Fractional Spacetime and the Emergence of the Dark Sector

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Abstract

It is well known that both Newtonian gravity and General Relativity can be built starting from the classical Poisson equation in 3-dimensional space. Here we speculate that, at least in principle, the concept of 3-dimensional space equipped with minimal fractality enables a qualitative explanation of both rotation curves of disk galaxies and cosmological expansion. Our approach bridges the gap between particle and gravitational interpretations of Dark Matter and suggests a unified picture of the Dark Sector. It offers a basis for explaining away Modified Newtonian Gravity (MOND) and its theoretical ramifications.

Key words: fractional space, minimal fractal manifold, Dark Matter, cosmological expansion, cosmological parameters, MOND.

1. Newtonian potential in 3-dimensional space

Consider a massive scalar field $u(\mathbf{x})$ embedded in ordinary 3-dimensional space whose Lagrangian is given by

$$L_0 = \frac{1}{2}[\partial_\mu \partial^\mu - m^2]\, u(\mathbf{x}) \tag{1}$$

(1) gives rise to the wave equation in free space

$$\partial_\mu \partial^\mu u(\mathbf{x}) + m^2 u(\mathbf{x}) = 0 \tag{2}$$

In the presence of a point-source interaction of strength g, (1) turns into

$$L = L_0 - g\,u(\mathbf{x})\delta(\mathbf{x}) \tag{3}$$

where $\delta(\mathbf{x})$ stands for the 3-dimensional delta function. Since $\delta(\mathbf{x})$ is time-independent, (3) yields

$$(-\Delta + m^2)u(\mathbf{x}) = g\,\delta(\mathbf{x}) \tag{4}$$

which is a particular embodiment of the Poisson equation with a non-vanishing mass term. Assuming spherical symmetry and applying the method of Fourier transform, the solution of (4) recovers the Yukawa potential [1]

$$u(r) = \frac{g}{4\pi}\frac{\exp(-mr)}{r} \tag{5}$$

where $r = |\mathbf{r}|$ is the radial coordinate and the propagators in real and momentum spaces are given by, respectively,

$$G(\mathbf{r}) = u(r) = \frac{g}{(2\pi)^3}\int d^3k\,\frac{\exp(i\mathbf{k}\cdot\mathbf{r})}{\mathbf{k}^2 + m^2} \tag{6}$$

$$G(\mathbf{k}) = \int d^3r\,G(\mathbf{r})\exp(-i\mathbf{k}\cdot\mathbf{r}) \tag{7}$$

Further assuming $m << 1/r$ in (5), produces the leading order approximation

$$u(r) \approx \frac{g}{4\pi}\frac{1 - mr}{r} = \frac{g}{4\pi}(\frac{1}{r} - m) \tag{8}$$

where the mass term acts as correction to the massless potential

$$u_0(r) = \frac{g}{4\pi}\frac{1}{r} \tag{9}$$

As it is known, the massless potential (9) enters the inverse square formulation of both Coulomb electrostatics and Newtonian gravity. The Poisson equation of Newtonian gravitaty in 3-dimensional space is written as

$$\Delta\varphi = 4\pi G\rho \tag{10}$$

where ρ stands for the mass density. An arbitrary distribution of non-relativistic masses is characterized by the potential

$$\varphi = -G\int \frac{\rho\, d^3 r}{r} \tag{11}$$

and total potential energy

$$U = \frac{1}{2}\int \rho\varphi\, d^3 r \tag{12}$$

The potential of a single point-source mass at radial distance r reads

$$\varphi = -\frac{G_N M}{r} \tag{13}$$

where $M = \int \rho\, d^3 r$. Newton's law follows from taking the gradient of (13), that is,

$$F_N = -\frac{\partial\varphi}{\partial r} = -G_N \frac{Mm}{r^2} \tag{14}$$

The $\mathbf{k}^2$ term in the propagator $G(\mathbf{r})$ comes from the Laplacian $\nabla^2 = \Delta$ in (4) and reflects the rotational symmetry of ordinary 3-dimensional space. Likewise, any deviation from the inverse square law (14) signals a breakdown of rotational symmetry, i.e. the onset of *anisotropic* 3-dimensional space [2].

