Some Studies on Neutrosophic Decision Making

Thesis

Submitted for the Degree of

Doctor of Philosophy (Science)

of

Indian Institute of Engineering, Science and Technology, Shibpur

by

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July, 2018

Dedicated To

My Beloved Parents

CERTIFICATE FROM THE SUPERVISOR(S)

This is to certify the thesis entitled **"Some Studies on Neutrosophic Decision Making''** submitted by Shyamal Dalapati who got his name registered on 21th March, 2016 (Registration No. PhD/R/2016/0028) for the award of Ph. D. (Science) degree from Indian Institute of Engineering, Science and Technology, Shibpur is absolutely based upon his own work under the supervision of Dr. Shariful Alam, Assistant Professor, Department of Mathematics, Indian Institute of Engineering, Science and Technology, Shibpur, P.O. Botanic Garden, Howrah - 711103 and Dr. Surapati Pramanik, Assistant Professor, Department of Mathematics, Nandalal Ghosh B.T. College, Panpur, P.O. – Narayanpur, Dist.-North 24 Parganas, Pin Code – 743126, and that neither this thesis nor any part of it has been submitted for any degree / diploma or any other academic award anywhere before**.**

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CONTENTS

Chapter 7: NC-cross entropy based MADM strategy in neutrosophic cubic set environment

Chapter 8: VIKOR based MAGDM strategy in bipolar neutrosophic set environment

Chapter 9: Conclusion

Acknowledgments

I dedicate this thesis to my parents who have always encouraged my intellectual potential and personal development. I owe a great debt of gratitude to Prof. Tapan Kumar Roy. My thesis advisors Dr. Shariful Alam and Dr. Surapati Pramanik were the two who believed in my ability to provide independent research at a very early stage of my career and have enabled this fruitful work. Dr. S. Pramanik has taught me importance of solving broad relevant problems and has, in many ways, guided me in academic matters. Much of the work in this thesis is the product of a very fruitful collaboration with my advisor and friend Dr. Surapati Pramanik. I admire Surapati Pramanik's enormous creative energy, deep professionalism and great personality. Surapati Pramanik has involved me into many exciting projects and has given me the opportunity to grow myself. I am indebted to Nahali, daughter of Surapati Pramanik, Manjira Saha wife Surapati Pramanik for their cooperation, support and kind attitude as most of the research work has done at Shyamnagar, residence of Surapati Pramanik. It has been a truly valuable and wonderful experience for me to work in an extremely creative and friendly environment in IIEST and Nandalal Ghosh B.T. College, Panpur, which I will never forget.

I thank Prof. Florentin Smarandache for his competence, motivation, and as a result, the great job we did on our projects and publications.

I acknowledge the support and care of the members of Indian Society for Neutrosophic Study (ISNS), Shyamnagar.

Finally, I acknowledged the support and care my beloved younger brother Kamal Dalapati, my elder sister Khuku Bhanj (Dalapati), Malati Das (Dalapati) and my all dear friends.

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List of Publications

1. NS-cross entropy based MAGDM under single valued neutrosophic set environment. Information, 9 (2), 2018, 37; https://doi.org/10.3390/info9020037.
2. IN-cross entropy based MAGDM strategy under interval neutrosophic set environment. Neutrosophic Sets and Systems, 18, 2017, 43-57.
3. NC-TODIM-based MAGDM under a neutrosophic cubic set environment. *Information*, *8*, 2017, 149. doi:10.3390/info8040149.
4. Neutrosophic cubic MCGDM method based on similarity measure. Neutrosophic Sets and Systems, 16, 2017, 44-56.
5. NC-VIKOR based MAGDM strategy under neutrosophic cubic set environment. Neutrosophic Sets and Systems, 20, 2018, 95-108.
6. NC-cross entropy based MADM strategy in neutrosophic cubic set environment. *Mathematics*, *6* (5), 2018, 67. https://doi.org/10.3390/math6050067.
7. VIKOR based MAGDM strategy under bipolar neutrosophic set environment. Neutrosophic Sets and Systems, 19, 2018, 57-69.

Lists of Tables

List of Figures

List of Acronyms

DM	**:**	Decision Maker
MCDM	**:**	Multiple Criteria Decision Making
MCGDM	**:**	Multiple Criteria Group Decision Making
MADM	**:**	Multi-Attribute Decision Making
MAGDM	**:**	Multi-Attribute Group Decision Making
TOPSIS	**:**	Techniques for Order Preference by Similarity to Ideal Solution
IFS	**:**	Intuitionistic Fuzzy Set
NS	**:**	Neutrosophic Set
SVNS	**:**	Single Valued Neutrosophic Set
IVNS	**:**	Interval Valued Neutrosophic Set
INN	**:**	Interval Neutrosophic Number
NCS	**:**	Neutrosophic cubic set
NCN	**:**	Neutrosophic cubic number
INNWA	**:**	Interval-Valued Neutrosophic Weighted Aggregation
BNS	**:**	Bipolar neutrosophic set
BNN	**:**	Bipolar neutrosophic number
BNNWA	**:**	Bipolar Neutrosophic Numbers Weighted Aggregation
TODIM	**:**	Tomada de decisao interativa e multicritévio
VIKOR	**:**	VIseKriterijumska Optimizacija I Kompromisno Resenje
SAW	**:**	Simple Additive Weighting
AHP	**:**	Analytic hierarchy process
TOPSIS	**:**	Technique for Order of Preference by Similarity to Ideal Solution

ELECTRE : ELimination Et Choix Traduisant la REalité (ELimination and Choice Expressing REality)

COPRAS : Complex Proportional Assessment

ARAS : Additive Ratio Assessment

PROMETHEE : Preference Ranking Organization Method for Enrichment of Evaluations

SWARA : Step-wise Weight Assessment Ratio Analysis

WASPAS : Weighted Aggregated Sum Product Assessment

MACBETH : Measuring Attractiveness by a Categorical Based Evaluation Technique

MULTIMOORA: Multiple Objective Optimization on the basis of Ratio Analysis

FARE : Factor Relationship

Preface

The thesis investigates some strategies for solving Multi-Attribute Decision Making (MADM) and Multi-Attribute Group Decision Making (MAGDM) problems in neutrosophic environment. It consists of the following **nine chapters.**

Chapter 1 describes a brief discussion on decision making problems in neutrosophic set environment. It presents some basic definitions and operations of neutrosophic sets, single valued neutrosophic sets, interval neutrosophic sets, bipolar neutrosophic sets and neutrosophic cubic sets.

Chapter 2 proposes a new cross entropy measure in single-valued neutrosophic set environment, namely NS-cross entropy, and proves its basic properties. It also defines weighted NS-cross entropy measure and investigates its basic properties. It develops a novel MAGDM strategy that is free from the drawback of asymmetrical behavior and undefined phenomena.

Chapter 3 proposes IN-cross entropy measure and proves its basic properties. It also develops weighted IN-cross entropy measure and investigates its basic properties. Based on the weighted IN-cross entropy measure, it develops a novel MAGDM strategy in interval neutrosophic environment. It solves an illustrative example of MAGDM to show the feasibility, validity and efficiency of the proposed MAGDM strategy.

Chapter 4 proposes the score function and accuracy function for neutrosophic cubic sets and prove their basic properties. It also develops a strategy for ranking of neutrosophic cubic numbers based on the score function and accuracy function. In this chapter, it firstly develops a TODIM (Tomada de decisao interativa e multicritévio) in the neutrosophic cubic set (NC) environment, namely the NC-TODIM. It establishes a new NC-TODIM strategy for solving MAGDM in neutrosophic cubic set environment. It illustrates the proposed NC-TODIM strategy for solving an MAGDM problem to show the applicability and effectiveness of the developed strategy. It also conducts sensitivity analysis to show the impact of ranking order of the alternatives for different values of the attenuation factor of losses for the proposed MAGDM strategy.

Chapter 5 defines similarity measure for neutrosophic cubic sets and proves some of its basic properties. It presents a new MAGDM strategy with linguistic variables in neutrosophic cubic set environment. Finally, it presents a numerical example to demonstrate the usefulness and applicability of the proposed strategy.

Chapter 6 proposes VIKOR strategy in neutrosophic cubic set environment, namely NC-VIKOR. It first defines NC-VIKOR strategy in neutrosophic cubic set environment to handle MAGDM problems. Actually, it combines the VIKOR with neutrosophic cubic numbers to deal with MAGDM problems. Finally, it solves an MAGDM problem to show the feasibility, applicability and effectiveness of the proposed NC-VIKOR strategy. Further, it presents sensitivity analysis to show the impact of different values of the decision making mechanism coefficient on ranking order of the alternatives.

Chapter 7 introduces a new cross entropy measure in a neutrosophic cubic set environment, namely, NC-cross entropy measure. It proves its basic properties. It also proposes weighted NC-cross entropy and investigates its basic properties. It develops a novel MADM strategy based on a weighted NC-cross entropy measure. To show the feasibility and applicability of the proposed MADM strategy, it solves an illustrative numerical example.

Chapter 8 extends the VIKOR (VIsekriterijumska optimizacija i KOmpromisno Resenje) strategy to MAGDM with bipolar neutrosophic set environment. It first defines VIKOR strategy in bipolar neutrosophic set environment to handle MAGDM problems. It combines the VIKOR with bipolar neutrosophic numbers to deal with MAGDM. It solves an MAGDM problem in bipolar neutrosophic set environment. Further, it presents sensitivity analysis to show the impact of different values of the decision making mechanism coefficient on ranking order of the alternatives.

Chapter 9 concludes the thesis with some future scope of research.

Chapter 1

Introduction

1.1 Decision making

Multi-attribute decision-making (MADM) refers to a cognitive process that involves evaluating and classifying data to find and select the best alternative from a set of feasible alternative with respect to some specific conflicting criteria. Each MADM problem consists of four components namely: (1) Expert/ decision maker, (2) alternatives, (3) criteria, (4) weight of each criterion. When decision making problem involves multiple decision makers, it is called a multi attribute group decision making (MAGDM) problem. MADM and MAGDM are widely employed in the fields of economy, engineering, management systems, and so on.

Decision making in every sphere of life has become the ultimate purpose of rational thinking of human being. In everywhere, human beings of every level have to make decision according to the demand of environment surrounding them. In the practical decision making context, decision making involves the application of elegant mathematical tools.

We may encounter decision making situation, where we have information with uncertainty and hesitancy and indeterminacy. So MADM environments are different. So to deal with MADM problems in different environments, elegant mathematical tools need to be developed. There are two types of decision making environments that we can classify. There are:

- MADM in exact environment/ crisp environment
- MADM in uncertain environment

1.1.1 MADM in exact environment/ crisp environment

MADM as a methodology officially has started since 1968 when simple additive weighting (SAW) strategy was introduced (Mac Crimon, 1968). Many MADM strategies have been developed in classical environment such as: Analytic hierarchy process (AHP) (Saaty, 1980), Technique for order preference by similarity to the ideal solution TOPSIS (Hwang & Yoon, 1981), ELimination Et ChoixTraduisant la REalité (ELECTRE) (Roy, 1968), COPRAS (Zavadskas & Kaklauskas, 1996), ARAS (Zavadskas & Turskis, 2010), VIKOR (Opricovic, 1998), SWARA (Keršulienė *et al.,* 2010), WASPAS (Zavadskas *et al.*,2012), MACBETH (Costa et al., 1994), PROMETHEE (Mareschal *et al.*,1984), MOORA (Brauers & Zavadskas, 2006), MULTIMOORA (Brauers & Zavadskas, 2010), etc.

