A four circle problem and division by zero

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Abstract. We generalize a problem involving four circles and a triangle, and consider some limiting cases of the problem by division by zero.

Keywords. four circle problem, division by zero.

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1. Introduction

We generalize the following problem involving four circles and a triangle in [20]. The same sangaku problem was proposed in 1826 and cited in [19] and [1] with no solution. Some limiting cases of the problem will be considered by division by zero [6].

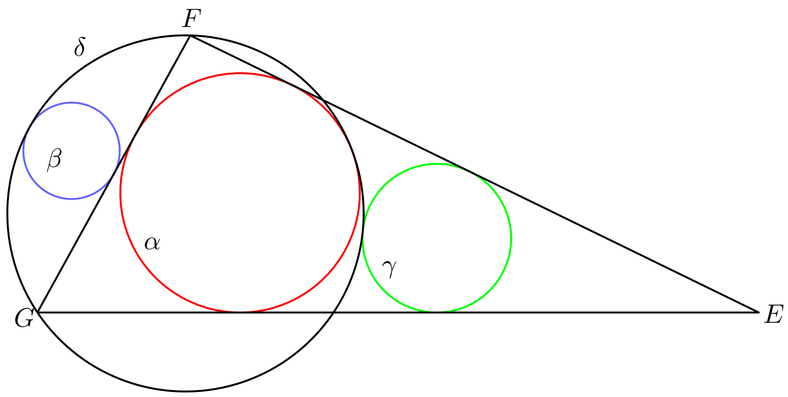

Figure 1.

Problem 1. For a triangle EFG with incircle α, δ is the circle passing through E and F and touching α, γ is the incircle of the curvilinear triangle made by δ and the sides EF and GE, and β is the circle touching δ and FG at the midpoint from the side opposite to α. Let a, b and c be the radii of α, β and γ, respectively. Show $a^2 = 4bc$.

A similar sangaku problem considering the case $|EF| = |GE|$ can be found in [2, p. 302].

2. Generalization

The problem assumes that α is the incircle of EFG, but we show that the condition is inessential. We consider the following figure (see Figure 2): For a chord FG of a circle δ, M is the midpoint of FG, β is a circle touching δ and FG at M, α is a circle touching δ and the chord FG from the side opposite to β, f and g are the tangents of α from the points F and G, respectively, γ is the circle lying on the same side of FG as α and touching δ externally and f and g from the same side as α. Let a, b, c and d be the radii of α, β, γ and δ, respectively. We denote this configuration by $\mathcal{S}$.

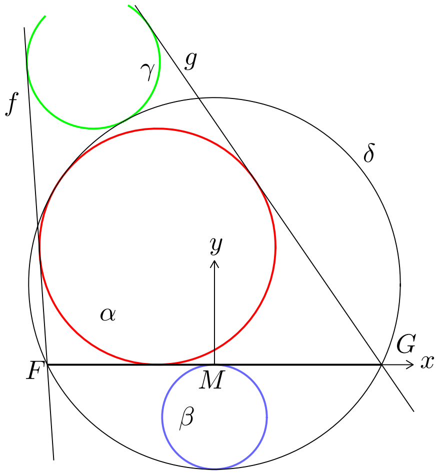

Figure 2: The configuration $\mathcal{S}$.

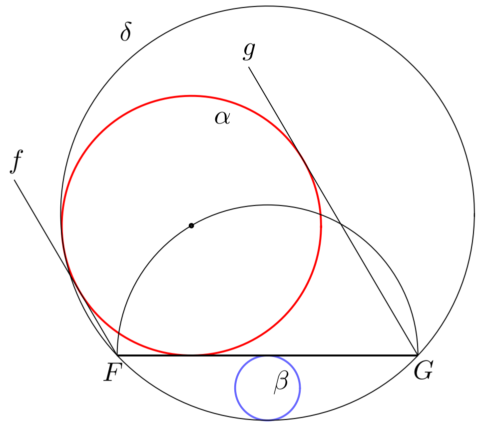

Figure 3: $4b = a = c$

We use a rectangular coordinate system with origin M such that the center of α has coordinate (x_a, a) for a real number x_a. Firstly we consider a special case in which f and g are parallel (see Figure 3).

Theorem 1. *The following statements are equivalent for $\mathcal{S}$.*
(i) *The lines f and g are parallel.*
(ii) *The center of α lies on the circle of diameter FG.*
(iii) $a = 4b$.

