PROOF OF THE RIEMANN HYPOTHESIS

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Abstract: The Riemann zeta function is one of the most Euler's important and fascinating functions in mathematics. By analyzing the material of Riemann's conjecture, we divide our analysis in the ζ(z) function and in the proof of the conjecture, which has very important consequences on the distribution of prime numbers. The proof of the Hypothesis of Riemann result from the simple logic, that when two properties are associated, (the resulting equations that based in two Functional equations of Riemann), if zero these equations, ie ζ(z) = ζ (1-z)= 0 and simultaneously they to have the proved property 1-1 of the Riemann function ζ(z).Thus, there is not margin for to non exist the Re (z) = 1/2 {because ζ (z) = ζ (1-z)=0 and also ζ(z) as and ζ(1-z) are 1-1}.This, as it stands, will gives the direction of all the non-trivial roots to be all in on the critical line, with a value in the real axis equal 1/2.

#1.Theorem 1.

The R-Hypothesis focuses on the point where we must prove that if s = Re(s) + Im(s)*I …
I) The functions ζ(s) and ζ(1-s) are 1-1 on the critical strip.
II) The common roots of the equations ζ(s)-ζ(1-s) = 0 they have Re(s)=Re(1-s)=1/2 within the interval (0,1) and determine unique position, which is called critical line.
Proof:

The functions ζ(s) and ζ(1-s) are 1-1 on the critical strip. For this we need to analyze when and where the exponential function n^z is 1-1 when $n \in Z^+$.

The exponential function n^z …

I) is 1-1 in each of these strips defined by the intervals 2πk/ln(n) and 2π(k+1)/ln(n) " where $k \in Z$. For this we must prove two cases… If $A \subset C \wedge f : A \to C$

then if $f(z_1) = f(z_2)$ then $z_1 = z_2$ that is..

$x_1 = x_2 \wedge y_1 = y_2$. Indeed,

if $z_1 = x_1 + i \cdot y_1$, $z_2 = x_2 + i \cdot y_2$ are two points

within into a such strip such that $e^{z_1} = e^{z_2}$ then,,

$n^{x_1+y_1 \cdot i} = n^{x_1+y_1 i} \Rightarrow |n^{z_1}| = |n^{z_2}| \Rightarrow |n^{x_1} n^{y_1 \cdot i}| = |n^{x_2} n^{y_2 \cdot i}| \Rightarrow$

$x_1 = x_2$ and since $x_1 = x_2$, the relation

$e^{z_1 \cdot \ln(n)} = e^{z_2 \cdot \ln(n)}$ gives $e^{i \cdot y_1 \ln(n)} = e^{i \cdot y_2 \cdot \ln(n)}$, so

$y_1 - y_2 = 2k\pi / \ln(n)$, is an integer multiple of 2π. But

z_1 and z_2 belong to the strip, so $|y_1 - y_2| < 2\pi / \ln(n)$

That is, the difference $y_1 - y_2$ is at the same time a multiple of 2π/ln(n) and at an absolute value of less than 2π/ln(n).The only case that this is true is when

$y_1 = y_2$. We finally conclude that $z_1 = z_2$, so n^z is

1-1 in the strip

$\{z : 2k\pi / \ln(n) \le \mathrm{Im}(z) < 2(k+1)\pi / \ln(n)\}$.**"lines down closed – open up".** We also notice that n^z is on

C-{0}.Because if $w \neq 0$ and we put

$z = \ln|w| / \ln(n) + i \arg w / \ln(n)$ then

$n^z = n^{\ln|w|/\ln(n) + i \arg w/\ln(n)} = n^{\ln|w|} n^{i \arg w/\ln(n)} = |w| \cdot n^{i \arg w/\ln(n)} = w$

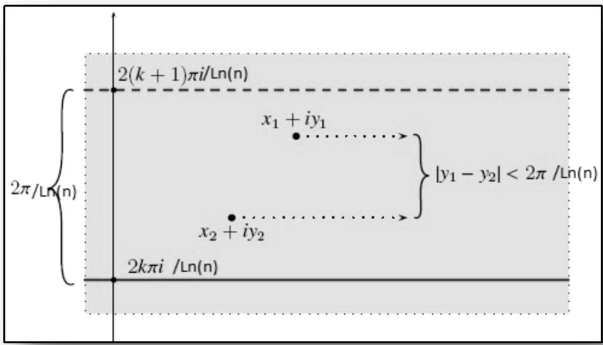


Fig.1 The z_1 and z_2 belong to the lane width $2k\pi / \ln(n)$

Formation of the strips 1-1 for ζ(s) and ζ(1-s).
a1.) If we accept the non-trivial zeroes on critical strip

of the Riemann Zeta Function ζ(s) as $s_1 = \sigma_1 + it_1$ and

$s_2 = \sigma_2 + it_2$ with $|s_2| > |s_1|$, and if we suppose that the real coordinates σ_1, σ_2 of each non-trivial zero of the Riemann Zeta, [1,2,3,4] function ζ(s) correspond to two imaginary coordinates t_1 and t_2 , then, we have the following equations group:

$$\zeta(\sigma_1 + \iota \cdot t_1) = \frac{1}{1^{\sigma_1+\iota\cdot t_1}} + \frac{1}{2^{\sigma_1+\iota\cdot t_1}} + \frac{1}{3^{\sigma_1+\iota\cdot t_1}} + ... + \frac{1}{n^{\sigma_1+\iota\cdot t_1}} + ... = 0$$

$$\zeta(\sigma_2 + \iota \cdot t_2) = \frac{1}{1^{\sigma_2+\iota\cdot t_2}} + \frac{1}{2^{\sigma_2+\iota\cdot t_2}} + \frac{1}{3^{\sigma_2+\iota\cdot t_2}} + ... + \frac{1}{n^{\sigma_2+\iota\cdot t_2}} + ... = 0$$

Taking the first equation and deducting the second, we obtain:

$$\zeta(\sigma_1 + \iota \cdot t_1) - \zeta(\sigma_2 + \iota \cdot t_2) = \sum_{n=1}^{\infty} \left(\frac{1}{n^{\sigma_1+\iota\cdot t_1}} - \frac{1}{n^{\sigma_2+\iota\cdot t_2}}\right) =$$

$$\sum_{n=1}^{\infty} \frac{n^{\sigma_2+\iota\cdot t_2} - n^{\sigma_1+\iota\cdot t_1}}{n^{\sigma_2+\iota\cdot t_2} \cdot n^{\sigma_1+\iota\cdot t_1}} = \sum_{n=1}^{\infty} \frac{n^{\sigma_2} e^{i\cdot t_2 \cdot \ln(n)} - n^{\sigma_1} e^{i\cdot t_1 \cdot \ln(n)}}{n^{\sigma_2+\iota\cdot t_2} \cdot n^{\sigma_1+\iota\cdot t_1}} = 0$$

From the previous relation, we conclude that if

$n^{\sigma_2+\iota\cdot t_2} \cdot n^{\sigma_1+\iota\cdot t_1} \neq 0, n > 1$ then

$\sigma_2 = \sigma_1 \wedge t_2 = t_1 \pm \frac{2k\pi}{\ln(n)}, (k = 1,2,...)$ .That is to say t_1 and t_2 can take any value, but according to the previous relation. So from Fig.1, that means **ζ (s) is 1-1 on the lane of critical Strip in Intervals such as defined, therefore and on the critical Line.** So n^{-z} is 1-1 in the strip and $\{z : 2k\pi / \ln(n) \leq \mathrm{Im}(z) < 2(k+1)\pi / \ln(n)\}$ **."lines down closed – open up"**

a2.) If we do the same work with $|s_2| > |s_1|$ for the case ζ(1-s) will we have:

$$\zeta(1 - \sigma_1 - \iota \cdot t_1) - \zeta(1 - \sigma_2 - \iota \cdot t_2) = \sum_{n=1}^{\infty} \left(\frac{1}{n^{1-\sigma_1-\iota\cdot t_1}} - \frac{1}{n^{1-\sigma_2-\iota\cdot t_2}}\right) =$$

$$\sum_{n=1}^{\infty} \frac{n^{1-\sigma_2-\iota\cdot t_2} - n^{1-\sigma_1-\iota\cdot t_1}}{n^{1-\sigma_2-\iota\cdot t_2} \cdot n^{1-\sigma_1-\iota\cdot t_1}} = \sum_{n=1}^{\infty} \frac{n^{1-\sigma_2} e^{-i\cdot t_2 \cdot \ln(n)} - n^{1-\sigma_1} e^{-i\cdot t_1 \cdot \ln(n)}}{n^{1-\sigma_2-\iota\cdot t_2} \cdot n^{1-\sigma_1-\iota\cdot t_1}} = 0$$

From the previous relation, we see that if

$$n^{1-\sigma_2-\iota\cdot t_2} \cdot n^{1-\sigma_1-\iota\cdot t_1} \neq 0, n > 1$$

$$\& (\{1 - \sigma_2 = 1 - \sigma_1\} \vee \sigma_2 = \sigma_1) \wedge$$

$$\wedge t_2 = t_1 \mp \frac{2k\pi}{\ln(n)}, (k = 1,2,...)$$

So, from Figure.1, it follows that **ζ (1-s) is 1-1 on the critical Strip,** So $n^{-(1-z)}$ is 1-1 in the strip and $\{z : 2k\pi / \ln(n) < \mathrm{Im}(z) \leq 2(k+1)\pi / \ln(n)\}$ **."lines up closed – down open"**

As we have seen before, the complex exponential form n^z but and n^{-z} is also periodic with a period of 2π/ln(n). For example $n^{z+2\cdot\pi\cdot i} = n^z \cdot n^{2\cdot\pi\cdot i / Log(n)} = n^z$ the n^z is repeated in all the **"lines down closed – open up"** horizontal strips with 2π/Ln(n) on Fig.2 below.

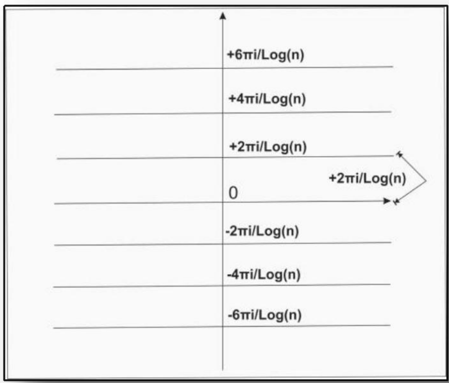


Fig 2. All the "bottom - closed upper "horizontal strips with 2π/ln(n) for $n^{-z}, n \geq 1, n \in Z$.

If we are asked to find the strips that are 1-1 of the function $\zeta(z) = \sum_{i=1}^{n} \frac{1}{n^z}$ it will be the union of the strips formed with period $2 \cdot \pi \cdot \kappa / \ln n$ where $\kappa, n \in \mathrm{Z}$. This results from the analysis we made for the cases a1, a2. Therefore, if we analytically assume that n = 2, 3, … then the exponential complex function **will be 1-1** in each of the strips, i.e. **in each one set** of the form…

$$S_n = \{z : 2k\pi / \ln(n) \leq \mathrm{Im}(z) < 2(k+1)\pi / \ln(n)\}, k \in Z$$

and therefore ***in each one subset of the unified expanded set***.... $S = \bigcup_{i=2}^{n\to\infty} S_i$.

The largest strip obviously contains all the rest it is wide, and has width $2 \cdot \pi / \ln 2$ i.e. with n = 2.

Overlapping contiguous Strips.

If we start from the upper bound of the strip with n = **2 and κ = 1, Fig.3 the upper ones will descend to zero by increasing** n with a width difference the interval δ = 2π/Ln(n-1) - 2π/Ln(n). In addition, we will prove that the strip i=n>=2 primary (κ=1) that generated in ascending order, the upper limit of that created of its double width enters the zone of the previous of (I = n-1, κ = 1) also primary.