1.1.2 MADM in uncertain environment

Three theories can be considered as the mathematical tools (Molodtsov, 1999) to deal with uncertainties, namely, theory of probability, theory of fuzzy sets (Zadeh, 1965) and interval mathematics. Fuzzy set (Zadeh, 1986), intuitionistic fuzzy set (Atanassov, 1986) and neutrosophic set (1998) studied uncertainty in non- stochastic sense while probability theory treats stochastic uncertainty. Neutrosophic set, (Smarandache, 1998), generalization of fuzzy set and intuitionistic fuzzy set, deals with uncertainty in terms of three independent membership functions namely, truth membership function, falsity membership function, and indeterminacy membership function. Dubois and Prade (1993) presented the correlation between fuzzy sets and probability theory, and established that fuzziness cannot be reduced to randomness. Similarly, neutrosophic fuzziness cannot be reduced to randomness. The concept of neutrosophic set enables formalization and reasoning of intangible internal characteristics, typically natural language-based and visual image information, as well as incomplete, indeterminate, inconsistent, unreliable, imprecise and vague performance and priority data. Elwahsh et al. (2017) used the neutrosophic set for MANETs data case study. Further, Elwahsh et al. (2018) proposed a novel approaches for classifying MANETs attacks with a neutrosophic intelligent system based on genetic algorithm. Salama et al. (2014) employed neutrosophic set to design and implement of neutrosophic data operations based on object oriented programming. Salama, El-Ghareeb et al., (2014) developed some software programs for dealing with neutrosophic sets. Interval is another non-probabilistic uncertainty formulation employed in MADM, where decision makers' preferences, criterion weights and performance value of alternatives are represented by the data ranges. In this thesis, neutrosophic set and its extension as well as neutrosophic hybrid sets such as interval neutrosophic sets (Wang et al., 2005), bipolar neutrosophic sets (Deli et al., 2015), neutrosophic cubic sets (Ali et al., 2016) have been employed to deal with indeterminate, inconsistent and incomplete information for MADM. In this thesis, uncertain environment is restricted to neutrosophic environment, interval neutrosophic environment, bipolar neutrosophic environment and neutrosophic cubic set environment.

Some classic MADM strategies such as TOPSIS (Biswas et al., 2016a), similarity measures (Pramanik et al., 2017, Mondal & Pramanik, 2015c), GRA (Biswas et al., 2014a, 2014b), TODIM (Zhang et al., 2016), cross entropy (Ye, 2013, 15b), VIKOR (Bausys & Zavadskas, 2015) have been studied in neutrosophic set environment. In this study, we have extended some of the strategies to uncertain environment, especially neutrosophic set environment, interval neutrosophic set environment, bipolar neutrosophic set environment and neutrosophic cubic set environment.

1.2 Preliminaries

In this section, we discuss briefly about neutrosophic sets, interval valued neutrosophic sets, bipolar neutrosophic sets, neutrosophic cubic sets.

1.2.1 Neutrosophic set (NS) (Smarandache, 1998)

Definition 1.1 Let U be a space of points (objects) with a generic element in U denoted by u i.e. $u \in U$. Then a neutrosophic set A in U is characterized by truth-membership function $T_A(u)$, an indeterminacy membership function $I_A(u)$ and falsity-membership function $F_A(u)$. Here $T_A(u)$, $I_A(u)$, $F_A(u)$ are the functions from U to $]^-0, 1^+[$ i.e. $T_A(u)$, $I_A(u)$, $F_A(u) : U \rightarrow\ ]^-0, 1^+[$ that means $T_A(u)$, $I_A(u)$, $F_A(u)$ are the real standard or non-standard subset of $]^-0, 1^+[$. Neutrosophic set can be expressed as A = {<u, ($T_A(u)$, $I_A(u)$, $F_A(u)$)>: $u \in U$}. Since $T_A(u)$, $I_A(u)$, $F_A(u)$ are the subset of $]^-0, 1^+[$, then the sum of $T_A(u)$, $I_A(u)$, $F_A(u)$ is lies

$^{-}0 \le T_A(u) + I_A(u) + F_A(u) \le 3^{+}$.

Definition 1.2 The complement of neutrosophic set A denoted by A^c and defined as $A^c = \{< u, T_{A^c}(u), I_{A^c}(u), F_{A^c}(u) >: u \in U\}$, where $T_{A^c}(u) = F_A(u)$, $I_{A^c}(u) = \{1^+\} - I_A(u)$, $F_{A^c}(u) = T_A(u)$.

Or,

Another definition for complement of neutrosophic set as follows:

$A^c = \{< u, T_{A^c}(u), I_{A^c}(u), F_{A^c}(u) >: u \in U\}$, where $T_{A^c}(u) = \{1^+\} - T_A(u)$, $I_{A^c}(u) = \{1^+\} - I_A(u)$, $F_{A^c}(u) = \{1^+\} - F_A(u)$.

Definition 1.3 A neutrosophic set A_1 is contained in another neutrosophic set A_2 i.e. $A_1 \subseteq A_2$ iff $T_{A_1}(u) \le T_{A_2}(u)$, $I_{A_1}(u) \ge I_{A_2}(u)$ and $F_{A_1}(u) \ge F_{A_2}(u)$, $\forall\, u \in U$.

Definition 1.4 Two neutrosophic sets A_1 and A_2 are equal iff $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$ i.e. $T_{A_1}(u) = T_{A_2}(u)$, $I_{A_1}(u) = I_{A_2}(u)$ and $F_{A_1}(u) = F_{A_2}(u)$, $\forall\, u \in U$.

Definition 1.5 The union of two neutrosophic sets R_1 and R_2 is a neutrosophic set R_3 (say) written as $A_3 = A_1 \cup A_2$.

$T_{A_3}(u) = \max\{T_{A_1}(u), T_{A_2}(u)\}$, $I_{A_3}(u) = \max\{I_{A_1}(u), I_{A_2}(u)\}$, $F_{A_3}(u) = \min\{F_{A_1}(u), F_{A_2}(u)\}$, $\forall\, u \in U$.

Definition 1.6 The intersection of two neutrosophic sets A_1 and A_2 denoted by A_4 and written as $A_4 = A_1 \cap A_2$ defined by $T_{A_4}(u) = \min\{T_{A_1}(u), T_{A_2}(u)\}$, $I_{A_4}(u) = \min\{I_{A_1}(u), I_{A_2}(u)\}$, $F_{A_4}(u) = \max\{F_{A_1}(u), F_{A_2}(u)\}$, $\forall\, u \in U$.

1.2.2 Single valued neutrosophic set (SVNS) (Wang et al., 2010)

Definition 1.7 Let U be a space of points (objects) with a generic element in U denoted by u. A single valued neutrosophic set H in U is expressed by $H = \{< u, (T_H(u), I_H(u), F_H(u)) >, u \in U\}$, where $T_H(u)$, $I_H(u)$, $F_H(u) : U \to [0, 1]$.

Therefore for each $u \in U, T_H(u)$, $I_H(u)$, $F_H(u) \in [0, 1]$ and $0 \le T_H(u) + I_H(u) + F_H(u) \le 3$

Definition 1.8 The complement of single valued neutrosophic set A denoted by A^c and defined as $A^c = \{< u, T_{A^c}(u), I_{A^c}(u), F_{A^c}(u) >: u \in U\}$, where $T_{A^c}(u) = F_A(u)$, $I_{A^c}(u) = \{1\} - I_A(u)$, $F_{A^c}(u) = T_A(u)$.

Another definition for complement of single valued neutrosophic set as follows:

$A^c = \{< u, T_{A^c}(u), I_{A^c}(u), F_{A^c}(u) >: u \in U\}$, where $T_{A^c}(u) = \{1\} - T_A(u)$, $I_{A^c}(u) = \{1\} - I_A(u)$, $F_{A^c}(u) = \{1\} - F_A(u)$.

Definition 1.9 A single valued neutrosophic set A_1 is contained in another single valued neutrosophic set A_2 i.e. $A_1 \subseteq A_2$ iff $T_{A_1}(u) \le T_{A_2}(u)$, $I_{A_1}(u) \le I_{A_2}(u)$ and $F_{A_1}(u) \ge F_{A_2}(u)$, $\forall u \in U$.

Definition 1.10 Two single valued neutrosophic sets A_1 and A_2 are equal iff $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$ i.e. $T_{A_1}(u) = T_{A_2}(u)$, $I_{A_1}(u) = I_{A_2}(u)$ and $F_{A_1}(u) = F_{A_2}(u)$, $\forall u \in U$.

Definition 1.11 The union of two single valued neutrosophic sets R_1 and R_2 is a neutrosophic set R_3 (say) written as $A_3 = A_1 \cup A_2$.

$T_{A_3}(u)$ = max $\{T_{A_1}(u), T_{A_2}(u)\}$, $I_{A_3}(u)$ = max $\{I_{A_1}(u), I_{A_2}(u)\}$, $F_{A_3}(u)$ = min $\{F_{A_1}(u), F_{A_2}(u)\}$, $\forall u \in U$.

Definition 1.12 The intersection of two single valued neutrosophic sets A_1 and A_2 denoted by A_4 and written as $A_4 = A_1 \cap A_2$ defined by $T_{A_4}(u)$ = min $\{T_{A_1}(u), T_{A_2}(u)\}$, $I_{A_4}(u)$ = min $\{I_{A_1}(u), I_{A_2}(u)\}$, $F_{A_4}(u)$ = max $\{F_{A_1}(u), F_{A_2}(u)\}$, $\forall u \in U$.

Definition 1.13

Let H_1 and H_2 be any two SVNSs. Then, operations are defined as:

i. $H_1 \oplus H_2 = \{T_{H_1}(u) + T_{H_2}(u) - T_{H_1}(u).T_{H_2}(u), I_{H_1}(u).I_{H_2}(u), F_{H_1}(u).F_{H_2}(u):\}$.

ii. $H_1 \otimes H_2 =$

$\{T_{H_1}(u).T_{H_2}(u), I_{H_1}(u)+I_{H_2}(u)-I_{H_1}(u).I_{H_2}(u),\ F_{H_1}(u)+F_{H_2}(u)-F_{H_1}(u).F_{H_2}(u)\}$

iii. $\lambda H_1 = \{1-(1-T_{H_1}(u))^{\lambda}, (I_{H_1}(u))^{\lambda}, (F_{H_1}(u))^{\lambda} : \lambda > 0\}$.

iv. $H_1^{\lambda} = \{(T_{H_1}(u))^{\lambda}, 1-(1-I_{H_1}(u))^{\lambda}, 1-(1-F_{H_1}(u))^{\lambda} : \lambda > 0\}$.