Proof. We may assume that the point G has coordinates $(k, 0)$, and f and g have equations $x + m_1 y + k = 0$ and $x + m_2 y - k = 0$, respectively for real numbers m_1 and m_2. Since f and g touch α, we have

$$m_1 = \frac{a^2 - (k + x_a)^2}{2a(k + x_a)}, \quad m_2 = -\frac{a^2 - (k - x_a)^2}{2a(k - x_a)}. \tag{1}$$

Notice that $k^2 - x_a^2 \neq 0$, since $k^2 - x_a^2 = 0$ implies that α touches FG at F or G. The lines f and g are parallel if and only if $m_1 = m_2$, which is equivalent to

$$a^2 + x_a^2 = k^2. \tag{2}$$

This proves the equivalence of (i) and (ii), since the left side equals the square of the distance between the center of α and M (see Figure 3). While the square of the distance between the centers of δ and α equals

$$x_a^2 + (d - 2b - a)^2 = (d - a)^2. \tag{3}$$

And the power of the origin with respect to δ equals

$$-2b(2d - 2b) = -k^2. \tag{4}$$

Eliminating d from (3) and (4), we get $xa^2 + 4ab = k^2$, which implies

$$a^2 + xa^2 - k^2 = a(a - 4b).$$

Hence (2) and $a = 4b$ are equivalent, i.e., (i) and (iii) are equivalent. □

Corollary 1. *One of the three relations $4b < a < c$, $4b = a = c$, $4b > a > c$ holds for $\mathcal{S}$.*

Figures 2, 3 and 4 show the cases $4b > a > c$, $4b = a = c$ and $4b < a < c$, respectively. The next theorem is a generalization of Problem 1.

Theorem 2. *The following statements hold.*
(i) *The relation $a^2 = 4bc$ holds.*
(ii) *One of the internal common tangents of α and γ is parallel to FG.*

Proof. We use the same notation as in the proof of Theorem 1. If f and g are parallel, we get $a = c$. Therefore we get $a^2 = 4bc$ by Theorem 1. We assume that f and g intersect. We denote the point of intersection by E, which has coordinates

$$(x_e, y_e) = \left(\frac{k(m_1 + m_2)}{m_1 - m_2}, \frac{-2k}{m_1 - m_2}\right). \tag{5}$$

Substituting (1) in (5), we get

$$(x_e, y_e) = \left(x_a - \frac{2a^2 x_a}{a^2 - k^2 + x_a^2}, 2a - \frac{2a^3}{a^2 - k^2 + x_a^2}\right). \tag{6}$$

The square of the distance between the centers of δ and γ equals

$$x_c^2 + (d - 2b - y_c)^2 = (c + d)^2, \tag{7}$$

where (x_c, y_c) are the coordinates of the center of γ. Since E is the external center of similitude of α and γ, we get

$$\frac{-cx_a + ax_c}{a - c} = x_e, \quad \frac{-ca + ay_c}{a - c} = y_e. \tag{8}$$

Eliminating x_a, x_c, y_c, x_e, y_e and d from (3), (4), (6), (7), (8), we get

$$(a^2 - 4bc)j(k) = 0,$$

where $j(k) = 4(a - 4b)b^2 - (4b - c)k^2$. If $j(k) = 0$, we have $k^2 = 4(a - 4b)b^2/(4b - c) > 0$. This implies $a < 4b < c$ or $c < 4b < a$. However this contradicts Corollary 1. Therefore we get $j(k) \neq 0$, which implies $a^2 = 4bc$.

We prove (ii). If f and g are parallel, the centers of α, γ and M are collinear, i.e., $x_a/a = x_c/y_c$. Eliminating b, c, k, x_a, x_c from the equations $x_a/a = x_c/y_c$, $a = c$, $a = 4b$, (3), (4) and (7), we get

$$(3a - y_c)((a + 4d)a + (4d - a)y_c) = 0.$$

Therefore we get $y_c = 3a = 2a + c$, since $(4d - a)y_c > 0$. If f and g intersect, we eliminate b, k, x_a, x_c, x_e, y_e from (3), (4), (6), (7), (8). Then we get