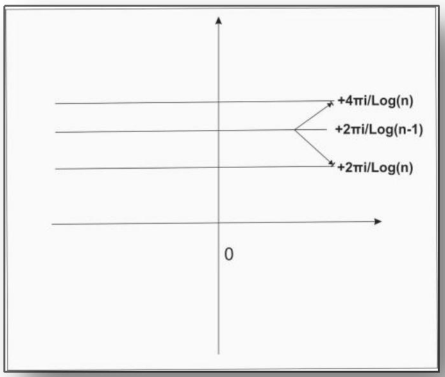


Fig3. The upper ones will go down to zero by increasing n.

Apply $2\cdot\pi/\ln(n-1) < 2\cdot 2\cdot\pi/\ln n \Rightarrow n < (n-1)^2 \Rightarrow$

$n^2 - 3\cdot n + 1 > 0 \Rightarrow n > \frac{3+\sqrt{5}}{2}$ and because we talked about integers then $n \geq 3$. The same phenomenon is also created at higher level strips.

The $\zeta(z) = \sum_{i=1}^{n} \frac{1}{n^z}$ **it is directly 1-1 from the comparison of 2 points** z_1, z_2**, when** $\zeta(z_1) = \zeta(z_2)$

Proof:

From the equality

$$\zeta(z_1) = \sum_{i=1}^{n} \frac{1}{n^{z1}} = \sum_{i=1}^{n} \frac{1}{n^{z2}} = \zeta(z_2) \Rightarrow$$

$$= 1 + 2^{-(x1+y1\cdot i)} + 3^{-(x1+y1\cdot i)} + \ldots n^{-(x1+y1\cdot i)} = \zeta(z_2) =$$

$$= 1 + 2^{-(x2+y2i)} + 3^{-(x2+y2i)} + \ldots n^{-(x2+y2\cdot i)} \Rightarrow$$

with comparison of similar terms the result arises.

$$x_1 = x_2 \wedge y_1 = y_2 - \frac{2\cdot\pi\cdot k}{\ln(n)}, k \in Z \Rightarrow$$

If we need $z_1 = z_2 \Rightarrow x_1 = x_2 \wedge y_1 = y_2$ therefore it will be 1-1 as shown in sections a1, a2 pages 1-2... Also according to Lagrange's generalized theory, each of the roots according to the obvious relation $\zeta(z) = \zeta(1-z)$ will result from the generalized theorem of Lagrange. From the equation that results from 3 Functional Equations, with condition of common roots. We take $\zeta(z) = \zeta(1-z)$ and for first approach we then follow the 3 first terms on each ζ-equation we will have in the Regular form $\zeta(z), \zeta(1-z)$ with the analysis below.

$$1^{-z} + 2^{-z} + 3^{-z} = 1^{-(1-z)} + 2^{-(1-z)} + 3^{-(1-z)} \Rightarrow 2^{-z} + 3^{-z} = \frac{1}{2}2^z + \frac{1}{3}3^z \Rightarrow$$

$$2^{-z} + 3^{-z} - 2^{-(1-z)} - 3^{-(1-z)} = 0 \Rightarrow 6^z + \frac{3}{2}4^z - 3 - 3(\frac{2}{3})^z = 0$$

With the replacement

$$6^z = y \Rightarrow z\cdot Log(6) = Log(y) + 2\cdot k\cdot\pi\cdot i \Rightarrow z = \frac{Log(y) + 2\cdot k\cdot\pi\cdot i}{Log(6)}$$

$$z_{in} = \frac{Log(y) + 2\cdot k\cdot\pi\cdot i}{Log(6)}$$

we consider it as initial value for the solution of Transcendental equation with the method of **Lagrange inversion** , then:

$$z = \frac{Log(y)+2\cdot k\cdot\pi\cdot i}{Log(6)} +$$

$$+\sum_{w=1}^{\infty} \frac{(-1)^w}{\Gamma(w+1)} \frac{d^{w-1}}{dy^{w-1}}\left(\left(\frac{1}{y\cdot Log(6)}\right)\cdot\left(\frac{3}{2}\cdot 4^{\frac{Log(y)+2\cdot k\cdot\pi\cdot i}{Log(6)}} - 3\cdot\left(\frac{2}{3}\right)^{\frac{Log(y)+2\cdot k\cdot\pi\cdot i}{Log(6)}}\right)^w\right)_{y\to 3}$$

Some solutions for $k \in Z$

1st 0.4985184869351 +/− 3.6072014300530 I,k = 1
2st 0.4997077392995 +/− 10.329023684711 I,k = 3
3st 0.5003426560276 +/− 13.990255078243 I,k = 4
4st 0.4993087825728 +/− 17.585326315022 I,k = 5
5st 0.5026747758005 +/− 21.304192633721 I, k = 6

And so on for the sequence of infinity. We now see that we approach the real roots and its complexities. With the additional method Newton or the Bisection method we approach more rapidly to them. And these roots will be contained in corresponding strips that we have previously defined and in closer relation with their position on the critical line. By this logic for the previous approximation equation in relation to its solutions of Zeta[z] the strips which determine the imaginary roots of the equation, will be between the intervals $2\cdot\pi\cdot\kappa/\ln 6, \kappa \in Z$ as defined by the analysis of the generalized Theorem of Lagrange. The strips of the imaginary part of the roots, as they appear for 6 consecutive intervals of the approximate equation

Strips – Imaginary Roots:

1st	(7.013424977055179	3.506712488527589)
2st	(10.52013746558276	7.013424977055179
3st	(14.02684995411035	10.52013746558276)
4st	(17.53356244263794	14.02684995411035)
5st	(21.04027493116553	17.53356244263794)
6st	(24.54698741969312	21.04027493116553)

It is clear the first root 3,607..i is located in 1st of strip (7.01-3.506), the second 10.3290..i in the 2st strip (7.013-10.52) etc. Now the root 13.99..i which approximates the first root of ζ(z) = ζ(1-z) = 0 i.e. the 14.1347..i, is in the 4st strip (14.02-17.533) with a lower limit the 14.026..i a value that is very close to the one required for the approximate and exclusive (1st root) when ζ(z) = 0 of the Riemann Hypothesis.

Corollary 1.

The only roots of the Zeta function not included in the set {z ∈C: 0 ≤ Re (z) ≤ 1} are the points -2, -4, -6..[11.page 47]

Proof..

On the functional equation…

$\zeta(1-z) = 2\cdot\zeta(z)\cdot\Gamma(z)\cdot\cos(\pi/2\cdot z)\cdot(2\pi)^{-z}$ we know that

for Re (z)> 1 the functions ζ(z) and Γ(z) do not equal zero(Proof-Th.3.p6).We also find that for Re (z)> 1 the 1 -Re (z) <0 and putting u = 1-z we will find all the roots of ζ (u) for Re (u) <0.Therefore we will have that ζ (u) will be zero , where it is zeroed the $\cos(\pi/2\cdot z)$ i.e. z = 3,5,7..Then the roots of ζ (u) for Re (u) <0 will be the points -2, -4, -6... and will be all the roots of the function ζ out of the strip {z ∈C: 0 ≤ Re (z) ≤ 1}.

Theorem Helpful.

The Riemann's Z-function has no roots on the lines Re(z) = 1 and Re(z) = 0.

The proof is detailed in the book [11] page 50-51.

II.)The common roots of the equations ζ(s) - ζ(1-s) = 0 they have Re(s)= Re(1-s)=1/2 within the interval (0,1) if moreover apply ζ(s)=0.(Refer to Theorem 3, page 5).

Proof: Let us assume z to be such that for complex $z = x_0 + iy_0, 0 < \mathrm{Re}(z) < 1 \wedge \mathrm{Im}(z) \neq 0$ and $\zeta(z) = \zeta(1-z) = 0$. According to the two equations, they must apply to both, because they are equal to zero that... $\zeta(z) = \zeta(\bar{z}) = 0 \wedge \zeta(1-z) = \zeta(\overline{1-z}) = 0$. But from [Theorem 1, II, a1, a2] the ζ(z) and ζ (1-z) are 1-1on the critical Strip.. If suppose generally that $\zeta(x_0 \pm y_0\cdot i) = \zeta(1 - x_0' \pm y_0'\cdot i) = 0$ then:

$y_0 = y_0' \wedge x_0 = 1 - x_0'$, but because are the 1-1 then we will apply two cases for complex z i.e.

$$z = [x_0' = 1 - x_0, x_0 \neq x_0', y_0 = y_0' \vee$$

$$\vee\, x_0' = 1 - x_0, x_0 = x_0' = \frac{1}{2}, y_0 = y_0']$$

[**Corollary.1 and Theorem{Helpful},because we suppose** $\zeta(z) = \zeta(1-z) = 0$.] We conclude that

0<x_0<1. Therefore we have two cases:

a) If $x_0 = x_0' = \frac{1}{2}$ we apply the obvious

i.e.

$\zeta(1/2 \pm y_0\cdot i) = \zeta(1 - 1/2 \pm y_0'\cdot i) = \zeta(1/2 \pm y_0'\cdot i) = 0 \wedge$

$\wedge\, y_0 = y_0'$. Which **fully meets the requirements of hypothesis!!**

b) If $0 < x_0 \neq x_0' \neq \frac{1}{2} < 1$ In this case if $x_0 < x_0' \wedge x_0 + x_0' = 1$ or $x_0 > x_0' \wedge x_0 + x_0' = 1$

then for the Functional equation **ζ(s) - ζ(1-s) = 0** simultaneously apply:

$\zeta(x_0 + y_0\cdot i) = \zeta(x_0 - y_0\cdot i) = 0 \wedge \zeta(x_0' + y_0'\cdot i) =$

$= \zeta(x_0' - y_0' i) = 0$. So let's **assume that**…

1^{st}. $x_0 < x_0' \wedge x_0 + x_0' = 1 => x_0 = \frac{1}{2} - a \wedge x_0' = \frac{1}{2} + b, a \neq b.$

But apply $x_0 + x_0' = 1 => 1 + b - a = 1 => b = a$. That is, a,b is symmetrical about of $\frac{1}{2}$. But the ζ(z) and ζ (1-z) are 1-1on the critical Strip then apply

$$\zeta(\frac{1}{2}-a\pm y_0\cdot i)=\zeta(\frac{1}{2}+a\pm y_0'\cdot i)\wedge y_0=y_0'\} =>$$

must $\frac{1}{2}-a=\frac{1}{2}+a => 2a=0 => a=0$

therefore $x_0=x_0'$.

2^{nd}. $x_0>x_0'\wedge x_0+x_0'=1 => x_0=\frac{1}{2}+a\wedge x_0'=\frac{1}{2}-\alpha$.

as before. Then because the ζ(z) and ζ (1-z) are 1-1on the critical Strip will apply…

$$\zeta(\frac{1}{2}+a\pm y_0\cdot i)=\zeta(\frac{1}{2}-a\pm y_0'\cdot i)\wedge y_0=y_0'\}$$ that

$\frac{1}{2}+a=\frac{1}{2}-a => 2a=0 => a=0$

therefore $x_0=x_0'$.We see that in all acceptable cases is true that "**The non-trivial zeros of ζ (z)- ζ (1-z)=0 have real part which is equal to ½ within the interval (0,1) .** In the end, we see still one partial case that appears in the roots of the equation $\zeta(z)-\zeta(1-z)=0.$

c) If $1<x_0\wedge x_0'<0$ Furthermore, in this case withthat is, it is symmetrical about $x_0+x_0'=1$then for the functional equations we apply

$$\zeta(x_0-y_0\cdot i)=\zeta(1-x_0-y_0\cdot i)\wedge\zeta(x_0+y_0\cdot i)=$$

$$=\zeta(1-x_0+y_0 i)\wedge y_0=y_0'$$ and for the three cases it is valid $y_0=0\vee y_0\neq 0)$. This case applies only when $\zeta(z)=\zeta(1-z)\neq 0\wedge x_0=1+a, x_0'=-a$, and **not for the equality of ζ function with zero**. But **in order to have common roots** of the two functions **ζ (z)** and **ζ (1-z) «because they are equal to zero and equal to each other»,** then first of all, it must be $x_0=x_0'$ and **also because the ζ (z) is 1-1** and for each case applies ($y_0=0$ and $y_0'=0$) or ($y_0\neq 0$ and $y_0'\cdot\neq 0$),

therefore we will apply $x_0=x_0'=\frac{1}{2}\wedge y_0=y_0'$ and with regard to the three cases **{a, b,c } the cases {b, c}** cannot happen, because they will have to be within (0,1) and will therefore be rejected.