1.2.3 Interval neutrosophic set (INS) (Wang et al., 2005)

Definition 1.13 Assume that U be a space of points (objects) with generic elements $u \in U$. An INSs J in U is characterized by a truth-membership measure $T_J(u)$, an indeterminacy-membership measure $I_J(u)$, and a falsity-membership measure $F_J(u)$, where, $T_J(u) = [T_J^-(u), T_J^+(u)]$, $I_J(u) = [I_J^-(u), I_J^+(u)]$, $F_J(u) = [F_J^-(u), F_J^+(u)]$ for all u in U. Therefore, a INS J can be expressed as J = {u, $[T_J^-(u), T_J^+(u)]$, $[I_J^-(u), I_J^+(u)]$, $[F_J^-(u), F_J^+(u)]$ | u∈U}. Where, $T_J^-(u), T_J^+(u)$, $I_J^-(u), I_J^+(u)$, $F_J^-(u), F_J^+(u) \subseteq [0,1]$.

Definition 1.14 Let J_1 = {u, $[T_{J_1}^-(u), T_{J_1}^+(u)]$, $[I_{J_1}^-(u), I_{J_1}^+(u)]$, $[F_{J_1}^-(u), F_{J_1}^+(u)]$ | u∈U} and J_2 = {u, $[T_{J_2}^-(u), T_{J_2}^+(u)]$, $[I_{J_2}^-(u), I_{J_2}^+(u)]$, $[F_{J_2}^-(u), F_{J_2}^+(u)]$ | u∈U} be any two INSs in U, then $J_1 \subseteq J_2$ iff $T_{J_1}^-(u) \le T_{J_2}^-(u)$, $T_{J_1}^+(u) \le T_{J_2}^+(u)$, $I_{J_1}^-(u) \ge I_{J_2}^-(u)$, $I_{J_1}^+(u) \ge I_{J_2}^+(u)$, $F_{J_1}^-(u) \ge F_{J_2}^-(u)$, $F_{J_1}^+(u) \ge F_{J_2}^+(u)$ for all u∈U.

Definition 1.15 The complement J^c of an INS J = {u,$[T_J^-(u), T_J^+(u)]$, $[I_J^-(u), I_J^+(u)]$, $[F_J^-(u), F_J^+(u)]$ | u∈U} is defined as follows: J^c = {u, $[1-T_J^+(u), 1-T_J^-(u)]$, $[1-I_J^+(u), 1-I_J^-(u)]$, $[1-F_J^+(u), 1-F_J^-(u)]$ | u∈U}.

Definition 1.16 Let J_1 = {u,$[T_{J_1}^-(u), T_{J_1}^+(u)]$, $[I_{J_1}^-(u), I_{J_1}^+(u)]$, $[F_{J_1}^-(u), F_{J_1}^+(u)]$ | u∈U} and J_2 = {u, $[T_{J_2}^-(u), T_{J_2}^+(u)]$, $[I_{J_2}^-(u), I_{J_2}^+(u)]$, $[F_{J_2}^-(u), F_{J_2}^+(u)]$ | u∈U} be any two INSs in U, then $J_1 = J_2$ iff $T_{J_1}^-(u) = T_{J_2}^-(u)$, $T_{J_1}^+(u) = T_{J_2}^+(u)$, $I_{J_1}^-(u) = I_{J_2}^-(u)$, $I_{J_1}^+(u) = I_{J_2}^+(u)$, $F_{J_1}^-(u) = F_{J_2}^-(u)$, $F_{J_1}^+(u) = F_{J_2}^+(u)$ for all u∈U.

1.2.4 Bipolar neutrosophic set (BNS) (Deli et al., 2015)

Definition 1.17 Let U be a space of points (objects) with a generic element in U denoted by u. A bipolar neutrosophic set H in U is defined as an object of the form

$$H=\{u,<T_H^+(u),\ I_H^+(u),\ F_H^+(u), T_H^-(u), T_H^-(u),\ F_H^-(u)>: u\in U\},$$

where, $T_H^+(u), I_H^+(u), F_H^+(u): U\to[0,1]$ and $T_H^-(u), I_H^-(u), F_H^-(u): U\to[-1,0]$.

We denote $H=\{\ u, <T_H^+(u), I_H^+(u), F_H^+(u), T_H^-(u), I_H^-(u), F_H^-(u)>|\ u\in U\}$ simply

H = $<T_H^+, I_H^+, F_H^+, T_H^-, I_H^-, F_H^->$ as a bipolar neutrosophic number (BNN).

Definition 1.18 Let $H_1=\{\ u, <T_1^+(u), I_1^+(u), F_1^+(u), T_1^-(u), I_1^-(u), F_1^-(u)>|\ u\in U\}$

and $H_2=\{\ u, <T_2^+(u), I_2^+(u), F_2^+(u), T_2^-(u), I_2^-(u), F_2^-(u)>| u\in U\}$ be any two bipolar neutrosophic sets in U. Then $H_1\subseteq H_2$ iff $T_1^+(u)\le T_2^+(u)$, $I_1^+(u)\ge I_2^+(u)$, $F_1^+(u)\ge F_2^+(u)$ and $T_1^-(u)\ge T_2^-(u)$, $I_1^-(u)\le I_2^-(u)$, $F_1^-(u)\le F_2^-(u)$ for all $u\in U$.

Definition 1.19 Let $H_1=\{\ u, <T_1^+(u), I_1^+(u), F_1^+(u), T_1^-(u), I_1^-(u), F_1^-(u)>: u\in U\}$

and $H_2=\{u, <T_2^+(u), I_2^+(u), F_2^+(u), T_2^-(u), I_2^-(u), F_2^-(u)>: u\in U\}$ be any two bipolar neutrosophic sets in U. Then, $H_1=H_2$ iff $T_1^+(u)=T_2^+(u)$, $I_1^+(u)=I_2^+(u)$, $F_1^+(u)=F_2^+(u)$ and $T_1^-(u)=T_2^-(u)$, $I_1^-(u)=I_2^-(u)$, $F_1^-(u)=F_2^-(u)$ for all $u\in U$.

Definition 1.20 Let $H_1=\{u, <T_1^+(u), I_1^+(u), F_1^+(u), T_1^-(u), I_1^-(u), F_1^-(u)>| u\in U\}$ and $H_2=\{u, <T_2^+(u), I_2^+(u), F_2^+(u), T_2^-(u), I_2^-(u), F_2^-(u)>| u\in U\}$ be any two bipolar neutrosophic sets in U. Then, their union is defined as follows:

$$\begin{aligned}H_3(u)=H_1(u)\cup H_2(u)=\{u, <\max(T_1^+(u), T_2^+(u)), \min(I_1^+(u), I_2^+(u)), \min(F_1^+(u), F_2^+(u)),\\ \min(T_1^-(u), T_2^-(u)),\ \max(I_1^-(u), I_2^-(u)), \max(F_1^-(u), F_2^-(u))>|\ u\in U\}, \text{for all } u\in U.\end{aligned}$$

Definition 1.21 Let $H_1 = \{u, < T_1^+(u), I_1^+(u), F_1^+(u), T_1^-(u), I_1^-(u), F_1^-(u) > | u \in U\}$ and $H_2 = \{u, < T_2^+(u), I_2^+(u), F_2^+(u), T_2^-(u), I_2^-(u), F_2^-(u) > | u \in U\}$ be any two bipolar neutrosophic sets in U. Then, their intersection is defined as follows:

$$H_4(u) = H_1(u) \cap H_2(u) = \{u, < \min(T_1^+(u), T_2^+(u)), \max(I_1^+(u), I_2^+(u)), \max(F_1^+(u), F_2^+(u)), \max(T_1^-(u), T_2^-(u)), \min(I_1^-(u), I_2^-(u)), \min(F_1^-(u), F_2^-(u)) > | u \in U\} \text{ for all } u \in U.$$

Definition 1.22 Let $H_1 = \{u, < T_1^+(u), I_1^+(u), F_1^+(u), T_1^-(u), I_1^-(u), F_1^-(u) >: u \in U\}$ be a bipolar neutrosophic set in U. Then the complement of H_1 is denoted by H_1^c and is defined by

$$H_1^c = \{u, < 1 - T_1^+(u), 1 - I_1^+(u), 1 - F_1^+(u), \{-1\} - T_1^-(u), \{-1\} - I_1^-(u), \{-1\} - F_1^-(u) >: u \in U\}$$
for all $u \in U$.

Definition 1.23 Let $h_1 = < T_1^+, I_1^+, F_1^+, T_1^-, I_1^-, F_1^- >$ and $h_2 = < T_2^+, I_2^+, F_2^+, T_2^-, I_2^-, F_2^- >$ be any two BNNs in U. Then Hamming distance measure between h_1 and h_2 is denoted by $D(h_1, h_2)$ and defined as follows:

$$D(h_1, h_2) = \frac{1}{6}[|T_1^+ - T_2^+| + |I_1^+ - I_2^+| + |F_1^+ - F_2^+| + |T_1^- - T_2^-| + |I_1^- - I_2^-| + |F_1^- - F_2^-|] \tag{1.1}$$

Definition 1.24

In decision making situation cost type attribute and benefit type attribute may exist simultaneously. Assume that, h_{ij} be a BNN to express the rating value of i-th alternative with respect to j-th attribute (c_j). If c_j belongs to the cost type attributes, then h_{ij} should be standardized by employing the complement of BNN h_{ij}. When the attribute c_j belongs to benefit type attributes, h_{ij} does not need to be standardized, we use the following formula of normalization as follows:

$$h_{ij}^* = < \{1\} - T_{ij}^+, \{1\} - I_{ij}^+, \{1\} - F_{ij}^+, \{-1\} - T_{ij}^-, \{-1\} - I_{ij}^-, \{-1\} - F_{ij}^- > \tag{1.2}$$

1.2.5 Neutrosophic cubic set (NCS) (Ali et al., 2016)

Definition 1.25 Assume that U is a space of points (objects) with generic elements $u_i \in U$. A NCS Q in U is a hybrid structure of INS and SVNS that can be expressed as follows:

$$Q=\{u_i, <[T_Q^-(u_i), T_Q^+(u_i)], [I_Q^-(u_i), I_Q^+(u_i)], [F_Q^-(u_i), F_Q^+(u_i)], (T_Q(u_i), I_Q(u_i), F_Q(u_i) > | u_i \in U\}.$$

Here, $([T_Q^-(u_i), T_Q^+(u_i)], [I_Q^-(u_i), I_Q^+(u_i)], [F_Q^-(u_i), F_Q^+(u_i)])$ and $(T_Q(u_i), I_Q(u_i), F_Q(u_i))$ are INS and SVNS, respectively, in U. NCS can be simply presented as

$$<[T_Q^-(u), T_Q^+(u)], [I_Q^-(u), I_Q^+(u)], [F_Q^-(u), F_Q^+(u)], (T_Q(u), I_Q(u), F_Q(u) > \quad (1.3)$$

Equation (1.3) represents neutrosophic cubic number (NCN).

