Pappus chain and division by zero

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Abstract. We consider a Pappus chain using division by zero.

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1. Introduction

Let O be a point on the segment AB such that $|AO| = 2a$ and $|BO| = 2b$ (see Figure 1). For an arbelos configuration formed by three circles α, β and γ with diameters AO, BO and AB, respectively, we consider circles touching two of the three circles by division by zero [2]:

$$\frac{z}{0} = 0 \quad \textit{for any real number } z. \tag{1}$$

We consider with a rectangular coordinates system with origin O such that the coordinates of A and B are $(2a, 0)$ and $(-2b, 0)$, respectively.

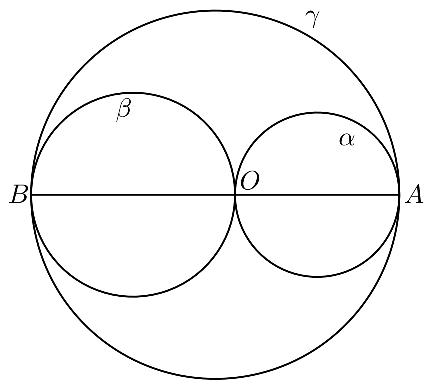

Figure 1.

2. Circles touching two of α, β and γ

If a circle touches one of given two circles internally and the other externally, we say that the circle touches the two circles in the opposite sense, otherwise in the same sense. Let $c = a + b$.

Theorem 1. *The following statements hold.*
(i) *A circle touches the circles β and γ in the opposite sense if and only if its has radius and center of coordinates*

$$r_{\alpha z} = \frac{abc}{a^2z^2 + bc} \quad \textit{and} \quad (x_{\alpha z}, y_{\alpha z}) = \left(-2b + \frac{bc(b+c)}{a^2z^2 + bc}, 2zr_{\alpha z}\right)$$

for a real number z.
(ii) *A circle touches the circles* γ *and* α *in the opposite sense if and only if its has radius and center of coordinates*

$$r_{\beta z} = \frac{abc}{b^2z^2+ca} \quad \textit{and} \quad (x_{\beta z}, y_{\beta z}) = \left(2a - \frac{ca(c+a)}{b^2z^2+ca}, 2zr_{\beta z}\right)$$

for a real number z.
(iii) *A circle touches the circles* α *and* β *in the same sense if and only if its has radius and center of coordinates*

$$r_{\gamma z} = \frac{abc}{|c^2z^2-ab|} \quad \textit{and} \quad (x_{\gamma z}, y_{\gamma z}) = \left(\frac{ab(b-a)}{c^2z^2-ab}, \frac{2abcz}{c^2z^2-ab}\right)$$

for a real number $z \neq \pm\sqrt{ab}/c$.

Proof. We denote the circle of radius and center described in (i) by δ. Since the square of the distance between the centers of α and δ equals the square of the sum of the radii of α and δ, α and δ touch externally. Similarly γ and δ touch internally. This proves (i). The rest of the theorem can be proved similarly. □

We denote the circle of radius $r_{\alpha z}$ and center of coordinates $(x_{\alpha z}, y_{\alpha z})$ by α_z. The circle α_z has an equation $(x-x_{\alpha z})^2 + (y-y_{\alpha z})^2 = r_{\alpha z}^2$, which is arranged as

$$\alpha_z(x,y) = \frac{bc((x-a)^2+y^2-a^2) - 4abcyz + a^2((x+2b)^2+y^2)z^2}{a^2z^2+bc} = 0.$$

Therefore we get $(x-a)^2+y^2=a^2$, $y=0$ and $(x+2b)^2+y^2=0$ in the case $z=0$ from $\alpha_z(x,y)=0$, $\alpha_z(x,y)/z=0$, and $\alpha_z(x,y)/z^2=0$, respectively by (1). They represent the circle $\alpha=\alpha_0$, the line AB and the point B, respectively. We denote the point B and the line AB by α_∞ and $\alpha_{\overline{\infty}}$, respectively, and consider that they also touch α and γ (see Figure 2). Someone may consider that $\alpha_{\overline{\infty}}$ is orthogonal to α and γ and does not touch them. But (1) implies $\tan(\pi/2)=0$. Therefore we can still consider that $\alpha_{\overline{\infty}}$ touches α and γ. We also consider that α_∞ and $\alpha_{\overline{\infty}}$ touch. Notice that $\alpha_0=\alpha$.