2. Newtonian potential in 3 ± ε space dimensions

How does one go about expanding the Poisson equation and Newtonian gravity in fractional spacetime equipped with minimal fractality? A couple of premises can assist us in this effort:

1) a sensible starting point is to compare the behavior of the Poisson equation in 3 and 4-dimensional space. Analysis shows that the propagator changes from $-1/r$ to $-1/r^2$ when passing from 3 to 4- dimensional space endowed with spherical symmetry [3]. Thus one can reasonably expect a behavior of the type $-1/r^{1\pm\varepsilon}$ from *out-of-equilibrium dimensional fluctuations* driven by $\varepsilon \ll 1$.

2) Fractional derivative and integrals are typically used to describe dynamics on fractional spacetime, which is characterized by *anisotropic* and *non-local* properties. Assuming nearly conservative settings set by $\varepsilon \ll 1$ [4], we choose below to work with Caputo derivative operators of non-integer order α (denoted $D_C^{\alpha} f(r)$) and promote (14) to

$$F_N = -\frac{\partial \varphi(r)}{\partial r} \Rightarrow f_N = \begin{cases} -D_C^{1-\varepsilon}\varphi'(r) \\ -D_C^{1+\varepsilon}\varphi'(r) \end{cases} \tag{15}$$

with $\alpha = 1 \pm \varepsilon$, $\varepsilon \ll 1$ and where the modified Newtonian potential is given by

$$\varphi'(r)=\begin{cases}-\dfrac{GM}{r^{1-\varepsilon}}\\ -\dfrac{GM}{r^{1+\varepsilon}}\end{cases} \tag{16}$$

Consider the first case in (15) in which $D_C^{1-\varepsilon}\varphi'(r)$ subject to the constraint

$$\varepsilon\cdot r<<1,\ \ 1<<r<\infty \tag{17}$$

The Caputo operator may be conveniently expanded as [5]

$$D_C^{1-\varepsilon}\varphi'(r)\ \approx\ \partial^{(1)}\varphi'(r)+\varepsilon D_1^1\varphi'(r)+... \tag{18}$$

in which

$$D_1^1\varphi'(r)=\partial^{(1)}\varphi'(0)\ln r+\gamma\partial^{(1)}\varphi'(r)+\int_0^r\partial^{(2)}\varphi'(r')\ln(r-r')\,dr' \tag{19}$$

The connection with gravitational physics on cosmic scales corresponds to applying (15) to the regime of arbitrarily large radial distances. To this end, we introduce the following observations and assumptions regarding the passage to $r\to\infty$:

1) the potential (16) is well approximated by its classical expression (13) in ordinary 3-dimensional space

$$\varphi'(r)\ \approx\ \varphi\ =-\frac{GM}{r}\ ,\quad \varepsilon<<1 \tag{20}$$

1) the first term of (19) becomes singular as in

$$I_1=(\frac{1}{r^2})\Big|_{r=0}\ln r\ \to\infty \tag{21}$$

2) the second term of (19) goes to zero as in

$$\gamma \partial^{(1)} \varphi'(r) = \gamma \frac{GM}{r^2} \to 0 \tag{22}$$

3) to simplify the derivation, the definite integral in the third term of (19) is approximated with

$$I_3 = \int_0^r \partial^{(2)} \varphi'(r') \ln(r - r')\, dr' \;\approx\; \left\langle \partial^{(2)} \varphi'(r') \ln(r - r') \right\rangle \cdot r \tag{23}$$

where $\langle \ldots \rangle$ denotes the average of the integrand. Then (23) leads to

$$I_3 \;\approx\; \frac{\ln\left(r/2 \right)}{r^2} \tag{24}$$

which goes to zero as $r \to \infty$.

4) when $\varepsilon \to 0$, the second term of (18) contains an undefined product of the form $0 \cdot \infty$. To make progress, we introduce an autonomous time-like parameter τ and consider the case when $D_1^1 \varphi'(r)$ approaches infinity faster than the rate at which ε approaches zero. This amounts to the assumption

$$\left| \frac{d}{d\tau} (D_1^1 \varphi'(r)) \right| >> \left| \frac{d\varepsilon}{d\tau} \right| \tag{25}$$

Under these conditions, it is apparent that the contribution of the second term of (18) has the effect of *strengthening* the ordinary Newtonian potential (13) in the limit $r \to \infty$, a condition matching the observed surge of gravitational pull in the rotation curves of galactic matter. Likewise, retracing the same steps for the case $D_C^{1+\varepsilon} \varphi'(r) = D_C^{1-(-\varepsilon)} \varphi'(r)$, the corresponding second term in (18) flips its sign, a condition matching the apparent *weakening* of gravitational attraction observed in the accelerated expansion of the Universe.

