (and thus kernel functions too) become more and more oscillatory, which further worsens numerical uncertainties. One can also notice, that with increasing precision order, kernel functions become numerically large.

Numerical errors are difficult to handle: a reliable mathematical treatment of the float-point arithmetic in computer registers is presumably a very hard task. If a reasonable precision order (not very high) is used one can hope for comparable numerical effects in the HO and LS approaches: both are based on convolution integrals with polynomial kernels and (for not very different degrees of polynomials) there is no *a priori* reason for the HO approach to be so much more sensitive to numerical errors as to end up with a larger total error. These quantitative arguments are supported by the numerical tests from the previous section: whatever the behavior of numerical errors is, the HO differentiation provides in studied examples a significantly smaller overall (total) error.

A rather obvious lesson can be learnt from previous observations: if a

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[8]This behavior is also observed in finite-difference methods when comparing formulas with different accuracy order.

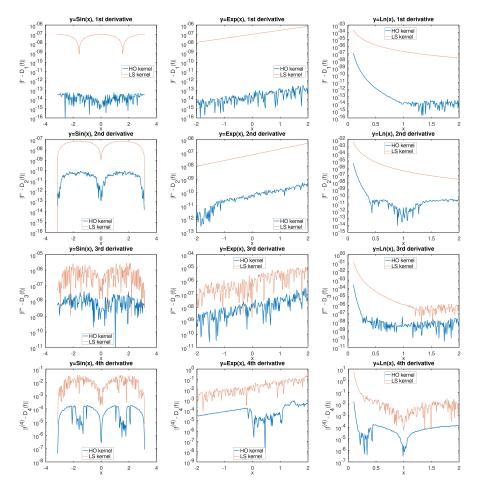[9]By an appropriate choice of kernel function, one can make vanish arbitrary high powers of h.

Figure 2: Absolute errors of the HO approach (full blue line) and LS-based methods (dotted red line) for 3 functions (in columns) and 4 derivative orders (rows).

function-related quantity is to be extracted in a numerical way from the function behavior on the interval $[x_0 - h, x_0 + h]$, where h is small enough and can play a role of a power-expansion parameter, then the method which reaches a higher precision with respect to this expansion is to be the preferred one, usually providing higher overall precision in real-life applications.

5   Summary and conclusion

In finite difference methods, the higher precision approach is a well established standard. Despite clear advantages, such approach was not well studied for the DbI up to now. Using straightforward calculations we constructed a system of linear equations (13) which leads to arbitrary precision order kernel functions for arbitrary derivative in the DbI procedure. Numerical tests show that these kernels significantly over-perform (with respect to precision) standard least-squares kernels based on the Legendre polynomials, which were most widely used and discussed up to now. If a DbI method is to be implemented on a computer, then the results provide strong evidence for preferring higher precision order kernels.

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Appendix

List of higher accuracy order weight and kernel functions. Kernel functions k are to be used in formula (7), notation from the end of Sec. 2 is adopted.

- First derivative

$$\omega_1^{[2]}(x) = \frac{45}{32}(1-x)(1+x)\left(1 - \frac{7}{3}x^2\right)$$

$$k_1^{[2]}(x) = \frac{1}{8}\left(105x^3 - 75x\right)$$

$$\omega_1^{[4]}(x) = \frac{525}{256}(1-x)(1+x)\left(1 - 6x^2 + \frac{33}{5}x^4\right)$$

$$k_1^{[4]}(x) = -\frac{1}{128}\left(10395x^5 - 13230x^3 + 3675x\right)$$

$$\omega_1^{[6]}(x) = \frac{11025}{4096}(1-x)(1+x)\left(1 - 11x^2 + \frac{143}{5}x^4 - \frac{143}{7}x^6\right)$$

$$k_1^{[6]}(x) = \frac{1}{512}\left(225225x^7 - 405405x^5 + 218295x^3 - 33075x\right)$$

$$\omega_1^{[8]}(x) = \frac{218295}{65536}(1-x)(1+x)\left(1 - \frac{52}{3}x^2 + 78x^4 - \frac{884}{7}x^6 + \frac{4199}{63}x^8\right)$$

$$k_1^{[8]}(x) = -\frac{1}{32768}\left(72747675x^9 - 168468300x^7 + 133783650x^5 - 41621580x^3 + 4002075x\right)$$

$$\omega_1^{[10]}(x) = \frac{2081079}{524288}(1-x)(1+x)\left(1 - 25x^2 + 170x^4 - \frac{3230}{7}x^6 + \frac{1615}{3}x^8 - \frac{7429}{33}x^{10}\right)$$

$$k_1^{[10]}(x) = \frac{1}{131072}(1405485081x^{11} - 3972023055x^9 + 4161167010x^7 - 1971079110x^5 + 405810405x^3 - 27054027x)$$

- Second derivative

$$\omega_2^{[2]}(x) = \frac{105}{64}(1-x)^2(1+x)^2\left(1 - 3x^2\right)$$

$$k_2^{[2]}(x) = -\frac{1}{32}\left(4725x^4 - 4410x^2 + 525\right)$$

$$\omega_2^{[4]}(x) = \frac{4725}{2048}(1-x)^2(1+x)^2\left(1 - \frac{22}{3}x^2 + \frac{143}{15}x^4\right)$$

$$k_2^{[4]}(x) = \frac{1}{256}\left(315315x^6 - 467775x^4 + 178605x^2 - 11025\right)$$