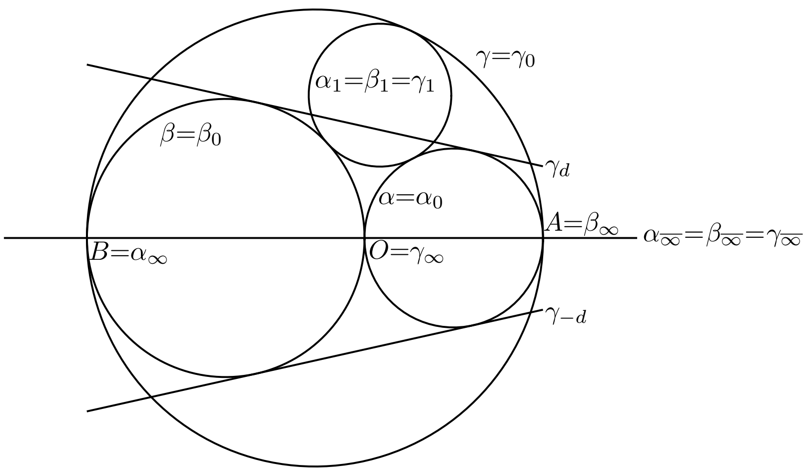

Figure 2.

Similarly we denote the circle of radius $r_{\beta z}$ and the center of coordinates $(x_{\beta z}, y_{\beta z})$ by β_z. We have $\beta_0=\beta$, and denote the point A and the line AB by β_∞ and $\beta_{\overline{\infty}}$, respectively. Also we denote the circle of radius $r_{\gamma z}$ and the center of coordinates $(x_{\gamma z}, y_{\gamma z})$ by γ_z. Notice that $\alpha_1 = \beta_1 = \gamma_1$ is the incircle of the arbelos in the regin $y \geq 0$. We have $\gamma_0 = \gamma$, and denote the point O and the line AB by γ_∞ and $\gamma_{\overline{\infty}}$, respectively. Let $d = \sqrt{ab}/c$. The external common tangent of α and β has an equation [16, 17]

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0, \tag{2}$$

which is denoted by $\gamma_{\pm d}$.

3. Pappus chain

Let $r_{\mathrm{A}} = ab/(a+b)$.

Theorem 2. *Let w and z be real numbers. Then each of the two circles of the three pairs α_z, α_w; β_z, β_w; γ_z, γ_w touch if and only if $|w-z| = 1$.*

Proof. If $|w| \neq d$ and $|z| \neq d$, we get

$$(x_{\gamma w} - x_{\gamma z})^2 + (y_{\gamma w} - y_{\gamma z})^2 - (r_{\gamma w} + r_{\gamma z})^2 = \frac{4a^2b^2c^2((w-z)^2-1)}{(c^2w^2-ab)(c^2z^2-ab)}.$$

Hence γ_w and γ_z touch if and only if $|w-z| = 1$. While γ_d and $\gamma_{d\pm1}$ have only one point in common, whose coordinates equal

$$\left(2r_{\mathrm{A}}\frac{(\sqrt{a} \mp \sqrt{b})^2}{-(a-b)}, \pm 2r_{\mathrm{A}}\right). \tag{3}$$

Therefore they touch. Since the figure is symmetric in AB, γ_{-d} and $\gamma_{-d\pm1}$ also touch. The rest of the theorem is proved in a similar way. □

Circles of radius r_{A} are said to be Archimedean. The next corollary is given by (3).

Corollary 1. *If $z = d$ or $z = -d$, then the smallest circle passing through the point of tangency of γ_z and $\gamma_{z\pm1}$ and touching AB is Archimedean.*

4. Division by zero calculus

For the Laurent expansion of a function $f(z)$ around $z = a$:

$$f(z) = \sum_{n=-1}^{-\infty} C_n(z-a)^n + C_0 + \sum_{n=-1}^{\infty} C_n(z-a)^n,$$

the definition $f(a) = C_0$ is called the division by zero calculus [13], [18].