$$(2a + c - y_c)((a + 4d)c + (4d - a)y_c) = 0.$$

Therefore we get $y_c = 2a + c$. This proves (ii). $\square$

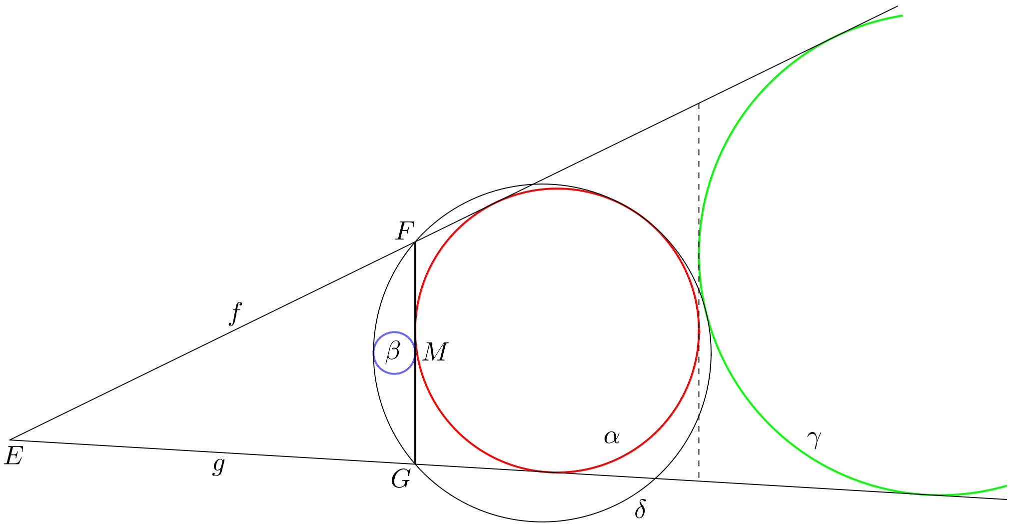

Figure 4: The configuration $\mathcal{S}$ in the case $4b < a < c$.

There are several sangaku problems stating the next corollary [2, p. 312, p. 317, p. 419] (see Figure 5).

Corollary 2. *For a semicircle δ with diameter FG, let α be the circle of radius a touching δ and FG at the midpoint. If c is the inradius of the curvilinear triangle made by δ and the tangents of α from the points E and F, then $a = 4c$.*

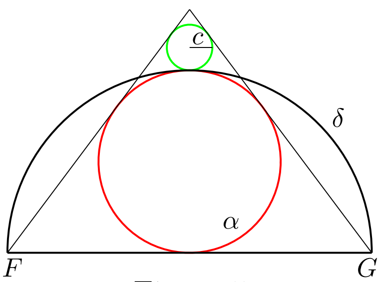

Figure 5.

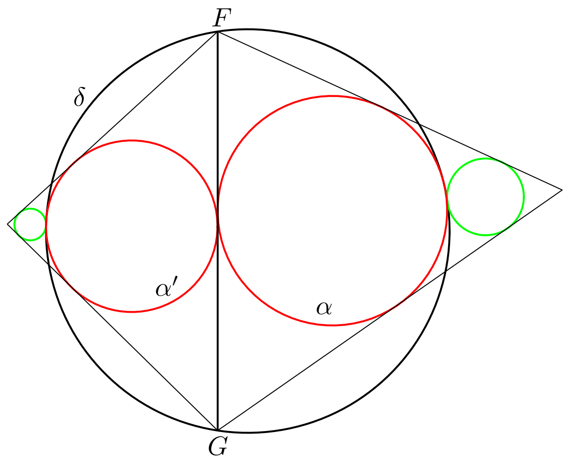

Figure 6.

The next corollary can be found in the sangaku hung in 1830 [3, p. 40], which is incorrectly cited in [1, p. 34] (see Figure 6).

Corollary 3. *For the configuration $\mathcal{S}$, let α' be a circle of radius a' touching the circle δ and its chord FG from the side opposite to α. If the inradius of the curvilinear triangle made by δ and the tangents of α' from F and G equals c', then $a^2a'^2 = cc'|FG|^2$.*

Proof. Let b' be the radius of the circle touching δ and FG at the midpoint from the side opposite to α'. Then we have $a'^2 = 4b'c'$, while $|FG|^2 = 16bb'$ and $a^2 = 4bc$. Eliminating b and b' from the three equations, we get $a^2a'^2 = |FG|^2cc'$. □

3. Limitimg cases with division by zero

In this section we fix the circle δ for $\mathcal{S}$ and consider the case where one of the circles α and β has radius 0 with the definition of division by zero [6]:

$$\frac{z}{0} = 0 \textit{ for a complex number } z. \tag{9}$$

Notice that the definition implies that lines have radius 0 as circles [17].