Therefore if ζ(z) = ζ(1 – z) = 0 then because "these two equations of zeta function are 1-1 on the lane of critical strip" as shown in theorem 1, is "sufficient condition" that all non - trivial zeros are on the critical line Re(z)=1/2. That means that the real part of z of ζ(z) = 0, equals to 1/2 ". This, like Theorem 1, has been proved and it helps the Theorem 3 below.

2#. Theorem 2.

For **the non- trivial zeroes** of the Riemann Zeta Function ζ(s) apply

i)**There exists an upper-lower bound** of Re s of the Riemann Zeta Function ζ(s) and more specifically in the **closed interval** $\left[\frac{\ln 2}{\ln 2\pi},\frac{\ln\pi}{\ln 2\pi}\right]$**.** The non-trivial zeroes of the Riemann Zeta function ζ(s) of **the upper-lower** bound **are distributed symmetrically** on the straight **line Re s = 1/2**.

ii).**The average value of the upper lower bound of Re s = 1/2.**

Proof:

i). Here, we formulate two of the functional equations from **E - q.**

Set

$$\zeta(1-s)/\zeta(s)=2(2\pi)^{-s}Cos(\pi s/2)\Gamma(s),\mathrm{Re}s>0$$
$$\zeta(s)/\zeta(1-s)=2(2\pi)^{s-1}Sin(\pi s/2)\Gamma(1-s),\mathrm{Re}s<1$$

We look at each one equation individually in order to identify the set of values that we want each time.

a). For the first equation and for real values with Re s > 0 and by taking the logarithm of two sides of the equation [5], we have…

$$\zeta(1-s)/\zeta(s)=2(2\pi)^{-s}Cos(\pi s/2)\Gamma(s) =>$$

$Log[\zeta(1-s)/\zeta(s)] = Log[2] - sLog[2\pi] +$
$+ Log[Cos(\pi s/2)\Gamma(s)] + 2k\pi i$ but solving for s and if $f(s) = Cos(\pi s/2)\Gamma(s)$ and from **Lemma 2[6,p6, Part1]** if $\lim_{s\to s_0} \zeta(1-s)/\zeta(s) = 1$ or

$Log[\lim_{s\to s_0} \zeta(1-s)/\zeta(s)] = 0$ we get:

$s = \dfrac{Log[2]}{Log[2\pi]} + \dfrac{Log[f(s)] + 2k\pi i}{Log[2\pi]}$ with

$\dfrac{Log[f(s)] + 2k\pi i}{Log[2\pi]} >= 0$ Finally, because we need real s

we will have $\operatorname{Re} s \geq \dfrac{Log[2]}{Log[2\pi]} = 0.3771$. This is the lower bound, which gives us the first ζ(s) of Riemann's Zeta Function.

b). For the second equation, for real values with Re s < 1 and by taking the logarithm of the two parts of the equation, we will have:

$\zeta(s)/\zeta(1-s) = 2(2\pi)^{s-1} Sin(\pi s/2)\Gamma(1-s) \Rightarrow$

$Log[\zeta(s)/\zeta(1-s)] = Log[2] + (s-1)Log[2\pi] +$

$+ Log[Sin(\pi \cdot s/2)\Gamma(1-s)] + 2k\pi i$ but solving for s

and if $f(s) = Sin(\pi s/2)\Gamma(1-s)$ and from

Lemma 2[6,{p6,Part I}] if $\lim_{s\to s_0} \zeta(s)/\zeta(1-s) = 1$ or

$Log[\lim_{s\to s_0} \zeta(s)/\zeta(1-s)] = 0$ we

get… $s = \dfrac{Log[\pi]}{Log[2\pi]} - \dfrac{Log[f(s)] + 2k\pi i}{Log[2\pi]}$ with

$\dfrac{Log[f(s)] + 2k\pi i}{Log[2\pi]} >= 0$

In the following, because we need real s we will take Re $s \leq \dfrac{Log[\pi]}{Log[2\pi]} = 0.62286$. This is the upper bound, which gives us the second of Riemann's Zeta Function of ζ(s). So, we see that the lower and the upper bound exist for Re s and they are well defined.

c). Assuming that $s_k = \sigma_{low} + it_k$ and

$s'_\kappa = \sigma_{upper} + it_k$ with $\sigma_{low} = \dfrac{Log[2]}{Log[2\pi]}$ and

$\sigma_{upper} = \dfrac{Log[\pi]}{Log[2\pi]}$ we apply $\sigma_1 = \sigma_{low}$ and $\sigma_2 = \sigma_{upper}$ [fig.4]. If we evaluate the difference and

$$\operatorname{Re}\Delta s'_\kappa = \operatorname{Re} s'_\kappa - 1/2 = \sigma_{upper} - 1/2 = 0.1228$$

$$\operatorname{Re}\Delta s_\kappa = 1/2 - \operatorname{Re} s_\kappa = 1/2 - \sigma_{low} = 0.1228$$

This suggests for our absolute symmetry

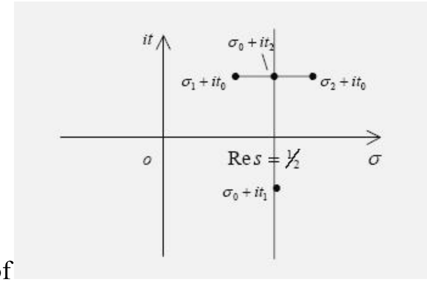
of 

Fig.4 Arrangement of low and upper of real part of zeros from 2 functional equations of Riemann.

ii) The average value of the upper lower bound is Re s = 1/2 because from (fig. 4):

$$\operatorname{Re} s = 1/2 \cdot \frac{Log[2] + Log[\pi]}{Log[2\pi]} = 1/2$$

#3. Theorem 3

The ***Riemann Hypothesis*** *states that* ***all the nontrivial zeros*** *of ζ (z) have real part* ***equal to 1/2.***

Proof: In any case, we Assume that: The Constant Hypothesis ζ (z) = 0, $z \in C$ **.** In this case, we use the two equations of **the Riemann zeta function, so if they apply what they represent the** ζ (z) and ζ (1-z) to equality. Before developing the method, we make the three following assumptions:

I). Analysis of specific parts of transcendental equations, which are detailed…

a). For $z \in C$,{ $a^z = 0 \Rightarrow a = 0 \wedge \operatorname{Re} z > 0$ } which refers to the inherent function similar to two **Riemann zeta functions as** $(2\pi)^{-z} = 0$ or $(2\pi)^{z-1} = 0$ and it seems that they do not have roots in C - Z, because $(2\pi) \neq 0$.

b). This forms Gamma(z) = 0 or Gamma(1-z) = 0 do not have roots in C-Z.

c). **Solution of Sin(π/2z)=0 or Cos(π/2z)=0**. More specifically **if z = x+yi,** then…

1. $Sin(\pi/2z) = 0 \Rightarrow \operatorname{Cosh}[(\pi \cdot y)/2]\operatorname{Sin}[(\pi \cdot x)/2] +$
$+ I \cdot \operatorname{Cos}[(\pi \cdot x)/2]\operatorname{Sinh}[(\pi \cdot y)/2] = 0$

If $k, m \in N(Integers)$ then all the solutions can be found with a program by the language of Mathematica.

$$i).\{x=(2+4\cdot k), y=2\cdot i\cdot(1+2\cdot m\}$$
$$ii).\{x=4\cdot k, y=2\cdot i\cdot(1+2\cdot m\}$$
$$iii).\{x=(-1+4\cdot k), y=i\cdot(-1+4\cdot m\}$$
$$iv).\{x=(-1+4\cdot k), y=i\cdot(1+4\cdot m\}$$
$$v).\{x=(1+4\cdot k), y=i\cdot(-1+4\cdot m\}$$
$$vi).\{x=(1+4\cdot k), y=i\cdot(1+4\cdot m\}$$
$$vii).\{x=(2+4\cdot k), y=i\cdot(4\cdot m\}$$
$$viii).\{x=(2+4\cdot k), y=2\cdot i\cdot(1+2\cdot m\}$$

From the generalized solution, it seems that in the pairs (x, y) will always arise integers which make impossible the case 0 < x <1, therefore there are no roots of the equation **Sin(πz/2)=0 in C-Z.**

Also, because as we see, all the roots given by the union of the sets $x \geq 1 \vee x = -1 \vee x = 0$ all are Integers**. So it cannot be true that for our case:** $Cos(\pi/2z) = 0 \Rightarrow$ Cosh[(π y)/2] Cos[(π x)/2] – - I Sin[($\pi\cdot$ x)/2] Sinh[(π y)/2] = 0

If $k, m \in N(Integers)$ and the solutions are:

$$i).\{x=4\cdot k, y=i\cdot(-1+4\cdot m)\}$$
$$ii).\{x=4\cdot k, y=i\cdot(1+4\cdot m\}$$
$$iii).\{x=(-1+4\cdot k), y=i\cdot(4\cdot m\}$$
$$iv).\{x=(-1+4\cdot k), y=2\cdot i\cdot(1+2\cdot m\}$$
$$v).\{x=(1+4\cdot k), y=4\cdot i\cdot m\}$$
$$vi).\{x=(1+4\cdot k), y=i\cdot(1+4\cdot m\}$$
$$vii).\{x=(2+4\cdot k), y=i\cdot(-1+4\cdot m\}$$
$$viii).\{x=(2+4\cdot k), y=i\cdot(1+4\cdot m\}$$

As we can see, again from the generalized solution, it seems that in the pairs (x, y) will always arise integers, which make impossible the case is 0 < x <1, therefore there are not roots receivers of the equation **Cos(πz/2) = 0.** Since all the roots are given by the union of the sets $x \geq 1 \vee x = -1 \vee x = 0$ it follows that they are Integers.

Great Result.

«With this three - cases analysis, we have proved that the real part of x, of the complex z = x+yi cannot be in the interval 0<x<1, and in particular in cases {a, b, c}, when they are zeroed. Therefore they cannot represent roots in the critical line».

II) Therefore, now we will analyze the two equations of **the Riemann zeta function** and we will try to find any common solutions.

1. For the first equation and for real values with Re z > 0 we apply:

Where $f(z) = 2(2\pi)^{-z} Cos(\pi\cdot z/2)\Gamma(z) \Rightarrow$

$\Rightarrow \zeta(1-z) = f(z)\cdot\zeta(z)$ but this means that the following two cases occur:

a). $\boldsymbol{\zeta(1-z)=\zeta(z)}$, where z is complex number. This assumption implies that $\zeta(z)\cdot(1-f(z)) = 0 \Rightarrow \zeta(z) = 0$. **In theorem 1(I, II),** (pages 1-2), we showed that the functions ζ(z) and ζ(1-z) are 1-1 and therefore if $\zeta(x_0 + y_0\mathrm{i}) = \zeta(x'_0 + y'_0\mathrm{i})$ then , but we also apply that $\zeta(x_0 + y_0\mathrm{i}) = \zeta(x_0 - y_0\mathrm{i}) = 0$, because we apply it for complex roots. The form $\zeta(1-z) = \zeta(z)$ means that if $z = x_0 + y_0\mathrm{i}$ we apply $1 - x_0 = x_0$ namely $x_0 = 1/2$ because in this case it will be verified that $\zeta(1-x_0-y'_0 i) = \zeta(1/2 - y'_0 i) = \zeta(1/2 + y'_0 i) =$ $\zeta(1-x_0+y'_0 i) = 0$ and

because $y_0 = y'_0$

$\zeta(x_0 - y_0 i) = \zeta(1/2 - y_0 i) = \zeta(1/2 + y_0 i) =$

$= \zeta(x_0 + y_0 i) = 0$ and these are the two forms of the ζ equation, they have common roots and therefore it can be verified by the definition of any complex equation, when it is equal to zero. Therefore if $z = x_0 + y_0\mathrm{i}$ then $x_0 = 1/2$ which verifies the equation $\zeta(z) = 0$.

b). $\boldsymbol{\zeta(1-z)\neq\zeta(z)}$, when z is a complex number.