Let

$$\gamma(z) = (x - x_{\gamma z})^2 + (y - y_{\gamma z})^2 - r_{\gamma z}^2.$$

If

$$\gamma(z) = \cdots + C_{-2}(z-d)^{-2} + C_{-1}(z-d)^{-1} + C_0 + \cdots$$

is the Laurent expansion of $\gamma(z)$ around $z = d$, then

$$C_{-1} = \frac{\sqrt{ab}}{a+b}((a-b)x - 2\sqrt{ab}y + 2ab).$$

Therefore we get half part of (2). Also from the Laurent expansion of $\gamma(z)$ around $z = -d$:

$$\gamma(z) = \cdots + C_{-2}(z+d)^{-2} + C_{-1}(z+d)^{-1} + C_0 + \cdots,$$

we get

$$C_{-1} = -\frac{\sqrt{ab}}{a+b}((a-b)x + 2\sqrt{ab}y + 2ab).$$

Therefore we get the rest part of (2).

For more applications of division by zero to circle geometry, see [1], [3], [4, 5, 6, 7, 8, 9, 10, 11, 12] [13, 14, 15, 16], and an extensive reference see [18].

REFERENCES

[1] Y. Kanai, H. Okumura, A three tangent congruent circle problem, Sangaku J. Math., **1** (2017) 16–20.

[2] M. Kuroda, H. Michiwaki, S. Saitoh, M. Yamane, New meanings of the division by zero and interpretations on $100/0 = 0$ and on $0/0 = 0$, Int. J. Appl. Math., **27**(2) (2014) 191–198.

[3] T. Matsuura, H. Okumura, S. Saitoh, Division by zero calculus and Pompe's theorem, Sangaku J. Math., **3** (2019) 36–40.

[4] H. Okumura, Remarks on Archimedean circles of Nagata and Ootoba, Sangaku J. Math., **3** (2019) 119–122.

[5] H. Okumura, The arbelos in Wasan geometry: Ootoba's problem and Archimedean circles, Sangaku J. Math., **3** (2019) 91–97.98–104.

[6] H. Okumura, Remarks on Archimedean circles of Nagata and Ootoba, Sangaku J. Math.,**3** (2019) 119–122.

[7] H. Okumura, The arbelos in Wasan geometry: Ootoba's problem and Archimedean circles, Sangaku J. Math.,**3** (2019) 91–97.

[8] H. Okumura, A characterization of the golden arbelos involving an Archimedean circle, Sangaku J. Math., **3** (2019) 67–71.

[9] H. Okumura, An analogue of Pappus chain theorem with division by zero, Forum Geom., **18** (2018) 409–412.

[10] H. Okumura, Solution to 2017-1 Problem 4 with division by zero, Sangaku J. Math., **2** (2018) 27–30.

[11] H. Okumura, Wasan geometry with the division by 0, Int. J. Geom., **8**(1)(2018), 17-20.

[12] H. Okumura, Is it really impossible to divide by zero?, Biostat Biometrics Open Acc J. **7**(1) (2018): 555703. DOI: 10.19080/BBOJ.2018.07.555703.

[13] H. Okumura, S. Saitoh, Wasan geometry and division by zero calculus, Sangaku J. Math., **2** (2018) 57–73.

[14] H. Okumura and S. Saitoh, Applications of the division by zero calculus to Wasan geometry, Glob. J. Adv. Res. Class. Mod. Geom., **7**(2) (2018) 44–49.

[15] H. Okumura and S. Saitoh, Harmonic mean and division by zero, Forum Geom., **18** (2018) 155–159.

[16] H. Okumura and S. Saitoh, Remarks for The Twin Circles of Archimedes in a Skewed Arbelos by Okumura and Watanabe, Forum Geom., **18** (2018) 97–100.

[17] H. Okumura and M. Watanabe, The twin circles of Archimedes in a skewed arbelos, Forum Geom., **4** (2004) 229–251.

[18] S. Saitoh, Division by zero calculus (draft), 2019.