Definition 1.26

Let $Q_1=\{u_i, <[T_{Q_1}^-(u_i), T_{Q_1}^+(u_i)], [I_{Q_1}^-(u_i), I_{Q_1}^+(u_i)], [F_{Q_1}^-(u_i), F_{Q_1}^+(u_i)],$

$(T_{Q_1}(u_i), I_{Q_1}(u_i), F_{Q_1}(u_i) > | u_i \in U\}.$ and

$Q_2=\{u_i, <[T_{Q_2}^-(u_i), T_{Q_2}^+(u_i)], [I_{Q_2}^-(u_i), I_{Q_2}^+(u_i)], [F_{Q_2}^-(u_i), F_{Q_2}^+(u_i)],$
$(T_{Q_2}(u_i), I_{Q_2\backslash}(u_i), F_{Q_2}(u_i) > | u_i \in U\}$ be any two NCSs in U. Then, $Q_1 \subseteq Q_2$ iff

$T_{Q_1}^-(u_i) \le T_{Q_2}^-(u_i)$, $T_{Q_1}^+(u_i) \le T_{Q_2}^+(u_i)$, $I_{Q_1}^-(u_i) \ge I_{Q_2}^-(u_i)$, $I_{Q_1}^+(u_i) \ge I_{Q_2}^+(u_i)$,

$F_{Q_1}^-(u_i) \ge F_{Q_2}^-(u_i)$, $F_{Q_1}^+(u_i) \ge F_{Q_2}^+(u_i)$ and

$T_{Q_1}(u_i) \le T_{Q_2}(u_i)$, $I_{Q_1}(u_i) \ge I_{Q_2}(u_i)$, $F_{Q_1}(u_i) \ge F_{Q_2}(u_i)$ for all $u_i \in U$.

Definition 1.27

Let

$Q_1=\{u_i, <[T_{Q_1}^-(u_i), T_{Q_1}^+(u_i)], [I_{Q_1}^-(u_i), I_{Q_1}^+(u_i)], [F_{Q_1}^-(u_i), F_{Q_1}^+(u_i)], (T_{Q_1}(u_i), I_{Q_1}(u_i), F_{Q_1}(u_i) > | u_i \in U\}.$
and

$Q_2=\{u_i, <[T_{Q_2}^-(u_i), T_{Q_2}^+(u_i)], [I_{Q_2}^-(u_i), I_{Q_2}^+(u_i)], [F_{Q_2}^-(u_i), F_{Q_2}^+(u_i)], (T_{Q_2}(u_i), I_{Q_2\backslash}(u_i), F_{Q_2}(u_i) > | u_i \in U\}$
be any two NCSs [59] in U. Then $Q_1 = Q_2$ iff $T_{Q_1}^-(u_i) = T_{Q_2}^-(u_i)$, $T_{Q_1}^+(u_i) = T_{Q_2}^+(u_i)$,

$I^{-}_{Q_1}(u_i) = I^{-}_{Q_2}(u_i)$, $I^{+}_{Q_1}(u_i) = I^{+}_{Q_2}(u_i)$, $F^{-}_{Q_1}(u_i) = F^{-}_{Q_2}(u_i)$, $F^{+}_{Q_1}(u_i) = F^{+}_{Q_2}(u_i)$ and

$T_{Q_1}(u_i) = T_{Q_2}(u_i), I_{Q_1}(u_i) = I_{Q_2}(u_i), F_{Q_1}(u_i) = F_{Q_2}(u_i)$ for all $u_i \in U$.

Definition 1.28

Assume that

$Q = \{u_i, < [T^{-}_{Q}(u_i), T^{+}_{Q}(u_i)], [I^{-}_{Q}(u_i), I^{+}_{Q}(u_i)], [F^{-}_{Q}(u_i), F^{+}_{Q}(u_i)], (T_Q(u_i), I_Q(u_i), F_Q(u_i) > | u_i \in U\}$ be any NCS in U. Then, complement Q^c of Q is defined as follows:

$$Q^c = \{u_i, < [1 - T^{+}_{Q}(u_i), 1 - T^{-}_{Q}(u_i)], [1 - I^{+}_{Q}(u_i), 1 - I^{-}_{Q}(u_i)],$$
$$[1 - F^{+}_{Q}(u_i), 1 - F^{-}_{Q}(u_i)], (1 - T_Q(u_i), 1 - I_Q(u_i), 1 - F_Q(u_i) > | u_i \in U\}.$$

1.2.6 Neutrosophic crisp set (Salama & Smarandache, 2015)

Definition 1.29 Assume that U is a non-empty fixed sample space. A neutrosophic crisp set B is an object having the form B = $< B_1, B_2, B_3 >$ where B_1, B_2, B_3 are subsets of U.

Definition 1.30

The object having the form B = $< B_1, B_2, B_3 >$ is called:

(i) A neutrosophic crisp set of type 1 if satisfying $B_1 \cap B_2 = \phi$, $B_1 \cap B_3 = \phi$ and $B_2 \cap B_3 = \phi$.

(ii) A neutrosophic crisp set of type 2 if satisfying $B_1 \cap B_2 = \phi$, $B_1 \cap B_3 = \phi$, $B_2 \cap B_3 = \phi$ and $B_1 \cup B_2 \cup B_3 = U$.

(iii). A neutrosophic crisp set of type 3 if satisfying $B_1 \cap B_2 \cap B_3 = \phi$ and $B_1 \cup B_2 \cup B_3 = U$.

1.2.7 Cross entropy measure

Cross entropy measure is one of the best way to calculate the divergence of any variable from the priori one variable. Majumdar and Samanta (2014) defined an entropy measure and presented an MADM strategy in SVNS environment. Ye (2013) proposed cross entropy measure in SVNS environment. Ye (2015b) defined improved cross entropy measures for SVNSs and INSs. Assume that H_1 and H_2 be any two SVNSs in U = $\{u_1, u_2, u_3, ..., u_n\}$. The single-valued cross-entropy of $H_1$and H_2 is denoted by CE (H_1, H_2), CE (H_1, H_2) satisfies the following four properties (Ye, 2013) such as:

i. $CE(H_1, H_2) \geq 0, \forall u_i \in U$

ii. $CE(H_1, H_2) = 0$ if and only if $T^-_{H_1}(u_i) = T^-_{H_2}(u_i)$, $T^+_{H_1}(u_i) = T^+_{H_2}(u_i)$, $I^-_{H_1}(u_i) = I^-_{H_2}(u_i)$, $I^+_{H_1}(u_i) = I^+_{H_2}(u_i)$, $F^-_{H_1}(u_i) = F^-_{H_2}(u_i)$, $F^+_{H_1}(u_i) = F^+_{H_2}(u_i)$ for all $\forall u_i \in U$.

iii. $CE(H_1, H_2) = CE(H^c_1, H^c_2)$, $\forall u_i \in U$

iv. $CE(H_1, H_2) = CE(H_2, H_1)$, $\forall u_i \in U$

1.2.8 Similarity measure

Similarity measure is a vital topic in fuzzy set theory. Chen and Hsiao (1995) presented comparisons of similarity measures of fuzzy sets. Pramanik and Mondal (2015a) studied weighted fuzzy similarity measure based on tangent function for medical diagnosis. Hwang and Yang (2013) constructed a new similarity measure between intuitionistic fuzzy sets based on lower, upper and middle fuzzy sets. Mondal and Pramanik (2015a) developed tangent similarity measures in intuitionistic fuzzy environment to deal with medical diagnosis. Ren and Wang (2015) proposed similarity measures in interval- valued intuitionistic fuzzy environment and applied it to MADM problems. Baccour et al. (2013) presented survey of similarity measures for intuitionistic fuzzy sets. Broumi and Smarandache (2013b) discussed several similarity measures of neutrosophic sets. Majumdar and Samanta (2014) introduced some measures of similarity and entropy of single valued neutrosophic sets. Aydogdu (2015a) proposed similarity and entropy measure of single valued neutrosophic sets. Mondal and Pramanik (2015c) extended the concept of intuitionistic tangent similarity measure to neutrosophic tangent similarity. Biswas et al. (2015) studied cosine similarity measure with trapezoidal fuzzy neutrosophic numbers to deal with MADM problems.

Aydogdu (2015b) also defined entropy and similarity measures of interval neutrosophic sets.Ye (2014a) proposed a similarity measures under interval neutrosophic domain using Hamming distances and Euclidean distances. Mondal et al. (2018b) proposed hybrid binary logarithm similarity measure and established an MAGDM strategy in SVNS environment. Mondal et al. (2018a) proposed hyperbolic sine similarity measure and developed an MADM strategy in SVNS environment.

1.2.9 VIKOR strategy

The VIKOR is a multi-criteria decision analysis strategy to solve a multi-criteria optimization problem. It focuses on ranking and selecting the best alternatives from a set of feasible alternatives in the presence of conflicting criteria for a decision problem. The compromise solution (Opricovic, 1998; Opricovic & Tzeng, 2004) is the closest to the ideal solution, and a compromise means an agreement established by mutual concessions. Using the L_p–metric, Opricovic and Tzeng (2007) defined

$$L_{pi} = \left\{ \sum_{j=1}^{n} \left[\left(\Omega_j^+ - \Omega_{ij} \right) / \left(\Omega_j^+ - \Omega_j^- \right) \right]^p \right\}^{\frac{1}{p}} 1 \le p \le \infty; i = 1, 2, 3, ..., m.$$

In the VIKOR strategy, L_{1i} (as S_i) and $L_{\infty i}$, (as R_i) are utilized to formulate ranking measure. The solution obtained by min S_i reflects the maximum group utility (‘‘majority” rule), and the solution obtained by min R_i indicates the minimum individual regret of the “opponent”.

Suppose that each alternative is evaluated by each criterion function. The compromise ranking is prepared by comparing the measure of closeness to the ideal alternative. The m alternatives are denoted as A_1, A_2, A_3, ...,A_m. For the alternative A_i, the rating of the j-th aspect is denoted by Ω_{ij}, i.e. Ω_{ij} is the value of jth criterion function for the alternative A_i; n is the number of criteria.

The compromise ranking algorithm of the VIKOR strategy is presented using the following steps:

Step 1. Determine the best Ω_j^+ and the worst Ω_j^- values of all criterion functions j =1, 2,..., n . If the

j-th function represents a benefit then:

$$\Omega_j^+ = \max_i \Omega_{ij}, \ \Omega_j^- = \min_i \Omega_{ij}$$

Step 2. Compute the values S_i and R_i ;i = 1, 2,..., m, by these relations:

$$S_i = \sum_{j=1}^{n} w_j \left(\Omega_j^+ - \Omega_{ij}\right) / \left(\Omega_j^+ - \Omega_j^-\right),$$

$$R_i = \max_j w_j \left(\Omega_j^+ - \Omega_{ij}\right) / \left(\Omega_j^+ - \Omega_j^-\right),$$

Here, w_j is the weight of the criterion that expresses its relative importance.