To verify this case must be:

$\zeta(z) = \zeta(1-z)\cdot f(z) \Rightarrow \zeta(z) = 0 \wedge f(z) = 0$.

But this case is not possible, because as we have shown in **Section I(a, b, c**), the individual functions of f(z) cannot be zero when z is a complex number.

2. **For the second equation and for any real values with Re z < 1 we apply**

$$\zeta(z)/\zeta(1-z) = 2(2\pi)^{z-1} Sin(\pi\cdot z/2)\Gamma(1-z) \Rightarrow$$

$\Rightarrow \zeta(z) = f(z)\cdot\zeta(1-z)$, where

$f(z) = 2(2\pi)^{z-1} Sin(\pi\cdot z/2)\Gamma(1-z)$ but this also means that two cases occur:

a). $\boldsymbol{\zeta(1-z) = \zeta(z)}$, when z is a complex number.

This case is equivalent to **1.a,** and therefore if $z = x_0 + y_0 i$ then $x_0 = 1/2$ in order to verify the equation $\zeta(z) = 0$ we follow exactly the same process algebraically.

b). $\boldsymbol{\zeta(1-z) \neq \zeta(z)}$, where z is a complex number.

Similarly, the above case is equivalent to **1.b** and therefore it cannot be happening, as it has been proved.

#4.Forms of the Riemann ζ Functional Equations

The Riemann ζ function has three types of zeros: [7]

$$\begin{cases}\zeta(1-s) = 2\cdot(2\cdot\pi)^{-s}\cdot\cos(\pi\cdot s/2)\cdot\Gamma(s)\cdot\zeta(s), \mathrm{Re}s>0 \\ \zeta(s) = 2\cdot(2\cdot\pi)^{s-1}\cdot\sin(\pi\cdot s/2)\cdot\Gamma(1-s)\cdot\zeta(1-s), \mathrm{Re}s<1\end{cases} \quad \text{(I,II)}$$

Eq. Set And

$$\zeta(s) = \pi^{s-1/2}\cdot\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s) \quad \text{(III)}$$

Usually referred to as the trivial zeros, and non-trivial complex zeros.Therefore has been proved that any non-trivial zero **lies in the open strip** $\{s \in C : 0 < \mathrm{Re}(s) < 1\}$ that is called the **critical Riemann Strip**: And all complex zeros of the function ζ lie in the line $\{s \in C : \mathrm{Re}(s) = 1/2\}$ which is called the **critical line.**

Solving of the Riemann ζ Functional Equations.

To find the imaginary part we must solve the functional equations (Eq. set), and for the cases which the real part of the roots lies on the critical line. Cases # 4.1 & # 4.2 [6].

 # 4.1: 1st type roots of the Riemann zeta functions (1st equation from the Eq. set).

For the first category roots and by taking the logarithm of two sides of the equations, and thus we get … [5]

$$\zeta(1-z)/\zeta(z) = 2(2\pi)^{-z} Cos(\pi\cdot z/2)\Gamma(z) \Rightarrow$$
$$\Rightarrow Log[\zeta(1-z)/\zeta(z)] = Log[2] - zLog[2\pi] +$$
$$+ Log[Cos(\pi\cdot z/2)\Gamma(z)] + 2k\pi i$$

and the total form from the theory of **Lagrange inversion theorem**, [5] for the root is $p_1(z) = s$ which means that $f(s) = p_1^{-1}(s) = z$, but with an initial value for s which is

$$\mathbf{s_{in}} \rightarrow \frac{\mathbf{Log[2]}}{\mathbf{Log[2\pi]}} + \frac{\mathbf{2\cdot k\cdot\pi\cdot i}}{\mathbf{Log(2\pi)}} ,$$

by setting values for k we can simply calculate the roots of the (**1st equation from the Eq. set)…**

Therefore, for the first six roots for z we take:

1st 0.377145562795
2nd 0.377145562795 +/− 3.41871903296 I
3rd 0.3771455627955 +/− 6.83743806592 I
4rth 0.3771455627955 +/− 10.2561570988 I
5th 0.3771455627955 +/− 13.6748761318 I
6th 0.3771455627955 +/− 17.0935951646 I

A simple program in mathematica is..

```
k ≥ 4; t := (Log[2] + 2 * k1 * π * i) / Log[2 π];
z = s + Sum[((-1 / Log[2 π])^w / Gamma[w + 1]) * D[(s') * (Log[Cos[π * (s) / 2]] +
      + Log[Gamma[s]] - Log[Zeta[1 - s] / Zeta[s]])^w, {s, w - 1}], {w, 1, ∞}];
FQ = N[z /. s → t, 10]
```

With $k = 0, \pm1, \pm2, \ldots$

But because the infinite sum approaching zero, theoretically x gets the initial value

$$\mathbf{z} \rightarrow \frac{\mathbf{Log[2]}}{\mathbf{Log[2\pi]}} + \frac{\mathbf{2\cdot k\cdot\pi\cdot i}}{\mathbf{Log(2\pi)}}$$

So we have in this case, in part, the consecutive intervals with k = n and k = n+1 for any n >= 4 and for the imaginary roots.

#4.2:2nd type of roots of the Riemann zeta functional equations (2nd equation from the Eq. set).

Same as in the first category roots by taking the logarithm of two sides of the equations, and thus we get:

$$\zeta(z)/\zeta(1-z) = 2(2\pi)^{z-1} Sin(\pi \cdot z/2)\Gamma(1-z) =>$$
$$Log[\zeta(z)/\zeta(1-z)] = Log[2] + (z-1)Log[2\pi] +$$
$$+ Log[Sin(\pi \cdot z/2)\Gamma(1-z)] + 2 \cdot k \cdot \pi \cdot i$$

Now, we will have for total roots of 3 groups fields (but I have interest for the first group), and therefore for our case we will get $p_1(z) = s$ which means that:

$$f(s) = p_1^{-1}(s) = z\,,$$

but with an initial value for s that is

$$\mathbf{s_{in}} \rightarrow \frac{\mathbf{Log[\pi]}}{\mathbf{Log[2\pi]}} + \frac{\mathbf{2 \cdot k \cdot \pi \cdot i}}{\mathbf{Log(2\pi)}}$$

the overall form from the **Lagrange inverse theory** succeeds after replacing the above initial value and therefore for the **first six roots** after calculating them we quote the following table…

for z is:

1st 0.622854437204
2nd 0.622854437204 −/+ 3.41871903296 I
3rd 0.622854437204 −/+ 6.83743806592 I
4rth 0.622854437204 −/+ 10.2561570988 I
5th 0.622854437204 −/+ 13.6748761318 I
6th 0.622854437204 −/+ 17.0935951648 I

A simple program **in mathematica is..**

```
k ≥ 4; t := (Log[π] + 2 * k1 * π * i) / Log[2 π];
z = s + Sum[((-1 / Log[2 π])^w / Gamma[w + 1]) * D[(s ') * (Log[Sin[π * (s) / 2]] +
        + Log[Gamma[1 - s]] - Log[Zeta[s] / Zeta[1 - s]])^w, {s, w - 1}], {w, 1, ∞}];
FQ = N[z /. s → t, 10]
```

Therefore, with $k = 0, \pm 1, \pm 2, ...$

But because the infinite sum approaches zero, theoretically s gets initial value

$$\mathbf{z} \rightarrow \frac{\mathbf{Log[\pi]}}{\mathbf{Log[2\pi]}} + \frac{\mathbf{2 \cdot k \cdot \pi \cdot i}}{\mathbf{Log(2\pi)}}$$

So we have in this case, in part, the consecutive intervals with k = n and k = n+1

for any n >= 4 and for the imaginary roots. And for the cases we have for Im(s) the relationship

$$\text{Im}(z) = \frac{2 \cdot \pi \cdot k}{Log[2\pi]}, k \in Z$$

For the 3rd functional equation, one has been previously put on the (as in the other two in #4, page 7) and we take as imaginary part

$$\text{Re}(z) = \frac{1}{2} \wedge \text{Im}(z) = \frac{2 \cdot \pi \cdot k}{Log[\pi]}, k \in Z$$

Following to complete the roots of the two sets of the functional equations Eq. Set, we solve the functions as cosine or sin **according to its Generalized theorem of Lagrange [5]**

#4.3: Transcendental equations for zeros of the function (Explicit formula)

The main new results presented in the next few sections are transcendental equations satisfied by individual zeros of some L-functions. For simplicity, we first consider the Riemann-function, which is the simplest Dirichlet L-function.

***Asymptotic equation satisfied by the n-th zero on the critical line**. [9, 10, 11]

As above, let us define the function

$$\chi(z) \equiv \pi^{-z/2}\Gamma(z/2)\zeta(z).$$

which satisfies the functional equation

$$\chi(z) = \chi(1-z)$$

Now consider the Stirling's approximation

$$\Gamma(z) = \sqrt{2\pi} z^{z-1/2} e^{-z}\left(1 + O\left(z^{-1}\right)\right)$$

where z = x + iy, which is valid for large y. Under this condition, we also have

$$z^z = \exp\left(i\left(y \log y + \frac{\pi x}{2}\right) + x \log y - \frac{\pi y}{2} + x + O\left(y^{-1}\right)\right).$$

Therefore, using the polar representation

$$\zeta = |\zeta| e^{i \arg \zeta}$$

and the above expansions, we can write

were

$$A(x,y) = \sqrt{2\pi}\,\pi^{-x/2}\left(\frac{y}{2}\right)^{(x-1)/2} e^{-\pi y/4}|\zeta(x+iy)|\left(1 + O\left(z^{-1}\right)\right),$$
$$\theta(x,y) = \frac{y}{2}\log\left(\frac{y}{2\pi e}\right) + \frac{\pi}{4}(x-1) + \arg\zeta(x+iy) + O\left(y^{-1}\right).$$

The final transactions we end up with

$$\boxed{n = \frac{y}{2\pi}\log\left(\frac{y}{2\pi e}\right) - \frac{5}{8} + \lim_{\delta \to 0^+}\frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2} + \delta + iy\right).}$$

Establishing the convention that zeros are labeled by positive integers, $z_n = 1/2 + i \cdot y_n$

where n = 1, 2,3,4…, we must replace $n \to n-2$. Therefore, the imaginary parts of these zeros satisfy the transcendental
equation

$$\frac{t_n}{2\pi}\log\left(\frac{t_n}{2\pi e}\right) + \lim_{\delta\to 0^+}\frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+\delta+it_n\right) = n - \frac{11}{8}.$$

eq. A

Let us recall the definition used in, namely:

$$S(y) = \lim_{\delta\to 0^+}\frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+\delta+iy\right) = \lim_{\delta\to 0^+}\frac{1}{\pi}\Im\left[\log\zeta\left(\tfrac{1}{2}+\delta+iy\right)\right])$$

These points are easy to find, since they do not depend on the uctuating S(y).