We now consider a simple case in which the centers of α, β and γ are collinear for $\mathcal{S}$ and use a rectangular coordinate system with origin at the point of tangency of β and δ such that the center of δ has coordinates $(0, d)$. The point of tangency of γ and δ and the tangent of δ at the point are denoted by D and t, respectively (see Figure 7). Notice that $d = a + b$.

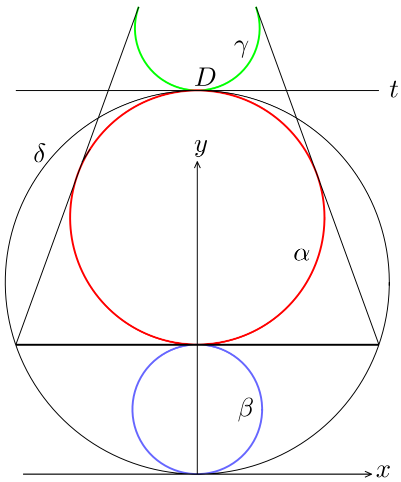

Figure 7.

3.1. **The case $b = 0$.** Firstly we consider the case $b = 0$. Then β is a point or a line. The circle α has an equation $x^2 + (y - (b + d))^2 = (b - d)^2$, which is arranged as

$$f_a(x, y) = (x^2 + (y - d)^2 - d^2) + 2b(2d - y) = 0. \tag{10}$$

From $f_a = 0$, we get $x^2 + (y - d)^2 = d^2$ in the case $b = 0$. Also from $f_a/b = 0$ we get $y = 2d$ in the case $b = 0$ by (9). Hence *α coincides with the circle δ or the line t* in the case $b = 0$.

The circle β has an equation

$$f_b(x, y) = (x^2 + y^2) - 2by = 0.$$

From $f_b = 0$ we get $x^2 + y^2 = 0$ in the case $b = 0$. Also from $f_b/b = 0$ we get $y = 0$ in the case $b = 0$ by (9). Hence *β coincides with the origin or the x-axis* in this case.

The circle γ has an equation $x^2 + (y - 2d - c)^2 = c^2$, where $c = (d - b)^2/(4b)$, which is arranged as.

$$f_c(x, y) = \frac{d^2}{2b}(2d - y) + \left(x^2 + \left(y - \frac{3d}{2}\right)^2 - \frac{d^2}{4}\right) + \frac{b}{2}(2d - y) = 0.$$

From $f_c = 0$ we get $x^2 + (y - 3d/2)^2 = (d/2)^2$ in the case $b = 0$. Also from each of $f_c b = 0$ and $f_c/b = 0$ we get $y = 2d$ in the case $b = 0$. Hence γ *coincides with the line* t *or the circle of radius* $d/2$ *touching* δ *at* D in this case.

When β approaches to the origin, the circles α and γ approach to δ and t, respectively. Therefore we can consider that α and γ coincide with δ and t, respectively when β degenerates to the origin, (see Figure 8). The relation $a^2 = 4bc$ does not holds in this case, but $a^2/b = 4c$ and $a^2/c = 4b$ hold by (9), since the radius of t equals 0.

When β coincides with the x-axis, we can thereby consider that α and γ coincides with t and the circle of radius $d/2$ touching δ at D, respectively as the remaining case (see Figure 9). The relation $a^2 = 4bc$ holds in this case.

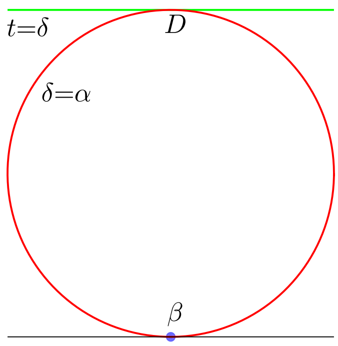

Figure 8.

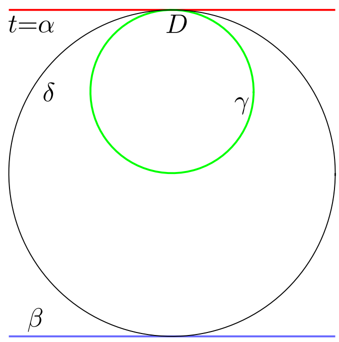

Figure 9.

case	α	β	γ	relation of the radii
1	δ	origin	t	$a^2/b = 4c$, $a^2/c = 4b$
2	t	x-axis	circle of radius $d/2$ touching δ at D	$a^2 = 4bc$

Table 1: $b = 0$.