Step 3. Compute the values Q_i: i = 1, 2,...,m, using the following relation:

$$Q_i = \nu\left(S_i - S^-\right) / \left(S^+ - S^-\right) + (1-\nu)\left(R_i - R^-\right) / \left(R^+ - R\right).$$ Here, $S^+ = \max_i S_i$, $S^- = \min_i S_i$

$R^+ = \max_i R_i$, $R^- = \min_i R_i$.

Here, v represents ''the decision making mechanism coefficient'' (or ''the maximum group utility''). Here we consider v = 0.5.

Step 4. Preference ranking order of the alternatives is done by sorting the values of S, Rand Q in decreasing order.

Step 5. Determine compromise solution

Obtain alternative A^1 as compromise solution, which is ranked as the best by the measure Q (Minimum) if the following two conditions are satisfied:

Condition 1. Acceptable stability: $Q(A^2) - Q(A^1) \geq \frac{1}{(m-1)}$, where A^1, A^2 are the alternatives with first and second position in the ranking list by Q; m is the number of alternatives.

Condition 2. Acceptable stability in decision making: Alternative A^1 must also be the best ranked by S or/and R. This compromise solution is stable within whole decision making process.

If one of the conditions is not satisfied, then a set of compromise solutions is proposed as follows:

◊ Alternatives A^1 and A^2 are compromise solutions if only condition 2 is not satisfied, or

◊ A^1, A^2, A^3,…, A^r are compromise solutions if condition 1 is not satisfied and A^m is decided by constraint $Q(A^m) - Q(A^1) \le \frac{1}{(m-1)}$ for maximum m.

1.3 Organization of the thesis

The proposed thesis consists of nine chapters.

Chapter 1: Introduction.

Chapter 2: NS-cross entropy-based MAGDM under single-valued neutrosophic set environment

Chapter 3: IN-cross entropy based MAGDM strategy in interval neutrosophic set environment

Chapter 4: NC-TODIM-based MAGDM in a neutrosophic cubic set environment

Chapter 5: Neutrosophic cubic MCGDM method based on similarity measure

Chapter 6: NC-VIKOR based MAGDM strategy in neutrosophic cubic set environment

Chapter 7: NC-cross entropy based MADM strategy in neutrosophic cubic set environment

Chapter 8: VIKOR based MAGDM strategy in bipolar neutrosophic set environment

Chapter 9: Conclusion.

1.4 Outline of the work

The thesis investigates some methods for solving MADM and MAGDM problems in neutrosophic environment with following **nine chapters.**

Chapter 1 describes a brief discussion on decision making problems in neutrosophic set environment. It presents some basic definitions and operations of neutrosophic sets, single valued neutrosophic sets, interval neutrosophic sets, bipolar neutrosophic sets and neutrosophic cubic sets.

Chapter 2 proposes a new cross entropy measure in single-valued neutrosophic set (SVNS) environment, namely NS-cross entropy, and proves its basic properties. It also defines weighted NS-cross entropy measure and investigates its basic properties. It develops a novel MAGDM strategy that is free from the drawback of asymmetrical behaviour and undefined phenomena.

Chapter 3 proposes IN-cross entropy measure and proves its basic properties. It also develops weighted IN-cross entropy measure and investigates its basic properties. Based on the weighted IN-cross entropy measure, it develops a novel MAGDM strategy in interval neutrosophic environment. It solves an illustrative example of MAGDM to show the feasibility, validity and efficiency of the proposed MAGDM strategy.

Chapter 4 proposes the score and accuracy functions for neutrosophic cubic sets and prove their basic properties. It also develops a strategy for ranking of neutrosophic cubic numbers based on the score and accuracy functions. It firstly develops a TODIM in the neutrosophic cubic set environment, which call the NC-TODIM. It established a new NC-TODIM strategy for solving MAGDM in neutrosophic cubic set environment. It illustrates the proposed NC-TODIM strategy for solving an MAGDM problem to show the applicability and effectiveness of the developed strategy. It also conducts a sensitivity analysis to show the impact of ranking order of the alternatives for different values of the attenuation factor of losses for multi-attribute group decision making strategies.

Chapter 5 defines similarity measure for neutrosophic cubic sets and proves some of its basic properties. It presents a new MADM strategy with linguistic variables in neutrosophic cubic set environment. Finally, it presents a numerical example to demonstrate the usefulness and applicability of the proposed strategy.

Chapter 6 proposes VIKOR strategy in neutrosophic cubic set environment, namely NC-VIKOR. It first defines NC-VIKOR strategy in neutrosophic cubic set environment to handle MAGDM problem, which means it combines the VIKOR with neutrosophic cubic number to deal with MAGDM problems. Finally, it solves an MAGDM problem using the newly proposed NC-VIKOR strategy to show the feasibility, applicability and effectiveness of the proposed strategy. Further, it presents sensitivity analysis to show the impact of different values of the decision making mechanism coefficient on ranking order of the alternatives.

Chapter 7 introduces a new cross entropy measure in a neutrosophic cubic set (NCS) environment, which is call NC-cross entropy measure. It proves its basic properties. It also proposes weighted NC-cross entropy and investigates its basic properties. It develops a novel MADM strategy based on a weighted NC-cross entropy measure. To show the feasibility and applicability of the proposed MADM strategy, it solves an MADM problem.

Chapter 8 extends the VIKOR strategy to MAGDM with bipolar neutrosophic set environment. It first defines VIKOR strategy in bipolar neutrosophic set environment to handle MAGDM problems, which means it combines the VIKOR with bipolar neutrosophic numbers to deal with MAGDM. Finally, it solves an MAGDM problem using the proposed VIKOR strategy in bipolar neutrosophic set environment. Further, it presents a sensitivity analysis to show the impact of different values of the decision making mechanism coefficient on ranking order of the alternatives.

Chapter 9 concludes the thesis with some future scope of research.

Chapter 2

NS-cross entropy-based MAGDM under single-valued neutrosophic set environment

2.1 Introduction

Majumdar and Samanta (2014) defined an entropy measure and presented an MADM strategy in SVNS environment. Ye (2013) proposed cross entropy measure in the SVNS environment, which is not symmetric straight forward and bears undefined phenomena. To overcome the asymmetrical behavior of the cross entropy measure, Ye (2013) used a symmetric discrimination information measure for single-valued neutrosophic sets. Ye (2015b) defined improve cross entropy measures for SVNSs to overcome the drawbacks of undefined phenomena of the cross entropy measure (Ye, 2013).

The object of the chapter is to define an NS-cross entropy measure and prove its basic properties.It also defines a weighted NS-cross entropy measure in the SVNS environment and proves its basic properties. The proposed NS-cross entropy is straightforward symmetric. It also bears no undefined behaviour. This chapterdevelops a new MAGDM strategy based on weighted NS-cross entropy measure to solve MAGDM problems with unknown weight of the attributes and unknown weight of decision-makers.

The chapter is organized as follows: Section 2.2 proposes a new NS-cross entropy measure between two SVNSs and investigates its basic properties. It also defines a weighted NS-cross entropy measure and proves its basic properties. Section 2.3develops a novel MAGDM strategy based on the proposed weighted NS-cross entropy with SVNS information.

The content of this chapter is based on the paper published in“**Information**” 2018, 9, 37; doi:10.3390/info9020037.

In Section 2.4 an illustrative example is solved to demonstrate the applicability and efficiency of the developed MAGDM strategy under SVNS environment.Section 2.5 presents comparative study and discussion. Section 2.6 offers conclusions and the future scope of research.

2.2 NS-cross entropy measure

In this section, we define a new single-valued neutrosophic cross-entropy measure for measuring the deviation of single-valued neutrosophic variables from an a priori one.

Definition 2.1 NS-cross entropy measure

Assume that $H_1$and H_2 be any two SVNSs in U = $\{u_1, u_2, u_3, ..., u_n\}$.Then, the single-valued cross-entropy of H1 and H_2 is denoted by CE_{NS} (H_1, H_2) and defined as follows:

$$CE_{NS}(H_1, H_2)$$

$$= \frac{1}{2}\left\{\sum_{i=1}^{n}\left\langle\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}+\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right]\right\rangle\right\} \tag{2.1}$$

Example 2.1 Assume that H_1= {u, (0.7, 0.3, 0.4)| u ∈U} and H_2= {u, (0.6, 0.4, 0.2)| u ∈U}. Using Equation (2.1), the cross entropy value of H_1 and H_2 is obtained as $CE_{NS}(H_1, H_2)$ = 0.707.

Theorem 2.1 Single-valued neutrosophic cross entropy $CE_{NS}(H_1, H_2)$ for any two SVNSs H_1, H_2, satisfies the following properties:

i. $CE_{NS}(H_1, H_2) \geq 0, \forall u_i \in U$

ii. $CE_{NS}(H_1, H_2)=0$ if and only if $T_{H_1}(u_i)=T_{H_2}(u_i)$, $I_{H_1}(u_i)=I_{H_2}(u_i)$, $F_{H_1}(u_i)=F_{H_2}(u_i)$, $\forall u_i \in U$.

iii. $CE_{NS}(H_1, H_2)=CE_{NS}(H_1^c, H_2^c)$, $\forall u_i \in U$.

iv. $CE_{NS}(H_1, H_2)=CE_{NS}(H_2, H_1)$, $\forall u_i \in U$.

Proof: i.For all values of $u_i \in U$, $\left|T_{H_1}(u_i)\right|\geq 0$, $\left|T_{H_2}(u_i)\right|\geq 0$, $\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|\geq 0$, $\sqrt{1+\left|T_{H_1}(u_i)\right|^2}\geq 0$, $\sqrt{1+\left|T_{H_2}(u_i)\right|^2}\geq 0$, $\left|(1-T_{H_1}(u_i))\right|\geq 0$, $\left|(1-T_{H_2}(u_i))\right|\geq 0$, $\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|\geq 0$, $\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}\geq 0$, $\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}\geq 0$.

Then,

$$\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right]\geq 0$$

Similarly,

$$\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}+\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right]\geq 0, \text{ and}$$

$$\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right]\geq 0$$

Therefore, $CE_{NS}(H_1, H_2)\geq 0$.