We have:

$$\zeta(\tfrac{1}{2}-iy) = \zeta\left(\tfrac{1}{2}+iy\right)\,G(y), \qquad G(y) = e^{2i\vartheta(y)}$$

The Riemann-Siegel # function is defined by

$$\vartheta(y) \equiv \arg\Gamma\left(\tfrac{1}{4}+\tfrac{i}{2}y\right) - y\log\sqrt{\pi},$$

Since the real and imaginary parts are not both zero, at y(+)n then G = 1,whereas at y(-)n then G = -1. Thus

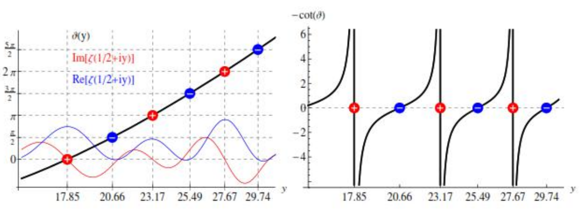
$$\Im\left[\zeta\left(\tfrac{1}{2}+iy_n^{(+)}\right)\right] = 0 \qquad \text{for } \vartheta(y_n^{(+)}) = (n-1)\pi,$$
$$\Re\left[\zeta\left(\tfrac{1}{2}+iy_n^{(-)}\right)\right] = 0 \qquad \text{for } \vartheta(y_n^{(-)}) = \left(n-\tfrac{1}{2}\right)\pi.$$

they can be written in the form of Eq. b

$$y_n^{(+)} = \frac{2\pi\,(n-7/8)}{W\left[e^{-1}(n-7/8)\right]}, \qquad y_n^{(-)} = \frac{2\pi\,(n-3/8)}{W\left[e^{-1}(n-3/8)\right]},$$

where above n = 1, 2… and the Labert – Function W denotes the principal branch W(0). The y(+)n are actually the Gram points. From the previous relation, we can see that these points (Fig. 5) are ordered in a regular manner…[8,12]

$$y_1^{(+)} < y_1^{(-)} < y_2^{(+)} < y_2^{(-)} < y_3^{(+)} < y_3^{(-)} < \cdots$$



Fig,5 The y(+/-)n are actually the Gram points. With this method, we are able to create intervals so as to approximate the correct values of imaginary part of non-trivial zeros. We did something similar in the cases #4.2. What it has left, is to find a method that approximates the values of imaginary part accurately.

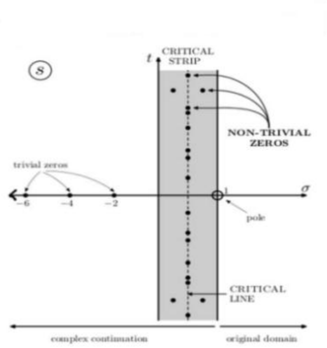


Fig.6

The black dots represent the zeros of ζ(s) function including possible zeros which do not lie on the critical line

The zeros of the Riemann Zeta function.

"PROGRAMMING"

#5.1The M Function – Bisection Method…

Knowing the time of the successive steps (k, k+1) of the relationship of imaginary parts

$$\mathrm{Im}(z) = \frac{2\cdot\pi\cdot k}{Log[2\pi]},$$

$\mathrm{Im}(z) = \dfrac{2\cdot\pi\cdot k}{Log[\pi]}$ (Eq I, II, III, page 7) with $k \in N$, we can calculate the roots by solving the equation $\zeta(1/2 + i\cdot y) = 0$ using the **Bisection Method.**

Bisection is the division of a given curve, figure, or interval into two equal parts (halves). A simple bisection procedure for iteratively converging on a solution, which is known to lie inside some interval $[a,b]$ proceeds by evaluating the function in question at the midpoint of the original interval $y = \dfrac{a+b}{2}$ and testing to see in which of the subintervals $[a,(a+b)/2]$ or $[(a+b)/2],b]$ the solution lies. The procedure is then repeated with the new interval as often as needed to locate the solution to the desired accuracy. Let a_n, b_n be the endpoints at the n th iteration (with $a_1 = a$ and $b_1 = b$) and let

r_n be the n th approximate solution. Then, the number of iterations required to obtain an error smaller than ε is found by noting that… $b_n - a_n = \frac{b-a}{2^{n-1}}$ and that r_n is defined by $r_n = \frac{1}{2}(a_n + b_n)$ In order for the error to be smaller than ε,

then… $|r_n - r| \le \frac{1}{2}(b_n - a_n) = 2^{-n}(b-a) < \varepsilon$ .Taking the natural logarithm of both sides gives:

$-n\ln 2 < \ln\varepsilon - \ln(b-a)$.Therefore, we have for steps:

$n > \frac{\ln(b-a) - \ln\varepsilon}{\ln 2}$.

#5.2: M-function of the Bisection Method..

We define the functions M_d^+, M_d^- on an interval (α,b) according to the scheme:

I. M_d^+, with M_{d-1} the Nearest larger of M_{d-m}

$$\frac{M_{d-1} + M_{d-m}}{2}, d \ge 2, d > m \ge d-2$$

$$M_1 = \frac{a+b}{2}, k = 1$$

II. M_d^-, with M_{d-1} the Nearest smaller of M_{d-m}

$$\frac{M_{d-1} + M_{d-m}}{2}, d \ge 2, d > m \ge d-2$$

$$M_1 = \frac{a+b}{2}, k = 1$$

For calculating the roots on solving the equation $\zeta(1/2 + i\cdot y) = 0$ take the limit r_S according to the scheme:

$r_s = \lim_{n\to\infty} M_n^+ = \lim_{n\to\infty} M_n^-, s \in N$ with M_d^+ and M_d^-, belong in the interval $(a,b), d \in N$ and also $(a = \frac{2\cdot\pi\cdot k}{Log[2\pi]}, b = \frac{2\cdot\pi\cdot(k+1)}{Log[2\pi]}), k \in N$ .k>=3 for the x of $\zeta(1/2 + i\cdot r_s) = 0$.

#5.3 Program in Mathematica for the Bisection method of $\zeta(1/2 + i\cdot y) = 0$.

Using the Intervals $(a = \frac{2\cdot\pi\cdot k}{Log[2\pi]}, b = \frac{2\cdot\pi\cdot(k+1)}{Log[2\pi]})$ and successive steps, we can compute all the roots of $\zeta(1/2 + i\cdot y) = 0$.

We can of course use three types such intervals more specifically in generall than

1. $a = \frac{2\cdot\pi\cdot k}{Log[2\pi]}, b = \frac{2\cdot\pi\cdot(k+1)}{Log[2\pi]}$ or

2. $a = \frac{2\cdot\pi\cdot k}{Log[\pi]}, b = \frac{2\cdot\pi\cdot(k+1)}{Log[\pi]}$ or

3. Were $a = y_n^-, b = y_n^+$.

$$y_n^{(+)} = \frac{2\pi\,(n - 7/8)}{W\left[e^{-1}(n - 7/8)\right]}, \qquad y_n^{(-)} = \frac{2\pi\,(n - 3/8)}{W\left[e^{-1}(n - 3/8)\right]},$$

We always prefer an interval that is shorter, in order to locate fewer non trivial zeros.

The most important is to calculate all the roots in each successive interval and therefore only then we will have the program for data {example: Integer k = 4, and $(a = \frac{8\cdot\pi}{Log[2\pi]}, b = \frac{10\cdot\pi}{Log[2\pi]})$ and Error approximate tol = 10^-6 and Trials n=22}..

"Programm for Bisection method"

A program relevant by dividing intervals…

```
Clear["`*"];
f[x_]:=Zeta[1/2+x*I];
k=Input["Epilogh k"];
a=(2*k)*π/Log[2π];
b=2*(k+1)*π/Log[2π];
tol=Input["Enter tolerance"];
n=Input["Enter total iteration"];
g=N[(-Log[tol]/Log[10]+4)];
If[Arg[f[c]]>0,{Print["No solution exists"] }];
Print[" n   a    b     c    .........  f(c)......"];
Do[{c=N[ (a+b)/2,g];
If[Arg[f[c]]<0,a= c,b=c], Print[PaddedForm[i,10],PaddedForm[N[a],{7,7}],
PaddedForm[N[b],{7,7}],PaddedForm[N[c],{7,7}],
PaddedForm[N[f[c]],{7,7}]]If[Abs[a-b]<tol && Abs[N[f[c]]]<tol*1000,
{Print["The solution is: ",N[c,g]] Exit[]}]},{i,1,n}];
Print["The maximum iteration failed,No solution exists"];
```

This program gives very good values as an approach to the roots we ask if we know the interval. Selecting the interval for the case $(a = \frac{2\cdot\pi\cdot k}{Log[2\pi]}, b = \frac{2\cdot\pi\cdot(k+1)}{Log[2\pi]}), k \in N$ .k>=3 the results are given below..

Calculation 1st Zetazero $z_1 = \frac{1}{2} + 14.1347216...$

	a	b	c
1	13.6748761300	17.0935951600	13.6748761300
2	13.6748761300	15.3842356500	15.3842356500
3	13.6748761300	14.5295558900	14.5295558900
4	14.1022160100	14.5295558900	14.1022160100
5	14.1022160100	14.3158859500	14.3158859500
6	14.1022160100	14.2090509800	14.2090509800
7	14.1022160100	14.1556335000	14.1556335000
8	14.1289247500	14.1556335000	14.1289247500
9	14.1289247500	14.1422791200	14.1422791200
10	14.1289247500	14.1356019400	14.1356019400
11	14.1322633500	14.1356019400	14.1322633500
12	14.1339326400	14.1356019400	14.1339326400
13	14.1339326400	14.1347672900	14.1347672900
14	14.1343499700	14.1347672900	14.1343499700
15	14.1345586300	14.1347672900	14.1345586300
16	14.1346629600	14.1347672900	14.1346629600
17	14.1347151300	14.1347672900	14.1347151300
18	14.1347151300	14.1347412100	14.1347412100
19	14.1347151300	14.1347281700	14.1347281700
20	14.1347216500	14.1347281700	14.1347216500

With final value 14.1347216500 , error near of 10^-7. This value is the approximate root of the $\zeta(1/2 + i \cdot y) = 0$ by the nearest error <10^-7.

In the event that we have two or more uncommon roots within the interval, we divide similar successive intervals in the order of finding of the first root either above or below. In such a case we have the k = 13, and at the interval (47.8620664 , 51.280785) the two roots are 48.0051088 and 49.7738324.

#5.4 Explicit formula and the Zeros of $\zeta(1/2 + i \cdot y) = 0$

Consider its leading order approximation, or equivalently its average since $< Arg\zeta(1/2 + i \cdot y) >= 0$ ..[11, 12]

Then we have the transcendental equation

$$\frac{t_n}{2\pi}\log\left(\frac{t_n}{2\pi e}\right) = n - \frac{11}{8}$$

Through the transformation

$$t_n = 2\pi\left(n - \frac{11}{8}\right)x_n^{-1}$$

this equation can be written a

$$x_n e^{x_n} = e^{-1}\left(n - \frac{11}{8}\right)$$

Comparing the previous results, we obtain

$$t_n = \frac{2\pi\left(n - \frac{11}{8}\right)}{W\left[e^{-1}\left(n - \frac{11}{8}\right)\right]}$$ where n = 1,2,3.....

 # 5.5 Programm by Newton's method, which finds the Zeros of $\zeta(1/2 + i \cdot y) = 0$

Using Newton's method we can reach the roots of the equation $\zeta(1/2 + i \cdot y) = 0$ at a very good initial value from $y \to t_n$ by the explicit formula.

