# Pappus chain and division by zero 

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Abstract. We consider a Pappus chain using division by zero.
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## 1. Introduction

Let $O$ be a point on the segment $A B$ such that $|A O|=2 a$ and $|B O|=2 b$ (see Figure 1). For an arbelos configuration formed by three circles $\alpha, \beta$ and $\gamma$ with diameters $A O, B O$ and $A B$, respectively, we consider circles touching two of the three circles by division by zero [2]:

$$
\begin{equation*}
\frac{z}{0}=0 \text { for any real number } z \tag{1}
\end{equation*}
$$



Figure 1.

## 2. Circles touching two of $\alpha, \beta$ and $\gamma$

If a circle touches one of given two circles internally and the other externally, we say that the circle touches the two circles in the opposite sense, otherwise in the same sense. Let $c=a+b$.

Theorem 1. The following statements hold.
(i) A circle touches the circles $\beta$ and $\gamma$ in the opposite sense if and only if its has radius and center of coordinates

$$
r_{\alpha z}=\frac{a b c}{a^{2} z^{2}+b c} \quad \text { and } \quad\left(x_{\alpha z}, y_{\alpha z}\right)=\left(-2 b+\frac{b c(b+c)}{a^{2} z^{2}+b c}, 2 z r_{\alpha z}\right)
$$

for a real number $z$.
(ii) A circle touches the circles $\gamma$ and $\alpha$ in the opposite sense if and only if its has radius and center of coordinates

$$
r_{\beta z}=\frac{a b c}{b^{2} z^{2}+c a} \text { and }\left(x_{\beta z}, y_{\beta z}\right)=\left(2 a-\frac{c a(c+a)}{b^{2} z^{2}+c a}, 2 z r_{\beta z}\right)
$$

for a real number $z$.
(iii) $A$ circle touches the circles $\alpha$ and $\beta$ in the same sense if and only if its has radius and center of coordinates

$$
r_{\gamma z}=\frac{a b c}{\left|c^{2} z^{2}-a b\right|} \text { and }\left(x_{\gamma z}, y_{\gamma z}\right)=\left(\frac{a b(b-a)}{c^{2} z^{2}-a b}, \frac{2 a b c z}{c^{2} z^{2}-a b}\right)
$$

for a real number $z \neq \pm \sqrt{a b} / c$.
Proof. We denote the circle of radius and center described in (i) by $\delta$. Since the square of the distance between the centers of $\alpha$ and $\delta$ equals the square of the sum of the radii of $\alpha$ and $\delta, \alpha$ and $\delta$ touch externally. Similarly $\gamma$ and $\delta$ touch internally. This proves (i). The rest of the theorem can be proved similarly.

We denote the circle of radius $r_{\alpha z}$ and center of coordinates $\left(x_{\alpha z}, y_{\alpha z}\right)$ by $\alpha_{z}$. The circle $\alpha_{z}$ has an equation $\left(x-x_{\alpha z}\right)^{2}+\left(y-y_{\alpha z}\right)^{2}=r_{\alpha z}^{2}$, which is arranged as

$$
\alpha_{z}(x, y)=\frac{b c\left((x-a)^{2}+y^{2}-a^{2}\right)-4 a b c y z+a^{2}\left((x+2 b)^{2}+y^{2}\right) z^{2}}{a^{2} z^{2}+b c}=0 .
$$

Therefore we get $(x-a)^{2}+y^{2}=a^{2}, y=0$ and $(x+2 b)^{2}+y^{2}=0$ in the case $z=0$ from $\alpha_{z}(x, y)=0, \alpha_{z}(x, y) / z=0$, and $\alpha_{z}(x, y) / z^{2}=0$, respectively by (1). They represent the circle $\alpha=\alpha_{0}$, the line $A B$ and the point $B$, respectively. We denote the point $B$ and the line $A B$ by $\alpha_{\infty}$ and $\alpha_{\bar{\infty}}$, respectively, and consider that they also touch $\alpha$ and $\gamma$ (see Figure 2). Someone may consider that $\alpha_{\bar{\infty}}$ is orthogonal to $\alpha$ and $\gamma$ and does not touch them. But (1) implies $\tan (\pi / 2)=0$. Therefore we can still consider that $\alpha_{\bar{\infty}}$ touches $\alpha$ and $\gamma$. We also consider that $\alpha_{\infty}$ and $\alpha_{\bar{\infty}}$ touch. Notice that $\alpha_{0}=\alpha$.