Hence complete the proof.□

ii. $$\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right]=0,$$

$\Leftrightarrow T_{H_1}(u_i) = T_{H_2}(u_i)$

$$\left[\frac{2\left|I_{H_1}(u_i) - I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2} + \sqrt{1+\left|I_{H_2}(u)\right|^2}} + \frac{2\left|(1-I_{H_1}(u_i)) - (1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2} + \sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right] = 0$$

$\Leftrightarrow I_{H_1}(u_i) = I_{H_2}(u_i)$, and

$$\left[\frac{2\left|F_{H_1}(u_i) - F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2} + \sqrt{1+\left|F_{H_2}(u_i)\right|^2}} + \frac{2\left|(1-F_{H_1}(u_i)) - (1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2} + \sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right] = 0,$$

$\Leftrightarrow F_{H_1}(u_i) = F_{H_2}(u_i)$

Therefore, $CE_{NS}(H_1, H_2) = 0$,iff $T_{H_1}(u_i) = T_{H_2}(u_i)$ $I_{H_1}(u_i) = I_{H_2}(u_i)$,

$F_{H_1}(u_i) = F_{H_2}(u_i)$, $\forall u_i \in U$.

Hence complete the proof.□

iii. Now, we have

$$CE_{NS}(H_1^c, H_2^c)$$

$$= \frac{1}{2}\left\{\sum_{i=1}^{n}\left\langle\left[\frac{2\left|(1-T_{H_1}(u_i)) - (1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2} + \sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}} + \frac{2\left|T_{H_1}(u_i) - T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2} + \sqrt{1+\left|T_{H_2}(u_i)\right|^2}}\right] + \right.\right.$$

$$\left[\frac{2\left|(1-I_{H_1}(u_i)) - (1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2} + \sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}} + \frac{2\left|I_{H_1}(u_i) - I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2} + \sqrt{1+\left|I_{H_2}(u)\right|^2}}\right] +$$

$$\left.\left.\left[\frac{2\left|(1-F_{H_1}(u_i)) - (1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2} + \sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}} + \frac{2\left|F_{H_1}(u_i) - F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2} + \sqrt{1+\left|F_{H_2}(u_i)\right|^2}}\right]\right\rangle\right\}$$

$$= \frac{1}{2}\left\{\sum_{i=1}^{n}\left\langle\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}+\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right]\right\rangle\right\}$$

$$=CE_{SN}\ (H_1, H_2)$$

Therefore, $CE_{NS}(H_1, H_2)=CE_{NS}\ (H_1^c, H_2^c)$.

Hence complete the proof.☐

iv. Since,

$$\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|=\left|T_{H_2}(u_i)-T_{H_1}(u_i)\right|,\ \left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|=\left|I_{H_2}(u_i)-I_{H_1}(u_i)\right|,$$

$$\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|=\left|F_{H_2}(u_i)-F_{H_1}(u_i)\right|,$$

$$\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|=\left|(1-T_{H_2}(u_i))-(1-T_{H_1}(u_i))\right|,$$

$$\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|=\left|(1-I_{H_2}(u_i))-(1-I_{H_1}(u_i))\right|,$$

$$\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|=\left|(1-F_{H_2}(u_i))-(1-F_{H_1}(u_i))\right|,\ \text{we have}$$

$$\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}=\sqrt{1+\left|T_{H_2}(u_i)\right|^2}+\sqrt{1+\left|T_{H_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u_i)\right|^2}=\sqrt{1+\left|I_{H_2}(u_i)\right|^2}+\sqrt{1+\left|I_{H_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}=\sqrt{1+\left|F_{H_2}(u_i)\right|^2}+\sqrt{1+\left|F_{H_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}=\sqrt{1+\left|(-T_{H_2}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2},$$

$$\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}=\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2},$$

$$\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}=\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}$$

$, \forall u_i \in U.$

Therefore, $CE_{NS}(H_1, H_2)=CE_{NS}(H_2, H_1)$.

Hence complete the proof. □

Definition 2.2 Weighted NS-cross entropy measure

We consider the weight $w_i(i = 1, 2, ..., n)$ for the element $u_i(i = 1, 2, .., n)$ with the conditions $w_i \in[0,1]$ and $\sum_{i=1}^{n} w_i =1$. Then the weighted cross entropy between SVNSs H_1 and H_2 is defined as:

$$CE_{NS}^{w}(H_1, H_2)$$

$$=\frac{1}{2}\left\langle \sum_{i=1}^{n} w_i \left\{ \left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}+\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right]\right\}\right\rangle \quad (2.2)$$

Theorem 2.2

Single-valued neutrosophic weighted NS-cross-entropy (defined in Equation (2.2)) satisfies the following properties:

i. $CE_{NS}^{w}(H_1, H_2)\geq 0, \forall u_i \in U$.

ii. $CE_{NS}^{w}(H_1, H_2)=0$, if and only if $T_{H_1}(u_i) =T_{H_2}(u_i)$ $I_{H_1}(u_i) =I_{H_2}(u_i)$, $F_{H_1}(u_i) =F_{H_2}(u_i), \forall u_i \in U.$

iii. $CE_{NS}^{w}(H_1, H_2)=CE_{NS}^{w}(H_1^c, H_2^c), \forall u_i \in U$.

iv. $CE_{NS}^{w}(H_1, H_2)= CE_{NS}^{w}(H_2, H_1), \forall u_i \in U$.

Proof. i. For all values of $u_i \in U$, $\left|T_{H_1}(u_i)\right| \geq 0$, $\left|T_{H_2}(u_i)\right| \geq 0$, $\left|T_{H_1}(u_i) - T_{H_2}(u_i)\right| \geq 0$, $\sqrt{1+\left|T_{H_1}(u_i)\right|^2} \geq 0$, $\sqrt{1+\left|T_{H_2}(u_i)\right|^2} \geq 0$, $\left|(1-T_{H_1}(u_i))\right| \geq 0$, $\left|(1-T_{H_2}(u_i))\right| \geq 0$, $\left|(1-T_{H_1}(u_i)) - (1-T_{H_2}(u_i))\right| \geq 0$, $\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2} \geq 0$, $\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2} \geq 0$, then

$$\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right] \geq 0$$

Similarly,

$$\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}+\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right] \geq 0 \text{ and}$$

$$\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right] \geq 0.$$

Since, $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, therefore, $CE_{NS}^{w}(H_1, H_2) \geq 0$.

Hence complete the proof.□

ii. Since,

$$\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right] = 0,$$

$\Leftrightarrow T_{H_1}(u_i) = T_{H_2}(u_i)$,

$$\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}+\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right] = 0,$$

$\Leftrightarrow I_{H_1}(u_i) = I_{H_2}(u_i)$,

$$\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right]=0,$$

$\Leftrightarrow F_{H_1}(u_i)=F_{H_2}(u_i)$ and $w_i\in[0,1]$, $\sum_{i=1}^{n} w_i=1$, $w_i\geq 0$. Therefore, $CE_{NS}^{w}(H_1, H_2)=0$

iff $T_{H_1}(u_i)=T_{H_2}(u_i)$, $I_{H_1}(u_i)=I_{H_2}(u_i)$, $F_{H_1}(u_i)=F_{H_2}(u_i)$, $\forall u_i\in U$.

Hence complete the proof.□

iii. Now, we have

$$CE_{NS}^{w}(H_1^c, H_2^c)$$

$$=\frac{1}{2}\left\{\sum_{i=1}^{n} w_i\left\langle\left[\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}+\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}+\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}+\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}\right]\right\rangle\right\}$$

$$=\frac{1}{2}\left\{\sum_{i=1}^{n} w_i\left\langle\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}+\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right]\right\rangle\right\}$$

$$=CE_{NS}^{w}\ (H_1, H_2)$$

Therefore, $CE_{NS}^{w}\ (H_1, H_2)= CE_{NS}^{w}\ (H_1^c, H_2^c)$.

Hence complete the proof.□

iv.Since $\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|=\left|T_{H_2}(u_i)-T_{H_1}(u_i)\right|, \left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|=\left|I_{H_2}(u_i)-I_{H_1}(u_i)\right|,$

$\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|=\left|F_{H_2}(u_i)-F_{H_1}(u_i)\right|,$

$\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|=\left|(1-T_{H_2}(u_i))-(1-T_{H_1}(u_i))\right|,$

$\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|=\left|(1-I_{H_2}(u_i))-(1-I_{H_1}(u_i))\right|,$

$\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|=\left|(1-F_{H_2}(u_i))-(1-F_{H_1}(u_i))\right|,$

we obtain,

$$\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}=\sqrt{1+\left|T_{H_2}(u_i)\right|^2}+\sqrt{1+\left|T_{H_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u_i)\right|^2}=\sqrt{1+\left|I_{H_2}(u_i)\right|^2}+\sqrt{1+\left|I_{H_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}=\sqrt{1+\left|F_{H_2}(u_i)\right|^2}+\sqrt{1+\left|F_{H_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}=\sqrt{1+\left|(-T_{H_2}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2},$$

$$\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}=\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2},$$

$$\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}=\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}$$

, $\forall u_i \in U$.

and $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1$.

Therefore, $CE_{NS}^{w}(H_1, H_2) = CE_{NS}^{w}(H_2, H_1)$.

Hence complete the proof. □

2.3 MAGDM Strategy using the proposed NS-cross entropy measure in SVNS environment

In this section, we develop a new MAGDM strategy using the proposed NS-cross entropy measure.

Description of the MAGDM Problem

Assume that $A = \{A_1, A_2, A_3, ..., A_m\}$ and $G = \{G_1, G_2, G_3, ..., G_n\}$ be the discrete set of alternatives and attributes respectively and $W = \{w_1, w_2, w_3, ..., w_n\}$ be the weight vector of attributes G_j(j = 1, 2, 3, …,*n*), where $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$. Assume that $E = \{E_1, E_2, E_3, ..., E_\rho\}$ be the set of decision-makers who are employed to evaluate the alternatives. The weight vector of the decision-makers $E_k\,(k = 1, 2, 3, ..., \rho)$ is denoted as $\lambda = \{\lambda_1, \lambda_2, \lambda_3, ..., \lambda_\rho\}$, where $\lambda_k \geq 0$ and $\sum_{k=1}^{\rho} \lambda_k = 1$.

Now, we describe the steps of the proposed MAGDM strategy (See Figure 2.1.).

Step 1. Formulate the decision matrices

For MAGDM with SVNSs information, the rating values of the alternatives $A_i\,(i=1,2,3,...,m)$ based on the attribute $G_j\,(j=1,2,3,...,n)$ provided by the *k*-th decision-maker can be expressed in terms of single valued neutrosophic numbers as $a_{ij}^{k} = < T_{ij}^{k}, I_{ij}^{k}, F_{ij}^{k} >$ (i = 1, 2, 3, …, m; j = 1, 2, 3, …, n; k = 1, 2, 3, …, ρ). We present these rating values of alternatives provided by the decision-makers in matrix form as follows:

$$M^k = \begin{pmatrix} & G_1 & G_2 & \ldots & .G_n \\ A_1 & a_{11}^k & a_{12}^k \cdots & & a_{1n}^k \\ A_2 & a_{21}^k & a_{22}^k & & a_{2n}^k \\ . & . & & \ldots & . \\ A_m & a_{m1}^k & a_{m2}^k \cdots & & a_{mn}^k \end{pmatrix} \tag{2.3}$$

Step 2. Formulate priori/ideal decision matrix

In the MAGDM, the a priori decision matrix has been used to select the best alternatives among the set of collected feasible alternatives. In the decision-making situation, we use the following decision matrix as a priori decision matrix.

$$P = \begin{pmatrix} & G_1 & G_2 & \ldots & .G_n \\ A_1 & a_{11}^* & a_{12}^* \cdots & & a_{1n}^* \\ A_2 & a_{21}^* & a_{22}^* & & a_{2n}^* \\ . & . & & \ldots & . \\ A_m & a_{m1}^* & a_{m2}^* \cdots & & a_{mn}^* \end{pmatrix} \tag{2.4}$$

where, $a_{ij}^* = < \max_i (T_{ij}^k), \min_i (I_{ij}^k), \min_i (F_{ij}^k) >$ for benefit attributes and

$a_{ij}^* = < \min_i (T_{ij}^k), \max_i (I_{ij}^k), \max_i (F_{ij}^k) >$ for cost attributes($i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$; $k = 1, 2, 3, \ldots, \rho$).