It follows a mathematica program for the first 50 roots by the Newton's method. This method determines and detects the roots at the same time in order to verify the relation ζ (z) = 0 and always according to the relation...

$$t_n = \frac{2\pi\left(n - \frac{11}{8}\right)}{W\left[e^{-1}\left(n - \frac{11}{8}\right)\right]}$$

where n = 1,2,3.... and W is the W-function.

```
Table[FindRoot[Zeta[s] == 0 ,
  {s, 1/2 + N[2*π*Exp[1]*(n - 11/8)/Exp[1]/LambertW[(n - 11/8)/Exp[1]],
     125]*I}] , {n, 50}]
```

1	{s -> 0.5 + 14.1347 I}	11	{s -> 0.5 + 52.9703 I}	21	{s -> 0.5 + 79.3374 I}	31	{s -> 0.5 + 103.726 I}	41	{s -> 0.5 + 124.257 I}
2	{s -> 0.5 + 21.0220 I}	12	{s -> 0.5 + 56.4462 I}	22	{s -> 0.5 + 82.9104 I}	32	{s -> 0.5 + 105.447 I}	42	{s -> 0.5 + 127.517 I}
3	{s -> 0.5 + 25.0109 I}	13	{s -> 0.5 + 59.3470 I}	23	{s -> 0.5 + 84.7355 I}	33	{s -> 0.5 + 107.169 I}	43	{s -> 0.5 + 129.579 I}
4	{s -> 0.5 + 30.4249 I}	14	{s -> 0.5 + 60.8318 I}	24	{s -> 0.5 + 87.4253 I}	34	{s -> 0.5 + 111.875 I}	44	{s -> 0.5 + 131.088 I}
5	{s -> 0.5 + 32.9351 I}	15	{s -> 0.5 + 65.1125 I}	25	{s -> 0.5 + 88.8091 I}	35	{s -> 0.5 + 111.875 I}	45	{s -> 0.5 + 133.498 I}
6	{s -> 0.5 + 37.5862 I}	16	{s -> 0.5 + 67.0798 I}	26	{s -> 0.5 + 92.4919 I}	36	{s -> 0.5 + 114.320 I}	46	{s -> 0.5 + 134.757 I}
7	{s -> 0.5 + 40.9187 I}	17	{s -> 0.5 + 69.5464 I}	27	{s -> 0.5 + 94.6513 I}	37	{s -> 0.5 + 116.227 I}	47	{s -> 0.5 + 138.116 I}
8	{s -> 0.5 + 43.3271 I}	18	{s -> 0.5 + 72.0672 I}	28	{s -> 0.5 + 95.8706 I}	38	{s -> 0.5 + 118.791 I}	48	{s -> 0.5 + 139.736 I}
9	{s -> 0.5 + 48.0052 I}	19	{s -> 0.5 + 75.7047 I}	29	{s -> 0.5 + 98.8312 I}	39	{s -> 0.5 + 121.370 I}	49	{s -> 0.5 + 141.124 I}
10	{s -> 0.5 + 49.7738 I}	20	{s -> 0.5 + 77.1448 I}	30	{s -> 0.5 + 101.318 I}	40	{s -> 0.5 + 122.947 I}	50	{s -> 0.5 + 143.112 I}

A very fast method that ends in the root very quickly and very close to the roots of ζ(s) = 0 as shown..

#5.6 Directly (from Explicit form) with the solution of this equation

[8]

$$\frac{t_n}{2\pi}\log\left(\frac{t_n}{2\pi e}\right) + \lim_{\delta\to 0^+}\frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2} + \delta + it_n\right) = n - \frac{11}{8}.$$

Using the initial value the relation

$$t_n = \frac{2\pi\left(n - \frac{11}{8}\right)}{W\left[e^{-1}\left(n - \frac{11}{8}\right)\right]}$$ the explicit formula..

As already discussed, the function $Arg\zeta(1/2+\delta+i\cdot y)$ oscillates around zero. At a zero it can be well-defined by the limit, which is generally not zero. For example, for the first Riemann zero y1 = 14.1347..the limit δ→0+ has value as...

$$\lim Arg\zeta(1/2+\delta+i\cdot y)=0.157873..$$

The argζ term plays an important role and indeed improves the estimate of the n-th zero.

We can calculate by Newton's method, and we locate the first 30 imaginary part roots, of the equation $\zeta(1/2+i\cdot y)=0$,on the bottom **of the table...**x=Im(1/2+yi)

```
Table[
 FindRoot[x/(2π)*Log[x/(2π*e)]+1/π*Arg[Zeta[1/2+x*I]] == n-11/8,
 {x, 1/2+N[2*π*Exp[1]*(n-11/8)/Exp[1]/LambertW[(n-11/8)/e], 20]}],
 {n, 1, 50}]
```

1	{x -> 14.1347}	11	{x -> 52.9703}	21	{x -> 79.3374}	31	{x -> 103.726}	41	{x -> 124.257}
2	{x -> 21.0220}	12	{x -> 56.4462}	22	{x -> 82.9104}	32	{x -> 105.447}	42	{x -> 127.517}
3	{x -> 25.0109}	13	{x -> 59.3470}	23	{x -> 84.7355}	33	{x -> 107.169}	43	{x -> 129.579}
4	{x -> 30.4249}	14	{x -> 60.8318}	24	{x -> 87.4253}	34	{x -> 111.030}	44	{x -> 131.088}
5	{x -> 32.9351}	15	{x -> 65.1125}	25	{x -> 88.8091}	35	{x -> 111.875}	45	{x -> 133.498}
6	{x -> 37.5862}	16	{x -> 67.0798}	26	{x -> 92.4919}	36	{x -> 114.320}	46	{x -> 134.757}
7	{x -> 40.9187}	17	{x -> 69.5464}	27	{x -> 94.6513}	37	{x -> 116.227}	47	{x -> 138.116}
8	{x -> 43.3271}	18	{x -> 72.0672}	28	{x -> 95.8706}	38	{x -> 118.791}	48	{x -> 139.736}
9	{x -> 48.0052}	19	{x -> 75.7048}	29	{x -> 98.8312}	39	{x -> 121.370}	49	{x -> 141.124}
10	{x -> 49.7738}	20	{x -> 77.1449}	30	{x -> 101.318}	40	{x -> 122.947}	50	{x -> 143.112}

Epilogue…

This analysis has proved the Riemann Hypothesis is correct,since the real part of the non-trivial zero-equation functions is always constant and equals 1/2. This proven by 3 independent methods. Theorems {Th 1.I-II-page 1-5, Theorem 2-pages 5-6,Theorem 3-pages 5-7}. This analysis has also demonstrated that the imaginary part of the non-trivial zero zeta function accepts certain values according to the intervals defined by the solution of the 3 based functional equations {#4-I,II,III page 5}.

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Distribution of primes numbers and Zeros

#1.Indrotuction

The distribution of prime numbers is most simply expressed as the (discontinuous) step function $\pi(x)$, where $\pi(x)$ [Fig.] is the number of primes less than or equal to x.

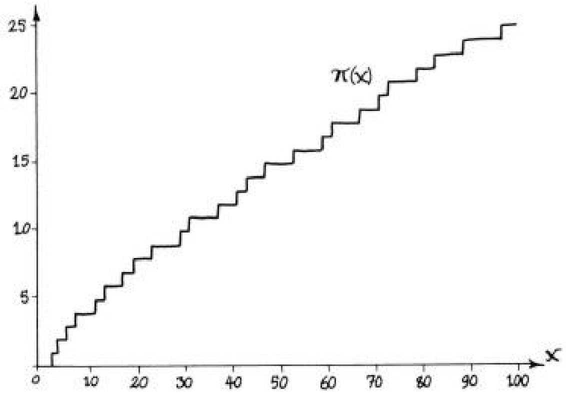

Fig 1.The function π(x) in relation to the random int. x.

It turns out that $\pi(x)$ can be expressed *exactly* as the limit of a sequence of smooth functions $R_n(x)$. To define $R_n(x)$ we first introduce the logarithmic integral function Li(x), which appears throughout the analytical theory of the prime distribution:

$$Li(x) = \int_2^x \frac{du}{\log u}$$

This is a smooth function which simply gives the area under the curve of the function 1/log u in the interval [2,x]. Don Zagier explains the reasoning behind the function Li in his excellent introductory article "The first 50 million prime numbers"from [2] , based on his inaugural lecture held at Bonn University, May 5, 1975): "A good approximation to $\pi(x)$, which was first given by Gauss is obtained by taking as starting point the empirical fact that the frequency of prime numbers near a very large number x is almost exactly 1/log x.

From this, the number of prime numbers up to x should be approximately given by the logarithmic sum...

$$\text{Li}(x) = 1/\log 2 + 1/\log 3 + \cdots + 1/\log x$$

or, what is essentially the same, by the logarithmic integral.

$$Li(x) = \int_2^x \frac{du}{\log u}$$

Using Li(x) we then define another smooth function, R(x), first introduced by Riemann in his original eight-page paper, and given by

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} Li(x^{1/n})$$

Riemann's research on prime numbers suggests that the probability for a large number x to be prime should be even closer to 1/log x if one counted not only the prime numbers but also the powers of primes, counting the square of a prime as half a prime, the cube of a prime as a third, *etc.* This leads to the approximation:

$$\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots \approx Li(x)$$

or, equivalently [*by means of the Möbius inversion formula*]

$$\pi(x) \approx Li(x) - \frac{1}{2}Li(x^{1/2}) - \frac{1}{3}Li(x^{1/3}) - \cdots$$

The function on the right side of this formula is denoted by R(x), in honour of Riemann. It represents an amazingly good approximation to $\pi(x)$ ..For those in the audience who know a little function theory, perhaps I might add that R(x) is an entire function of log x, given by the rapidly converging power series:

$$R(x) = 1 + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{k k! \zeta(k+1)}$$

where $\zeta(\kappa+1)$ is the Riemann zeta

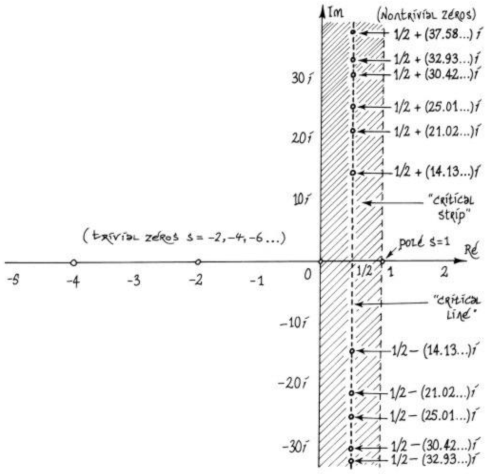

Fig.2 The non-trivial zeros in the Critical Line

Here we see the zeros of the Riemann zeta function in the complex plane. These fall into two categories, *trivial* and *nontrivial*zeros [Fig.2].Here are some tables on nontrivial zeros compliled by Andrew Odlyzko[4].The trivial zeros are simply the negative even integers. The nontrivial zeros are known to all **lie in the critical strip** that is 0 < Re[*s*] < 1, and always come in complex conjugate pairs. All known nontrivial zeros lie on the ***critical line*** Re[*s*] = 1/2. The Riemann Hypothesis states that they *all* lie on this line.The difference between the prime counting function and its "amazingly good approximation" R(*x*), *i.e.* the *fluctuations* in the distribution of primes, can be expressedin terms of the entire set of zeros of zeta, which we shall represent by ρ , via the function R itself:

$$R(x)-\pi(\chi)=\sum_{\rho} R(x^{\rho})$$

Obviously some of the x^{ρ} are complex values, so here R is the analytic continuation of the real-valued function R defined previously. This was mentioned above by Zagier, and is known as the Gram Series expansion:

$$R(x)=1+\sum_{k=1}^{\infty}\frac{(\ln x)^k}{k k!\zeta(k+1)}$$

The sum over ρ separates into two sums, over the trivial and nontrivial zeros, respectively. The former sum is of course just R(x^{-2}) + R(x^{-4}) + R(x^{-6}) + ···, and the latter can be written..

$$\sum_{k=1}^{\infty}[R(x^{\rho_k})+R(x^{\rho_{-k}})]$$

The contributions from the complex-conjugate pairs ρ_κ and $\rho_{-\kappa}=\overline{\rho}_\kappa$ cancel each others' imaginary parts, so

$$\pi(x)=R(x)-\sum_{m=1}^{\infty}R(x^{-2m})+\sum_{k=1}^{\infty}T_k(x)$$

Where

$$T_k(x)=-R(x^{\rho_k})-R(x^{\rho_{-k}})$$

are real-valued.We can now define the sequence of functions R$_n$(x) which approximate $\pi(x)$ in limit:

$$R_n(x)=R(x)-\sum_{m=1}^{\infty}R(x^{-2m})+\sum_{k=1}^{n}T_k(x)$$

$$f(x)=-2\,\mathrm{Re}\left(\sum_{k=1}^{\infty}\mathrm{Ei}(\rho_k\log(x))\right)+\int_x^{\infty}\frac{1}{(t^3-t)\log(t)}\,dt-\log(2)$$

where ρ_κ is the κ^{th} complex zero of the zeta function.