Figure 2.
Similarly we denote the circle of radius $r_{\beta z}$ and the center of coordinates $\left(x_{\beta z}, y_{\beta z}\right)$ by $\beta_{z}$. We have $\beta_{0}=\beta$, and denote the point $A$ and the line $A B$ by $\beta_{\infty}$ and $\beta_{\bar{\infty}}$, respectively. Also we denote the circle of radius $r_{\gamma z}$ and the center
of coordinates $\left(x_{\gamma z}, y_{\gamma z}\right)$ by $\gamma_{z}$. Notice that $\alpha_{1}=\beta_{1}=\gamma_{1}$ is the incircle of the arbelos in the regin $y \geq 0$. We have $\gamma_{0}=\gamma$, and denote the point $O$ and the line $A B$ by $\gamma_{\infty}$ and $\gamma_{\bar{\infty}}$, respectively. Let $d=\sqrt{a b} / c$. The external common tangent of $\alpha$ and $\beta$ has an equation $[16,17]$

$$
\begin{equation*}
(a-b) x \mp 2 \sqrt{a b} y+2 a b=0, \tag{2}
\end{equation*}
$$

which is denoted by $\gamma_{ \pm d}$.

## 3. Pappus chain

Let $r_{\mathrm{A}}=a b /(a+b)$.
Theorem 2. Let $w$ and $z$ be real numbers. Then each of the two circles of the three pairs $\alpha_{z}, \alpha_{w} ; \beta_{z}, \beta_{w} ; \gamma_{z}, \gamma_{w}$ touch if and only if $|w-z|=1$.

Proof. If $|w| \neq d$ and $|z| \neq d$, we get

$$
\left(x_{\gamma w}-x_{\gamma z}\right)^{2}+\left(y_{\gamma w}-y_{\gamma z}\right)^{2}-\left(r_{\gamma w}+r_{\gamma z}\right)^{2}=\frac{4 a^{2} b^{2} c^{2}\left((w-z)^{2}-1\right)}{\left(c^{2} w^{2}-a b\right)\left(c^{2} z^{2}-a b\right)} .
$$

Hence $\gamma_{w}$ and $\gamma_{z}$ touch if and only if $|w-z|=1$. While $\gamma_{d}$ and $\gamma_{d \pm 1}$ have only one point in common, whose coordinates equal

$$
\begin{equation*}
\left(2 r_{\mathrm{A}} \frac{(\sqrt{a} \mp \sqrt{b})^{2}}{-(a-b)}, \pm 2 r_{\mathrm{A}}\right) . \tag{3}
\end{equation*}
$$

Therefore they touch. Since the figure is symmetric in $A B, \gamma_{-d}$ and $\gamma_{-d \pm 1}$ also touch. The rest of the theorem is proved in a similar way.

Circles of radius $r_{\mathrm{A}}$ are said to be Archimedean. The next corollary is given by (3).
Corollary 1. If $z=d$ or $z=-d$, then the smallest circle passing through the point of tangency of $\gamma_{z}$ and $\gamma_{z \pm 1}$ and touching $A B$ is Archimedean.

## 4. Division by zero calculus

For the Laurent expansion of a function $f(z)$ around $z=a$ :

$$
f(z)=\sum_{n=-1}^{-\infty} C_{n}(z-a)^{n}+C_{0}+\sum_{n=-1}^{\infty} C_{n}(z-a)^{n}
$$

the definition $f(a)=C_{0}$ is called the division by zero calculus [13], [18].
Let

$$
\gamma(z)=\left(x-x_{\gamma z}\right)^{2}+\left(y-y_{\gamma z}\right)^{2}-r_{\gamma z}^{2} .
$$

If

$$
\gamma(z)=\cdots+C_{-2}(z-d)^{-2}+C_{-1}(z-d)^{-1}+C_{0}+\cdots
$$

is the Laurent expansion of $\gamma(z)$ around $z=d$, then

$$
C_{-1}=\frac{\sqrt{a b}}{a+b}((a-b) x-2 \sqrt{a b} y+2 a b) .
$$

Therefore we get half part of (2). Also from the Laurent expansion of $\gamma(z)$ aruond $z=-d$ :

$$
\gamma(z)=\cdots+C_{-2}(z+d)^{-2}+C_{-1}(z+d)^{-1}+C_{0}+\cdots,
$$

we get

$$
C_{-1}=-\frac{\sqrt{a b}}{a+b}((a-b) x+2 \sqrt{a b} y+2 a b)
$$

Therefore we get the rest part of (2).
For more applications of division by zero to circle geometry, see [1], [3], [4, 5, $6,7,8,9,10,11,12][13,14,15,16]$, and an extensive referece see [18].

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