Step 3. Determinate the weight of decision-makers

To determine the decision-makers' weights, we introduce a model based on the NS-cross entropy measure. The collective NS-cross entropy measure between M^k and P (Ideal matrix) is defined as follows:

$$CE_{NS}^c(M^k, P) = \frac{1}{m} \sum_{i=1}^{m} CE_{NS}\left((M^k(A_i), P(A_i)\right) \tag{2.5}$$

where, $CE_{NS}\left((M^k(A_i), P(A_i)\right) = \sum_{j=1}^{n} CE_{NS}\,(M^k(A_i(G_j)), P(A_i(G_j)))$

Now, we introduce the following weight model of the decision-makers:

$$\lambda_K = \frac{\left(1 \div CE_{NS}^c(M^k, P)\right)}{\sum_{k=1}^{\rho} \left(1 \div CE_{NS}^c(M^k, P)\right)} \tag{2.6}$$

where, $0 \le \lambda_K \le 1$ and $\sum_{k=1}^{\rho} \lambda_K = 1$ for k = 1, 2, 3, …, ρ.

Step 4. Formulate the weighted aggregated decision matrix

For obtaining one group opinion, we aggregate all the individual decision matrices (M^k) to an aggregated decision matrix (M) using single valued neutrosophic weighted averaging (SVNWA) operator (Ye, 2014b)as follows:

$$a_{ij} = SVNSWA_{\lambda}(a_{ij}^1, a_{ij}^2, a_{ij}^3, \cdots, a_{ij}^{\rho}) =$$

$$(\lambda_1 a_{ij}^1 \oplus \lambda_2 a_{ij}^2 \oplus \lambda_3 a_{ij}^3 \oplus \ldots \oplus \lambda_{\rho} a_{ij}^{\rho}) = <1 - \prod_{k=1}^{\rho}(1 - T_{ij}^k)^{\lambda_k}, \prod_{k=1}^{\rho}(I_{ij}^k)^{\lambda_k}, \prod_{k=1}^{\rho}(F_{ij}^k)^{\lambda_k}> \quad (2.7)$$

Therefore, the aggregated decision matrix is defined as follows:

$$M = \begin{pmatrix} & G_1 & G_2 & \ldots & .G_n \\ A_1 & a_{11} & a_{12} \ldots & & a_{1n} \\ A_2 & a_{21} & a_{22} & & a_{2n} \\ . & . & \ldots & . & \\ A_m & a_{m1} & a_{m2} \ldots & & a_{mn} \end{pmatrix} \quad (2.8)$$

where, $a_{ij} = < T_{ij}, I_{ij}, F_{ij} >$, ($i$ = 1, 2, 3, …, m; j = 1, 2, 3, …, n; k = 1, 2, 3, …, ρ)

Step 5. Determinate the weight of attributes

To find the weight of attributes, we introduce a model based on the NS-cross entropy measure. The collective NS-cross entropy measure between M (Weighted aggregated decision matrix) and P (Ideal matrix) for each attribute is defined as:

$$CE_{NS}^j(M, P) = \frac{1}{m} \sum_{i=1}^{m} CE_{NS}\Big((M(A_i(G_j)), P(A_i(G_j))\Big) \quad (2.9)$$

where, i = 1, 2, 3, …, m; j= 1, 2, 3, …, n.

Now, we define a weight model for attributes as follows:

$$w_j = \frac{\Big(1 \div CE_{NS}^j(M, P)\Big)}{\sum_{J=1}^{n}\Big(1 \div CE_{NS}^j(M, P)\Big)} \quad (2.10)$$

where, $0 \le w_j \le 1$ and $\sum_{j=1}^{n} w_j = 1$ for j = 1, 2, 3, …, n.

Step 6. Calculate the weighted NS-cross entropy measure

Using Equation (2.2), we calculate the weighted cross entropy value between weighted aggregated matrix and priori matrix. The cross entropy values can be presented in matrix form as follows:

$$ {}^{NS}M_{CE}^{w} = \begin{pmatrix} CE_{NS}^{w}\ (A_1) \\ CE_{NS}^{w}\ (A_2) \\ .... \\ .... \\ CE_{NS}^{w}\ (A_m) \end{pmatrix} \tag{2.11} $$

Step 7. Rank the priority

Smaller value of the cross entropy reflects that an alternative is closer to the ideal alternative. Therefore, the preference priority order of all the alternatives can be determined according to the increasing order of the cross entropy values $CE_{NS}^{w}(A_i)$(i = 1, 2, 3, …,m).The smallest cross entropy value indicates the best alternative and the greatest cross entropy value indicates the worst alternative.

Step 8. Select the best alternative

From the preference rank order (from step 7), we select the best alternative.

2.4 Illustrative example

In this section, we solve an illustrative example of MAGDM adapted from (He & Liu, 2013) to reflect the feasibility, applicability and efficiency of the proposed strategy under the SVNS environment.

Now, we use the example (He & Liu, 2013)for cultivation and analysis. A venture capital firm intends to make an evaluation and selection of the best enterprise from the five enterprises (alternatives) with the investment potential

The alternatives are:

(1) Automobile company (A_1)

(2) Military manufacturing enterprise (A_2)

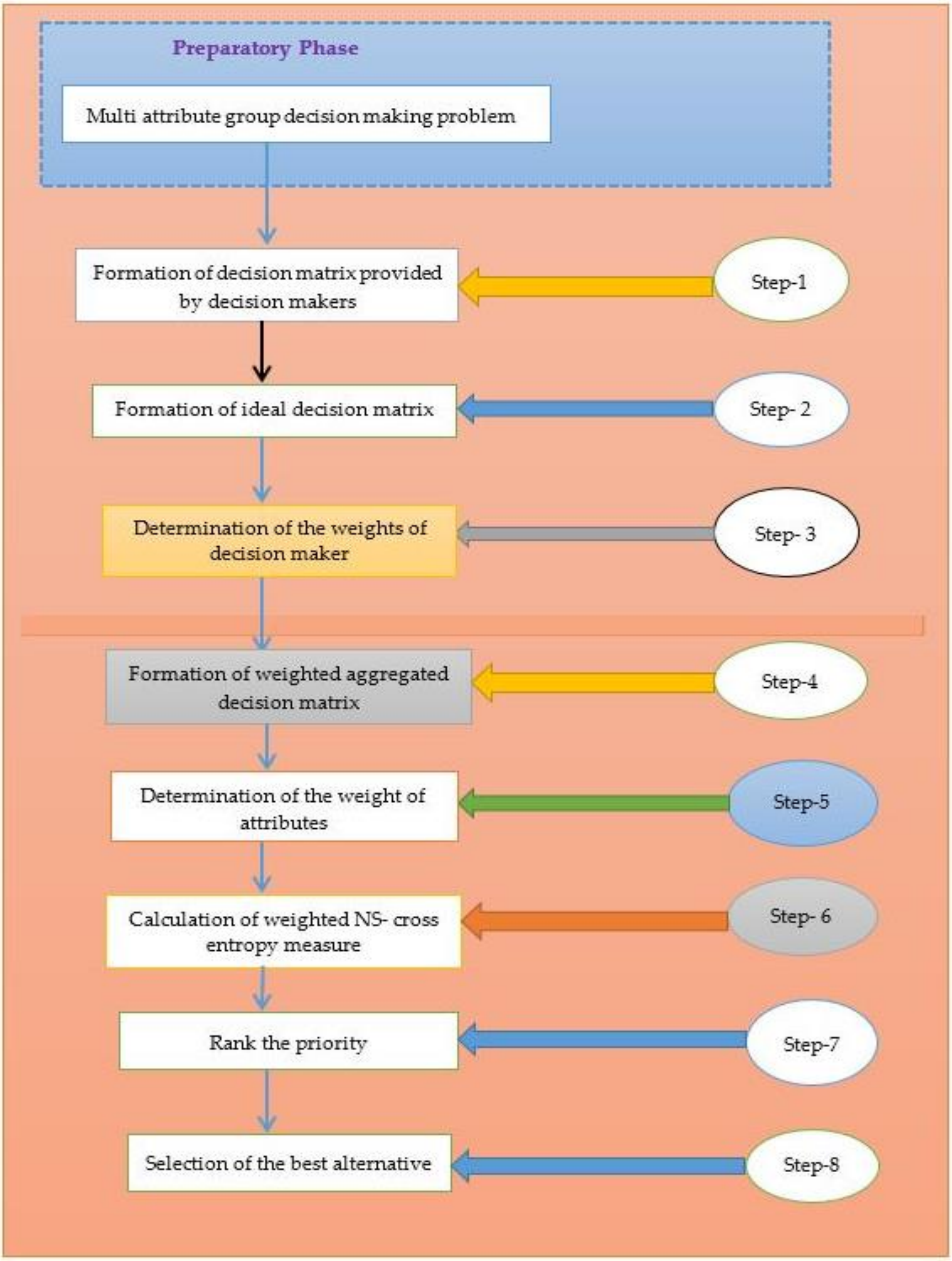

Figure 2.1 Decision-making procedure of the proposed MAGDM strategy

(3) TV media company (A_3)

(4) Food enterprises (A_4)

(5) Computer software company (A_5)

The four attributes are:

(1) Social and political factor (G_1)

(2) The environmental factor (G_2)

(3) Investment risk factor (G_3)

(4) The enterprise growth factor (G_4).

The investment firm makes a panel of three decision-makers.