In this formula, E$i(z)$ (*Mathematica*'s built-in function ExpIntegral E$i(z)$) is the generalization of the logarithmic integral to complex numbers. These equations come from references [1], [2], and [3]. First, let M be the smallest integer such that $x^{1/M}<2$. We need to add only the first M-1 terms (that is, $n=1,2,...M$) in the sum in equation (1). For each of these values of N, we use equation (2) to compute the value of $f(x^{1/n})$. However, we will add only the first N terms (that is, $\kappa=1,2,...\mathrm{N}$) in the sum in equation (2). Because the purpose of this Demonstration is to show how the jumps in the step function $\pi(x)$ can be closely approximated by adding to $R(x)$ a correction term that involves zeta zeros, we ignore the integral and the $\log 2$ in second equation; this speeds up the computation and will not noticeably affect the graphs, especially for x more than about 5.The more zeros we use, the closer we can approximate $\pi(x)$. For larger x , the correction term must include more zeros in order to accurately approximate $\pi(x)$

#2.The GUE hypothesis.

While many attempts to prove the RH had been made, a few amount of work has been devoted to the study of the distribution of zeros of the Zeta function. A major step has been done toward a detailed study of the distribution of zeros of the Zeta function by Hugh Montgomery [6], with the Montgomery pair correlation conjecture. Expressed in terms of the normalized spacing ..

$$\delta_n = (\gamma_{n+1} - \gamma_n)\frac{\log(\gamma_n/(2\pi))}{2\pi}$$
,

this conjecture is that, for M → ∞

$$\frac{1}{M}\#\{n : N+1 \le n \le N+M, \delta_n \in [\alpha, \beta]\}$$

$$\sim \int_a^b p(0,u)du \sim \int_a^b 1 - (\frac{\sin \pi u}{\pi u})^2 du .$$

In other words, the density of normalized spacing between non-necessarily consecutive zeros is $1-(\frac{\sin \pi u}{\pi u})^2$. It was first noted by the Freeman Dyson, a quantum physicist, during a now-legendary short teatime exchange with Hugh Montgomery[6], that this is precisely the pair correlation function of eigenvalues of random hermitian matrices with independent normal distribution [In figure 3&4] of its coefficients. Such random hermitian matrices are called the Gauss unitary ensemble (GUE). As referred by Odlyzko in [4] for example, this motivates the GUE hypothesis which is the conjecture that the distribution of the normalized spacing between zeros of the Zeta function is asymptotically equal to the distribution of the GUE eigen values. Where p(0, u) is a certain probability density function, quite complicated to obtain for an expression of it). As reported by Odlyzko in [4], we have the Taylor expansion around zero..

$$p(0,u) = \frac{\pi^2}{3}u^2 - 2\frac{\pi^4}{45}u^4 + ...$$

which under the GUE hypothesis entails that the proportion of δn less than a given small value δ is asymptotic to $\frac{\pi^2}{9}\delta^3 + O(\delta^5)$ .Thus very close pair of zeros are rare.

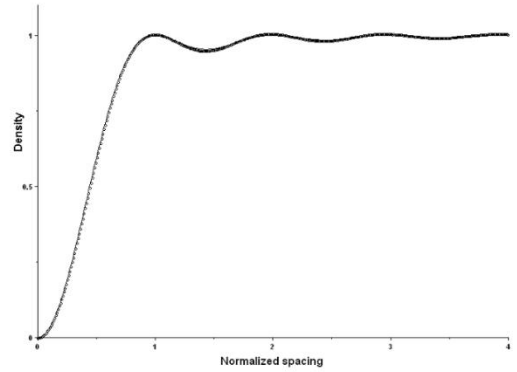

Fig3. Probability density of the normalized spacing between non-necessarily consecutive zeros and the GUE prediction[8]

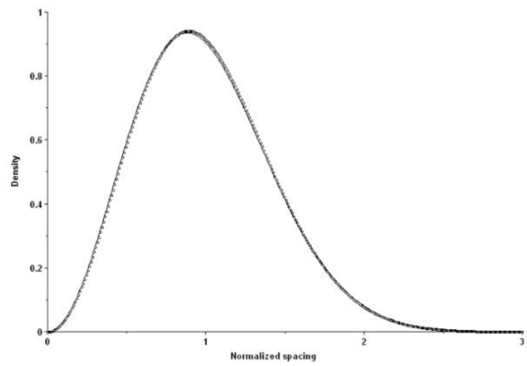

Fig 4.Probability density of the normalized spacing δn and the GUE prediction[8].

#3.Gaps between zeros..

The table below lists the minimum and maximal values of normalized spacing between zeros δn and of δn +δn+1, and compares this with what is expected under the GUE hypothesis. It can be proved that p(0, t) have the following Taylor expansion around 0

$$p(0,u) = \frac{\pi^2}{3}u^2 - 2\frac{\pi^4}{45}u^4 + \cdots$$

so in particular, for small delta

$$\text{Prob}(\delta_n < \delta) = \int_0^\delta p(0,u)\,du \sim \frac{\pi^2}{9}\delta^3$$

so that the probability that the smallest δn are less than δ for M consecutive values of δn is about..

$$1 - \left(1 - \frac{\pi^2}{9}\delta^3\right)^M \simeq 1 - \exp\left(-\frac{\pi^2}{9}\delta^3 M\right)$$

This was the value used in the sixth column of the table.
The result can be also obtained for the δn + δn+1..

$$\text{Prob}(\delta_n + \delta_{n+1} < \delta) \sim \frac{\pi^6}{32400}\delta^8,$$

from which we deduce the value of the last column..

Height	Mini δ_n	Maxi δ_n	Mini $\delta_n + \delta_{n+1}$	Maxi $\delta_n + \delta_{n+1}$	Prob min δ_n in GUE	Prob min $\delta_n + \delta_{n+1}$ in GUE
10^{13}	0.0005330	4.127	0.1097	5.232	0.28	0.71
10^{14}	0.0009764	4.236	0.1213	5.349	0.87	0.94
10^{15}	0.0005171	4.154	0.1003	5.434	0.26	0.46
10^{16}	0.0005202	4.202	0.1029	5.433	0.27	0.53
10^{17}	0.0006583	4.183	0.0966	5.395	0.47	0.36
10^{18}	0.0004390	4.194	0.1080	5.511	0.17	0.67
10^{19}	0.0004969	4.200	0.0874	5.341	0.24	0.18
10^{20}	0.0004351	4.268	0.1067	5.717	0.17	0.63
10^{21}	0.0004934	4.316	0.1019	5.421	0.23	0.50
10^{22}	0.0008161	4.347	0.1060	5.332	0.70	0.61
10^{23}	0.0004249	4.304	0.1112	5.478	0.15	0.75
10^{24}	0.0002799	4.158	0.0877	5.526	0.05	0.19

For very large spacing in the GUE, as reported by Odlyzko in [4], des Cloizeaux and Mehta [5] have proved that

$$\log p(0,t) \sim -\pi^2 t^2/8 \qquad (t \to \infty),$$

which suggests that

$$\max_{N+1 \le n \le N+M} \delta_n \sim \frac{(8 \log M)^{1/2}}{\pi}.$$

Statistics of False zetazeros, δ-intervals and count of primes.

#4.General equations.

With this statistic we find the crowd of individual parts intervals defined by consecutive Zetazeros in a fixed integer interval. Here we use δ = 1000. We have the general Equation **δ + Zetazero (kin) -Zetazero (kf) = 0** and after given initial value in kin, i calculate the **kf usually by the Newton method**. From the initial and final value of **kin, kf** and by performing the process of successive intervals, i calculate the number(count) of the primes ones that are within the intervals.In this way i will have consecutive intervals..

$$S_k = \{Zetazero(k+1) < p < Zetazero(k), k \in (kin, kf)\}$$

from where resulting the number of the primes in the given consecutive interval. The sum of the primes in the given interval δ = 1000 will be obvious..

$$S_\delta = \bigcup_{i=k_a}^{k_f} S_i$$

The False intervals will be in a normalized form

$F_\delta = 1000 - S_\delta$ over N>= 4000.

#5. The statistics

The Statistic looking for ways to show us which function is the most ideal to get closer the points of interest, uses the **NonlinearModelFit** method of the function

$$y = (a + b \cdot x)/(Log(d + c \cdot x)$$

and therefore after determining the variables{a, b, c, d} we are able to make statistical and probable prediction at higher levels of numbers .

This function is directly related to the function π(x) = x / logx which was reportedin the introduction that is, defining the number of primes numbers relative to x.

We will do a double statistic of the intervals **{δ, False intervals}** and **{δ, number of primes}** that we are ultimately interested in the statistics mainly the count of the primes inside at δ-intervals.

#6.The first statistic is about count of **False intervals** and the **count of the primes** that corresponding in them. The table below gives it in aggregate until the count of primes equal $p_\delta = 29$ corresponding to the count of False $F_\delta = 971$ and the Total range Integers for a interval D=1000 to $2 \cdot 10^{14}$ + 1000.

```
data1 = {{499, 162}, {735, 129}, {825, 121}, {885, 114},
   {887, 113}, {892, 108}, {895, 105}, {894, 106},
   {900, 100}, {903, 97}, {897, 103}, {891, 99}, {904, 96},
   {898, 102}, {908, 92}, {902, 98}, {912, 88}, {902, 98},
   {908, 92}, {902, 98}, {906, 94}, {902, 98}, {912, 88},
   {908, 92}, {905, 95}, {912, 88}, {908, 92}, {914, 86},
   {900, 90}, {908, 92}, {912, 88}, {925, 75}, {932, 68},
   {936, 64}, {930, 70}, {937, 63}, {934, 66}, {945, 55},
   {925, 75}, {940, 60}, {939, 61}, {930, 70}, {938, 62},
   {946, 54}, {949, 51}, {946, 54}, {945, 55}, {955, 45},
   {944, 56}, {942, 58}, {951, 49}, {956, 44}, {962, 38},
   {953, 47}, {956, 44}, {955, 45}, {961, 39}, {953, 47},
   {960, 40}, {955, 45}, {966, 34}, {963, 37}, {961, 39},
   {971, 29}, {974, 26}, {963, 37}, {965, 35}, {962, 38},
   {965, 35}, {970, 30}, {966, 34}, {964, 36}, {970, 30},
   {976, 24}, {973, 27}, {971, 29}};
```

The diagram given below[Fig.5] shows the arrangement of the points at the level (x, y) according to the data1[7]..

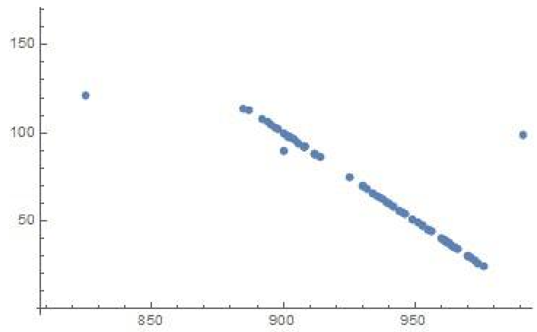

Fig 5.The depiction of the intervals False and the primes ones (file:data1) contained in the intervals N+δ-N = 1000.

Even more macroscopically [Fig.6] the points are shown by the line **y = 988.709 - 0.988372*x**, which, as they appear, are stacked on its lower right..