We summarize the results in Table 1. The case 1 described in Figure 8 is supposable without (9). But (9) enable us to get the case by algebraic manipulation. On the other hand, the case 2 described in Figure 9 can not be obtained without (9). In this case $d = a + b$ does not hold, but still can be considered that α and β touch. However we cannot attain a reasoned interpretation for this case at the current moment. A similar phenomenon, in which a circle of half the radius appears, can be found in [8].

3.2. **The case $a = 0$.** We now consider the case $a = 0$. Substituting $b = d - a$ in (10), we get

$$f_a = (x^2 + (y - 2d)^2) + 2a(y - 2d) = 0.$$

Hence we get $x^2 + (y - 2d)^2 = 0$ or $y = 2d$ in the case $a = 0$. Therefore α *coincides with* D *or* t in this case. Similarly we have

$$f_b = (x^2 + (y - d)^2 - d^2) + 2ay = 0.$$

Therefore we get $x^2 + (y - d)^2 = d^2$ or $y = 0$ in the case $a = 0$. Hence β *coincides with* δ *or the* x*-axis* in the case $a = 0$. Also we have

$$f_c = 2d(x^2 + (y - 2d)^2) + 2a(x^2 + (y - 2d)^2) + a^2(2d - y) = 0.$$

Therefore we get $x^2 + (y - 2d)^2 = 0$ or $y = 2d$ in the case $a = 0$. Hence γ *coincides with* D *or* t in this case.

When α approaches to D, β and γ approach to δ and D, respectively. Hence we consider that β and γ coincide with δ and D, respectively when α coincides with D (see Figure 10). As the remaining case β and γ coincide with the x-axis and t, respectively when α coincides with t (see Figure 11).

We summarize the results in Table 2. The case 3 described in Figure 10 is supposable without (9). On the other hand, the case 4 described in Figure 11 can not be obtained without (9). However we cannot attain a reasoned interpretation for this case at the current moment.

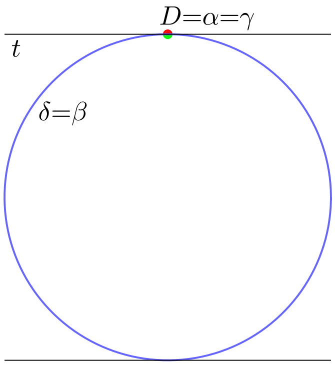

Figure 10.

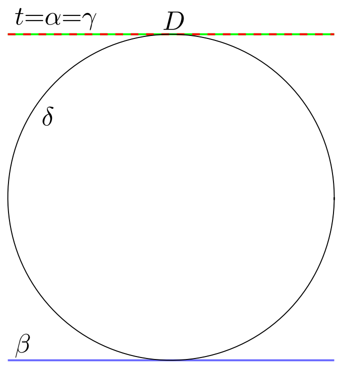

Figure 11.

case	α	β	γ	relation of the radii
3	D	δ	D	$a^2 = 4bc$
4	t	x-axis	t	$a^2 = 4bc$

Table 2: $a = 0$.

For a brief introduction of division by zero with Wasan geometry see [14], and its application to Wasan geometry see [4], [8], [9, 10, 11, 12, 13], [15]. For an extensive reference of division by zero and division by zero calculus, see [17].

4. Incorrect sangaku problems

In [16] we have considered two incorrect sangaku problems in [5, p. 69, p. 123], each of which can also be found in [7] and [21], respectively.

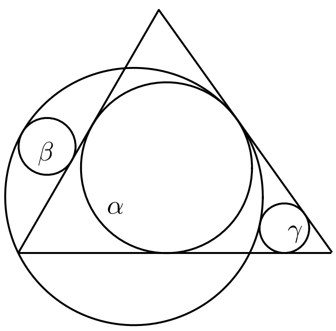

Figure 12: The figures in [5], [21].