These are our main results and are summarized in the table below:

$r >> 1$, $\varepsilon << 1$	Outcome
$f_N = D_C^{1-\varepsilon}\varphi'(r)$	Gravitational pull of Dark Matter, the flat rotation curves of galactic matter
$f_N = D_C^{1+\varepsilon}\varphi'(r)$	Cosmological expansion and the cosmological constant

3. Follow-up comments

1) Extending the approach to the equations of General Relativity is also beyond the scope of the paper.

2) By the very nature of fractional spacetime, *dimensional fluctuations* in one dimension are driven by $\varepsilon = \pm(1-D) << 1$ and arise as non-equilibrium phenomena on timescales $t << O(M_{EW}^{-1})$, where M_{EW} denotes the Fermi scale. These fluctuations cancel out and are thus un-observable on laboratory timescales $t \sim O(M_{EW}^{-1})$. In line with the principles of fractional dynamics, residual "memory" and spatial nonlocality are expected to carry over cosmological timescales $t >> O(M_{EW}^{-1})$ [4, 6]. As a result, Dark Matter may persist as leftover effect of dimensional fluctuations, whose low-energy manifestation is the process of *topological condensation* [7]. A helpful analogy here is the surviving imbalance of matter over antimatter arising from the Sakharov conditions for baryogenesis [8].

3) As a side note linked to the topic of spatial anisotropy above the Fermi scale, we recall that fractional dynamics breaks the isotropy of ordinary space and leads to violation of parity and time reversal symmetries in electroweak interactions [6]. Ordinary space regains its isotropic properties in the conformal limit $\varepsilon \to 0$. By the same token, it can be shown that spin is a continuous topological property of the *minimal fractal manifold* (MFM), yet it splits into integer states (bosons) and half-integer states (fermions) as $\varepsilon \to 0$ [4]. This is to say that, while the spin-statistics connection is broken on the MFM, it is restored in ordinary 3-dimensional space, where Lorentz and CPT symmetries retain their full validity.

4) Refer to the Newtonian potential in 3 ± ε space (16) and write it as

$$\varphi' \propto \frac{1}{r^{1\pm\varepsilon}} = \frac{1}{r} r^{\mp\varepsilon} \sim \frac{1}{r}(1 \mp \varepsilon \ln r) \ , \ \ \varepsilon << 1, \ \ r < \infty \tag{26}$$

Comparing (8) to (26) leads to the operational identification

$$-\frac{gm}{4\pi} \Rightarrow \mp \frac{\varepsilon \ln r}{r} \tag{27}$$

which sets up a direct mapping between the physical mass $m > 0$ and ε. The most straightforward interpretation of (27) is that mass arises from the minimal fractality of spacetime, an idea elaborated upon in [4, 9].

5) Following the same lines of inquiry, is the Coulomb interaction enhanced or inhibited at large distances? Can fractional spacetime account for the anomalous emission rates of some charged cosmic particles? Furthermore, it is tempting to speculate that the existence of fractional propagators in spacetime with continuous dimensionality may substantially

impact phenomenology at both ends of the energy scale. In this context, several research topics are bound to gain particular attention. Few examples are, but not limited to,

a) the LCDM model and non-standard cosmology,

b) asymptotic freedom and color confinement in quantum chromodynamics (QCD),

c) asymptotically safe gravity,

d) renormalization of gravity in less than 3 spatial dimensions,

e) black hole thermodynamics,

f) leptoquark models and flavor anomalies,

g) other physics scenarios beyond the Standard Model.

6) In line with [10], our approach suggests an integrated description of the Dark Sector and a unified picture of Dark Matter as dual manifestation of Cantor Dust and gravitational physics. It also explains away the premises on which MOND and related modified gravity theories are built.

References

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