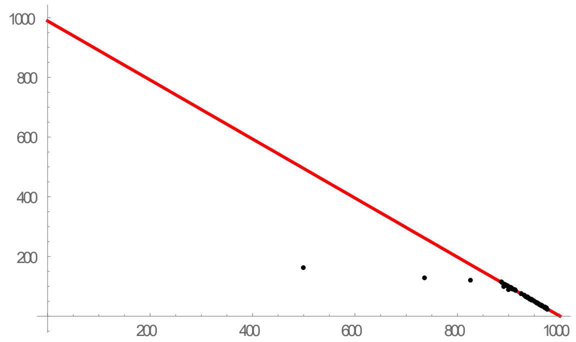

Fig.6 Graphic depiction of the line y = 988.709 - 0.988372*x with the archive(data1) points.

It is obvious that for {x = 999, y = 1.32} and for {y = 1000, x = 0.33}, a value located below the unit and means that it can in such a interval, and we are talking about this is for high order integers intervals so there is not one prime within the interval δ = 1000, chosen at random.

#7..The second statistic refers to the count of the primes ones that located in the intervals [N,N+ δ] with N = 1000 up to $2\cdot10^{14}$, range δ = 1000 and the **count of the primes within successive ZetaZeros** .After we found the roots kf of general Equation **δ + Zetazero (kin) -Zetazero (kf) = 0** of given initial value kin, **usually by the Newton method.** The data[7] that we have met with the above method are..

```
data = {{1000, 162}, {2000, 129}, {3000, 121},
  {4000, 114}, {5000, 113}, {6000, 108},
  {8000, 105}, {9000, 106}, {10 000, 106},
  {11 000, 100}, {12 000, 97}, {13 000, 103},
  {14 000, 99}, {15 000, 96}, {16 000, 102},
  {17 000, 92}, {18 000, 98}, {19 000, 88},
  {20 000, 98}, {21 000, 92}, {22 000, 98},
  {23 000, 94}, {24 000, 98}, {25 000, 88},
  {26 000, 92}, {27 000, 95}, {28 000, 88},
  {29 000, 92}, {30 000, 86}, {40 000, 90},
  {50 000, 92}, {60 000, 88}, {1 001 000, 75},
  {2 001 000, 68}, {3 001 000, 64}, {4 001 000, 70},
  {5 001 000, 63}, {6 001 000, 66}, {7 001 000, 55},
  {8 001 000, 75}, {9 001 000, 60}, {10 001 000, 61},
  {20 001 000, 70}, {40 001 000, 62}, {60 001 000, 54},
  {80 001 000, 51}, {100 001 000, 54},
  {200 001 000, 55}, {400 001 000, 45},
  {600 001 000, 56}, {800 001 000, 58},
  {1 000 001 000, 49}, {10 000 001 000, 44},
  {20 000 001 000, 38}, {30 000 001 000, 47},
  {40 000 001 000, 44}, {60 000 001 000, 45},
  {80 000 001 000, 39}, {100 000 001 000, 47},
  {200 000 001 000, 40}, {400 000 001 000, 45},
  {600 000 001 000, 34}, {1 000 000 001 000, 37},
  {2 000 000 001 000, 39}, {3 000 000 001 000, 29},
  {4 000 000 001 000, 26}, {5 000 000 001 000, 37},
  {6 000 000 001 000, 35}, {7 000 000 001 000, 38},
  {8 000 000 001 000, 35}, {9 000 000 001 000, 30},
  {10 000 000 001 000, 34}, {20 000 000 001 000, 36},
  {40 000 000 001 000, 36}, {70 000 000 001 000, 30},
  {90 000 000 001 000, 24}, {110 000 000 001 000, 27},
  {200 000 000 001 000, 29}};
```

Using the **NonlinearModelFit[7]** method of the function $y = (a+c*x)/(\mathrm{Log}[d+b*x]$ and after specifying the variables {a, b, c, d} the following result the function will be..

$$y= \frac{1108.254246288494-1.116325646187134\times10^{-12}x}{\mathrm{Log}[-4503.90336177023+5.425635171403081x]}$$

with very good approach and value performance in each pairing.The diagram [Fig.7] of the above equation is shown in more detail below with gravity in the latest data.

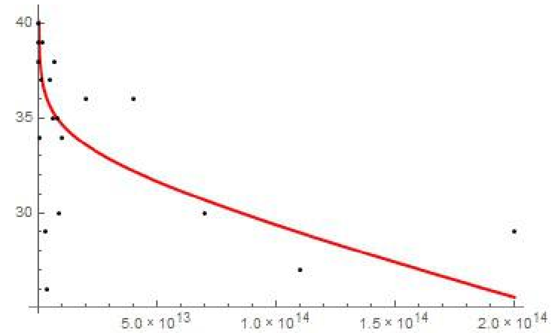

Fig 7.The graphic depiction of the function

$y = (a+c*x)/(\mathrm{Log}[d+b*x]$ .Thus a test value for x $=2\cdot10^{14}$ gives us y = 25.6, close to 29 we took with the analysis.

They follow **statistics ANOVA** and t-Statistic[7,8] and we get the tables ...

Analysis -ANOVA…

	DF	SS	MS
Model	4	441724.	110431.
Error	74	1329.65	17.9683
Uncorrected Total	78	443054.	
Corrected Total	77	71143.8	

Analysis - T-statistic…

	Estimate	Standard Error	t- Statistic	P- Value
a	1108.25	8.79256	126.044	4.12076×10^{-88}
b	-1.11633×10^{-12}	5.63966×10^{-13}	−1.97942	0.0514887
c	5.42564	0.186181	29.1417	2.68171×10^{-42}
d	−4503.9	0.153555	−29330.8	3.51117×10^{-263}

As we can see from the results, we have an important Standard Error only for **the variable a**. The other variables are observed to have a low statistical error and have good compatibility. By adapting the method as we see, we associate two lists of results of the number of primes and long intervals, which although disproportionately, work together with impeccable and good contact.

Ronald Fisher introduced the term variance and proposed its formal analysis in a 1918 article *The Correlation Between Relatives on the Supposition of Mendelian Inheritance*. His first application of the analysis of variance was published in 1921. Analysis of variance became widely known after being included in Fisher's 1925 book *Statistical Methods for Research Workers*. One of the attributes of ANOVA that ensured its early popularity was computational elegance. The structure of the additive model allows solution for the additive coefficients by simple algebra rather than by matrix calculations. In the era of mechanical calculators this simplicity was critical. The determination of statistical significance also required access to tables of the F function which were supplied by early statistics texts.

#.8. Standar –Error and Confidence –Interval

Part-1.

Where we observe, apart from 1-2 initial measurements**, the Standar –Error** as well and the **Confidence Interval** is stabilized at good and acceptable values..

Observed	Predicted	Standard Error	Confidence Interval
162	162.352	4.17901	{154.025, 170.679}
129	126.574	0.851191	{124.878, 128.27}
121	118.232	0.759511	{116.719, 119.745}
114	113.637	0.729151	{112.184, 115.09}
113	110.529	0.711538	{109.112, 111.947}
108	108.21	0.699052	{106.817, 109.602}
105	104.861	0.68137	{103.503, 106.219}
106	103.582	0.67462	{102.237, 104.926}
106	102.476	0.668765	{101.143, 103.808}
100	101.504	0.6636	{100.182, 102.826}
97	100.64	0.658982	{99.3265, 101.953}
103	99.862	0.65481	{98.5573, 101.167}
99	99.1565	0.651008	{97.8593, 100.454}
96	98.5114	0.647517	{97.2212, 99.8016}
102	97.9179	0.644293	{96.6341, 99.2017}
92	97.3687	0.6413	{96.0909, 98.6465}
98	96.858	0.638507	{95.5858, 98.1303}
88	96.3812	0.635891	{95.1142, 97.6482}
98	95.9342	0.633431	{94.6721, 97.1964}
92	95.5138	0.631112	{94.2563, 96.7713}
98	95.1171	0.628918	{93.864, 96.3703}
94	94.7418	0.626837	{93.4928, 95.9908}
98	94.3858	0.624859	{93.1408, 95.6309}
88	94.0474	0.622974	{92.8061, 95.2887}
92	93.7249	0.621176	{92.4872, 94.9627}
95	93.4171	0.619455	{92.1829, 94.6514}
88	93.1228	0.617807	{91.8918, 94.3538}
92	92.8408	0.616226	{91.613, 94.0687}
86	92.5703	0.614706	{91.3455, 93.7951}
90	90.3458	0.602126	{89.1461, 91.5456}
92	88.7017	0.592736	{87.5206, 89.8828}
88	87.4066	0.585287	{86.2403, 88.5728}
75	71.4688	0.490798	{70.4909, 72.4468}
68	68.4113	0.47218	{67.4704, 69.3521}
64	66.7409	0.461946	{65.8205, 67.6614}
70	65.6044	0.454957	{64.6979, 66.5109}
63	64.7492	0.449683	{63.8531, 65.6452}
66	64.0667	0.445467	{63.1791, 64.9544}

Part-2.

55	63.5009	0.441964	{62.6202, 64.3815}
75	63.0187	0.438976	{62.1441, 63.8934}
60	62.5995	0.436374	{61.73, 63.469}
61	62.2291	0.434073	{61.3642, 63.0941}
70	59.8979	0.419533	{59.062, 60.7338}
62	57.735	0.405955	{56.9261, 58.5439}
54	56.5407	0.398419	{55.7468, 57.3346}
51	55.7229	0.393243	{54.9393, 56.5064}
54	55.1046	0.389322	{54.3289, 55.8804}
55	53.2687	0.377631	{52.5163, 54.0212}
45	51.5512	0.366632	{50.8207, 52.2817}
56	50.5969	0.360493	{49.8786, 51.3152}
58	49.941	0.356262	{49.2311, 50.6509}
49	49.4438	0.353049	{48.7403, 50.1473}
44	44.8373	0.322996	{44.1937, 45.4809}
38	43.6138	0.314912	{42.9863, 44.2413}
47	42.9283	0.310357	{42.3099, 43.5467}
44	42.4548	0.307197	{41.8427, 43.0669}
45	41.8046	0.302835	{41.2012, 42.408}
39	41.355	0.299799	{40.7576, 41.9523}
47	41.0126	0.297474	{40.4199, 41.6053}
40	39.9828	0.290385	{39.4042, 40.5614}
45	38.9993	0.283419	{38.4345, 39.564}
34	38.4427	0.279364	{37.886, 38.9993}
37	37.7577	0.274288	{37.2112, 38.3043}
39	36.8486	0.267924	{36.3147, 37.3824}
29	36.3208	0.265411	{35.7919, 36.8496}

#9. Statistical comparison.

With statistical comparison it appears that as long as we moving to a higher order of integer-size that are within a given interval, we have chosen δ= 1000, the count of the primes diminishes to disappear or to there are 1-2 primes at high order intervals of more than 10^{20}. This also agrees with of **Gram's law**. In particular, the problem of distribution of the differences $t_{n+1} - t_n$ **(that is of difference of ZetaZeros)** is considered [9]. If we accept Gram's law then the order of this difference does not exceed the quantity ...

$$t_{n+1} - t_n \approx \frac{2\pi}{\ln(n)} \to 0, n \to \infty$$

for much larger integers [9,8] then mean and their mean value is close to,,

$$t_{n+1} - t_n \approx \frac{\ln\ln(n)}{\ln(n)} \to 0, n \to \infty$$

And as the above analysis of **- False intervals**- as we have shown, **it is compatible with this result of law Gram's**.

#.10. **From above analysis they arise 3 big conclusions:**

1st . The number(Count) of primes located within δ intervals is gradually decreasing with a higher order size n of 10^n .

2nd . The distribution of the number(Count) of the primes, within interval δ follows a Nonlinear correlation is expressed by the function $y = (a + b \cdot x)/(Log(d + c \cdot x)$ similar to the number (Count) of the primes $\pi(x) \approx x / Log(x)$, which is apply approximating for large numbers.

3rd .The measurement(Count) of false intervals and the count of the primes follows a linear correlation and it increases if the order of magnitude size of the integer increases.

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