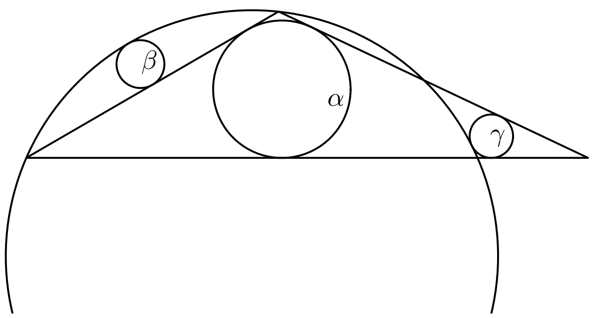

Figure 13: The figure in [5].

The problems and the answers are almost the same as Problem 1, i.e., they demand to show the relation $a^2 = 4bc$ for three circles α, β and γ of radii a, b and c, respectively. However the figures are slightly different as shown in Figures 12 and 13. The figure in [21] is also the same as Figure 12. It seems that those problems were correct and essentially the same as Problem 1 in the original context but the figures were incorrectly transcribed in [5] and [21]. While the figure in [7] is the same as Figure 1, therefore the problem is correct.

References

[1] H. Fukagawa, D. Pedoe, Japanese Temple Geometry Problems, 1989, Charles Babbage Research Centre, Winnipeg Canada.

[2] Fukushimaken Wasan Kenkyū Hozonkai (福島県和算研究保存会) ed., The Sangaku in Fukushima (福島の算額), Sōju Shuppan (蒼樹出版), 1989.

[3] A. Hirayama (平山諦), M. Ohno (大野政治), A. Mitsuhashi (三橋愛子) ed., The sangaku in Chiba prefecture (千葉県の算額), Naritasan Shiryōkan (成田山資料館), 1970.

[4] Y. Kanai, H. Okumura, A three tangent congruent circle problem, Sangaku J. Math., **1** (2017) 16–20.

[5] Kinki Sūgakushigakukai (近畿数学史学会) ed., The Sangaku in Kinki (近畿の算額), Osaka Kyōiku Tosho (大阪教育図書), 1992.

[6] M. Kuroda, H. Michiwaki, S. Saitoh, M. Yamane, New meanings of the division by zero and interpretations on $100/0 = 0$ and on $0/0 = 0$, Int. J. Appl. Math., **27**(2) (2014) 191–198.

[7] H. Kuwabara (桑原秀夫), R. Yoshida (吉田柳二), T. Yamaguchi (山口正) ed., The sangaku in Shiga (滋賀の算額), 1977.

[8] T. Matsuura, H. Okumura, S. Saitoh, Division by zero calculus and Pompe's theorem, Sangaku J. Math., **3** (2019) 36–40.

[9] H. Okumura, Remarks on Archimedean circles of Nagata and Ootoba, Sangaku J. Math., **3** (2019) 119–122.

[10] H. Okumura, The arbelos in Wasan geometry: Ootoba's problem and Archimedean circles, Sangaku J. Math., **3** (2019) 91–97.98–104.

[11] H. Okumura, A characterization of the golden arbelos involving an Archimedean circle, Sangaku J. Math., **3** (2019) 67–71.

[12] H. Okumura, Wasan geometry with the division by 0, Int. J. Geom., **7**(1) (2018) 17–20.

[13] H. Okumura, Solution to 2017-1 Problem 4 with division by zero, Sangaku J. Math., **2** (2018) 27–30.

[14] H. Okumura, S. Saitoh, Wasan geometry and division by zero calculus, Sangaku J. Math., **2** (2018) 57–73.

[15] H. Okumura and S. Saitoh, Applications of the division by zero calculus to Wasan geometry, Glob. J. Adv. Res. Class. Mod. Geom., **7**(2) (2018) 44–49.

[16] H. Okumura (奥村博), On a problem in a sangaku in Kinki (近畿の算額の問題について), Wasan (和算), (72) (1992) 13–15.

[17] S. Saitoh, Division by zero calculus (draft), 2019.

[18] Saitama prefectural library (埼玉県立図書館) ed., The sangaku in Saitama (埼玉の算額), 1969, Saitama Prefectural Library (埼玉県立図書館).

[19] Shiraishi (白石長忠) ed., Shamei Sampu (社盟算譜), 1826, Tohoku University Digital Collection.

[20] Suzuki (鈴木順吉良茂) ed., Sampō Tenshōhō Yōran (算法天生法要覧) volume 1, 1788, Tohoku University Digital Collection.

[21] I. Yamamoto (山本一郎) ed., The sangaku in Hyōgo (兵庫の算額), 1967.