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$$\omega_2^{[6]}(x) = \frac{24255}{8192}(1-x)^2(1+x)^2\left(1 - 13x^2 + 39x^4 - \frac{221}{7}x^6\right)$$

$$k_2^{[6]}(x) = -\frac{1}{4096}\left(34459425x^8 - 69369300x^6 + 44594550x^4 - 9604980x^2 + 363825\right)$$

$$\omega_2^{[8]}(x) = \frac{945945}{262144}(1-x)^2(1+x)^2\left(1 - 20x^2 + 102x^4 - \frac{1292}{7}x^6 + \frac{323}{3}x^8\right)$$

$$k_2^{[8]}(x) = \frac{1}{65536}(3360942585x^{10} - 8511477975x^8 + 7665307650x^6 - 2898645750x^4$$
$$+ 405810405x^2 - 10405395)$$

$$\omega_2^{[10]}(x) = \frac{4459455}{1048576}(1-x)^2(1+x)^2\left(1 - \frac{85}{3}x^2 + \frac{646}{3}x^4 - 646x^6 + \frac{7429}{9}x^8 - \frac{37145}{99}x^{10}\right)$$

$$k_2^{[10]}(x) = -\frac{1}{524288}(152260883775x^{12} - 463810076730x^{10} + 536223112425x^8$$
$$- 291281690700x^6 + 73915466625x^4 - 7304587290x^2 + 135270135)$$

- Third derivative

$$\omega_3^{[2]}(x) = \frac{945}{512}(1-x)^3(1+x)^3\left(1 - \frac{11}{3}x^2\right)$$

$$k_3^{[2]}(x) = \frac{1}{32}\left(72765x^5 - 85050x^3 + 19845x\right)$$

$$\omega_3^{[4]}(x) = \frac{10395}{4096}(1-x)^3(1+x)^3\left(1 - \frac{26}{3}x^2 + 13x^4\right)$$

$$k_3^{[4]}(x) = -\frac{1}{256}\left(6081075x^7 - 10405395x^5 + 5145525x^3 - 654885x\right)$$

$$\omega_3^{[6]}(x) = \frac{105105}{32768}(1-x)^3(1+x)^3\left(1 - 15x^2 + 51x^4 - \frac{323}{7}x^6\right)$$

$$k_3^{[6]}(x) = \frac{1}{4096}\left(800224425x^9 - 1791890100x^7 + 1352701350x^5 - 386486100x^3 + 31216185x\right)$$

$$\omega_3^{[8]}(x) = \frac{2027025}{524288}(1-x)^3(1+x)^3\left(1 - \frac{68}{3}x^2 + \frac{646}{5}x^4 - \frac{1292}{5}x^6 + \frac{7429}{45}x^8\right)$$

$$k_3^{[8]}(x) = -\frac{1}{65536}(91356530265x^{11} - 252070693875x^9 + 255344339250x^7$$
$$- 114979614750x^5 + 21739843125x^3 - 1217431215x)$$

$$\omega_3^{[10]}(x) = \frac{75810735}{16777216}(1-x)^3(1+x)^3\left(1 - \frac{95}{3}x^2 + 266x^4 - 874x^6 + \frac{10925}{9}x^8 - \frac{6555}{11}x^{10}\right)$$

$$k_3^{[10]}(x) = \frac{1}{524288}(4743512148375x^{13} - 15530610145050x^{11} + 19711928261025x^9$$
$$- 12154390548300x^7 + 3713841556425x^5 - 502625173050x^3 + 20696330655x)$$

- Fourth derivative

$$\omega_4^{[2]}(x) = \frac{2079}{1024}(1-x)^4(1+x)^4\left(1 - \frac{13}{3}x^2\right)$$

$$k_4^{[2]}(x) = -\frac{1}{64}\left(2837835x^6 - 4002075x^4 + 1403325x^2 - 72765\right)$$

$$\omega_4^{[4]}(x) = \frac{45045}{16384}(1-x)^4(1+x)^4\left(1 - 10x^2 + 17x^4\right)$$

$$k_4^{[4]}(x) = \frac{1}{2048}\left(1137161025x^8 - 2213511300x^6 + 1352701350x^4 - 267567300x^2 + 8513505\right)$$

$$\omega_4^{[6]}(x) = \frac{225225}{65536}(1-x)^4(1+x)^4\left(1 - 17x^2 + \frac{323}{5}x^4 - \frac{323}{5}x^6\right)$$

$$k_4^{[6]}(x) = -\frac{1}{8192}(43692253605x^{10} - 108030297375x^8 + 94074230250x^6$$
$$- 33817533750x^4 + 4347968625x^2 - 93648555)$$

$$\omega_4^{[8]}(x) = \frac{34459425}{8388608}(1-x)^4(1+x)^4\left(1 - \frac{76}{3}x^2 + \frac{798}{5}x^4 - \frac{1748}{5}x^6 + \frac{2185}{9}x^8\right)$$

$$k_4^{[8]}(x) = \frac{1}{613312}(1631366611875x^{12} - 4881048902730x^{10} + 5509545166125x^8$$
$$- 2854618767000x^6 + 645712192125x^4 - 46520223750x^2 + 237277755)$$

$$\omega_4^{[10]}(x) = \frac{160044885}{33554432}(1-x)^4(1+x)^4\left(1 - 35x^2 + 322x^4 - 1150x^6 + 1725x^8 - \frac{10005}{11}x^{10}\right)$$

$$k_4^{[10]}(x) = -\frac{1}{1048576}(334078784164125x^{14} - 1171647500648625x^{12} + 1622948760157725x^{10}$$
$$- 1123579910878425x^8 + 404133485730975x^6 - 70562989572075x^4$$
$$+ 4774939143975x^2 - 56175754635)$$