Step 1. Formulate the decision matrices

We represent the rating values of alternatives A_i (i = 1, 2, 3, 4, 5) with respect to the attributes G_j (j = 1, 2, 3, 4) provided by the decision-makers E_k (k = 1, 2, 3) in matrix form as follows:

Decision matrix for decision-maker E_1

$$M^1 = \begin{array}{cc} & \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} & \begin{pmatrix} (0.9,0.5,0.4) & (0.7,0.4,0.4) & (0.7,0.3,0.4) & (0.5,0.4,0.9) \\ (0.7,0.2,0.3) & (0.8,0.4,0.3) & (0.9,0.6,0.5) & (0.9,0.1,0.3) \\ (0.8,0.4,0.4) & (0.7,0.4,0.2) & (0.9,0.7,0.6) & (0.7,0.3,0.3) \\ (0.5,0.8,0.7) & (0.6,0.3,0.4) & (0.7,0.2,0.5) & (0.5,0.4,0.7) \\ (0.8,0.4,0.3) & (0.5,0.4,0.5) & (0.6,0.4,0.4) & (0.9,0.7,0.5) \end{pmatrix} \end{array} \tag{2.12}$$

Decision matrix for decision-maker E_2

$$M^2 = \begin{array}{cc} & \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} & \begin{pmatrix} (0.7,0.2,0.3) & (0.5,0.4,0.5) & (0.9,0.4,0.5) & (0.6,0.5,0.3) \\ (0.7,0.4,0.4) & (0.7,0.3,0.4) & (0.7,0.3,0.4) & (0.6,0.4,0.3) \\ (0.6,0.4,0.4) & (0.5,0.3,0.5) & (0.9,0.5,0.4) & (0.6,0.5,0.6) \\ (0.7,0.5,0.3) & (0.6,0.3,0.6) & (0.7,0.4,0.4) & (0.8,0.5,0.4) \\ (0.9,0.4,0.3) & (0.6,0.4,0.5) & (0.8,0.5,0.6) & (0.5,0.4,0.5) \end{pmatrix} \end{array} \tag{2.13}$$

Decision matrix for decision-maker E_3

$$M^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{pmatrix} G_1 & G_2 & G_3 & G_4 \\ (0.7,0.2,0.5) & (0.6,0.4,0.4) & (0.7,0.4,0.5) & (0.9,0.4,0.3) \\ (0.6,0.5,0.5) & (0.9,0.3,0.4) & (0.7,0.4,0.3) & (0.8,0.4,0.5) \\ (0.8,0.3,0.5) & (0.9,0.3,0.4) & (0.8,0.3,0.4) & (0.7,0.3,0.4) \\ (0.9,0.3,0.4) & (0.6,0.3,0.4) & (0.5,0.2,0.4) & (0.7,0.3,0.5) \\ (0.8,0.3,0.3) & (0.6,0.4,0.3) & (0.6,0.3,0.4) & (0.7,0.3,0.5) \end{pmatrix} \quad (2.14)$$

Step 2. Formulate priori/ideal decision matrix

A priori/ideal decision matrix

$$P = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{pmatrix} G_1 & G_2 & G_3 & G_4 \\ (0.9,0.2,0.3) & (0.7,0.4,0.4) & (0.9,0.3,0.4) & (0.9,0.4,0.3) \\ (0.7,0.2,0.3) & (0.9,0.3,0.3) & (0.9,0.3,0.3) & (0.9,0.1,0.3) \\ (0.8,0.3,0.4) & (0.9,0.3,0.2) & (0.9,0.3,0.4) & (0.7,0.3,0.3) \\ (0.9,0.3,0.3) & (0.6,0.3,0.4) & (0.7,0.2,0.4) & (0.7,0.3,0.4) \\ (0.9,0.3,0.3) & (0.6,0.4,0.3) & (0.8,0.3,0.4) & (0.9,0.3,0.5) \end{pmatrix} \quad (2.15)$$

Step 3. Determine the weight of decision-makers

Using Equations (2.5) and (2.6), we determine the weights of the three decision-makers as follows:

$$\lambda_1 = \frac{(1 \div 0.9)}{3.37} \approx 0.33, \; \lambda_2 = \frac{(1 \div 1.2)}{3.37} \approx 0.25, \; \lambda_1 = \frac{(1 \div .07)}{3.37} \approx 0.42.$$

Step 4. Formulate the weighted aggregated decision matrix

Using Equation (2.7), the weighted aggregated decision matrix is presented as follows:

Weighted Aggregated decision matrix

$$M = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{pmatrix} G_1 & G_2 & G_3 & G_4 \\ (0.8,0.3,0.4) & (0.6,0.4,0.4) & (0.8,0.4,0.4) & (0.7,0.4,0.5) \\ (0.7,0.3,0.4) & (0.8,0.3,0.4) & (0.8,0.4,0.4) & (0.8,0.2,0.3) \\ (0.8,0.4,0.4) & (0.8,0.3,0.3) & (0.9,0.5,0.5) & (0.7,0.3,0.4) \\ (0.7,0.5,0.5) & (0.6,0.3,0.4) & (0.6,0.2,0.4) & (0.7,0.4,0.5) \\ (0.8,0.4,0.4) & (0.6,0.4,0.4) & (0.7,0.4,0.4) & (0.8,0.5,0.5) \end{pmatrix} \quad (2.16)$$

Step 5. Determinate the weight of the attributes

Using Equations (2.9) and (2.10), we determine the weight of the attributes as follows:

$$w_1 = \frac{(1 \div 0.26)}{25} \approx 0.16, w_2 = \frac{(1 \div 0.11)}{25} \approx 0.37, w_3 = \frac{(1 \div 0.20)}{25} \approx 0.20, w_4 = \frac{(1 \div 0.15)}{25} \approx 0.27.$$

Step 6. Calculate the weighted SVNS cross entropy matrix

Using Equation (2.2) and weight of attributes, we calculate the weighted NS-cross entropy values between ideal matrix and weighted aggregated decision matrix.

$$^{NS}M_{CE}^{w} = \begin{pmatrix} 0.195 \\ 0.198 \\ 0.168 \\ 0.151 \\ 0.184 \end{pmatrix} \tag{2.17}$$

Step 7. Rank the priority

The cross entropy values of alternatives are arranged in increasing order as follows:

0.151 < 0.168 < 0.184 < 0.195 < 0.198.

Alternatives are then preference ranked as follows: $A_4 > A_3 > A_5 > A_1 > A_2$.

Step 8. Select the best alternative

From step 7, we identify that A_4 is the best alternative. Hence, Food enterprises (A_4) is the best alternative for investment.

In Figure 2.2, we draw a bar diagram to represent the cross entropy values of alternatives which shows that A_4 is the best alternative according our proposed strategy.

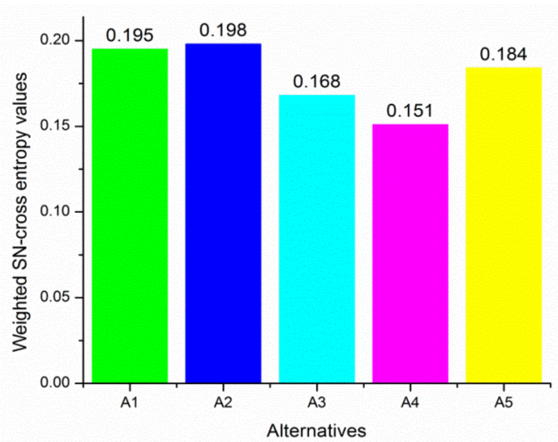

Figure 2.2 Bar diagram of alternatives versus weighted NS-cross entropy values of alternatives.

In Figure 2.3, we represent the relation between cross entropy values and acceptance value of alternatives. The range of acceptance level for five alternatives is taken by five points. The high acceptance level of alternatives indicates the best alternative for acceptance and low acceptance level of alternative indicates the worst acceptance alternative.

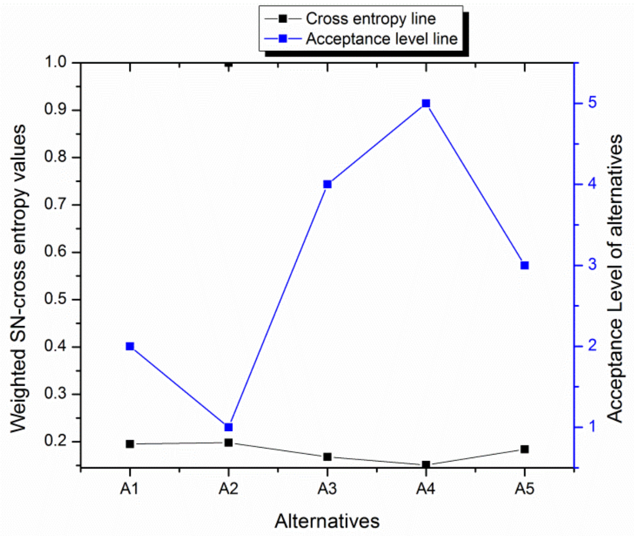

Figure 2.3 Relation between weighted NS-cross entropy values and acceptance level line of alternatives.

We see from Figure 2.3 that alternative A_4 has the smallest cross entropy value and the highest acceptance level. Therefore A_4 is the best alternative for acceptance. Figure 2.3 indicates that alternative A_2 has the highest cross entropy value and the lowest acceptance value that means A_2 is the worst alternative. Finally, we conclude that the relation between cross entropy values and acceptance value of alternatives is opposite in nature.

2.5 comparative study and discussion

In literature only two MADM strategies based on cross entropy (Ye, 2013, 2015b) have been proposed in SVNS environment. MADGM strategy based on cross entropy is yet to appear. So the proposed MAGDM is novel and non-comparable with the existing cross entropy under SVNS environment.

i. The MADM strategies (Ye, 2013, 2015b) are not applicable for MAGDM problems. The proposed MAGDM strategy is free from such drawbacks.

ii. Ye (2013) proposed cross entropy that does not satisfy straightforwardlythe symmetrical property and is undefined for some situations but the proposed strategy satisfies symmetric property and is free from undefined phenomenon.

iii. The strategies (Ye, 2013, 2015b) cannot deal with the unknown weight of the attributes whereas the proposed MADGM strategy can deal with the unknown weight of the attributes

iv. The strategies (Ye, 2013, 2015b) are not suitable for dealing with the unknown weight of decision-makers, whereas the essence of the proposed NS-cross entropy-based MAGDM is that it is capable of dealing with the unknown weight of the decision-makers.

2.6 Conclusion

In this chapter, we have defined a novel cross entropy measure in SVNS environment. The proposed cross entropy measure in SVNS environment is free from the drawbacks of asymmetrical behaviour and undefined phenomena. It is capable of dealing with the unknown weight of attributes and the unknown weight of decision-makers. We have proved the basic properties of the NS-cross entropy measure. We also defined weighted NS-cross entropy measure and proved its basic properties. Based on the weighted NS-cross entropy measure, we have developed a novel MAGDM strategy to solve neutrosophic MAGDM problems. We have at first proposed a novel MAGDM strategy based on NS-cross entropy measure with technique to determine the unknown weight of attributes and the unknown weight of decision-makers.

Other existing cross entropy measures (Ye, 2013, 2015b) can deal only with the MADM problem with single decision-maker and known weight of the attributes. So in general, our proposed NS-cross entropy-based MAGDM strategy is not comparable with the existing cross-entropy-based MADM strategies (Ye, 2013, 2015b) under the single-valued neutrosophic environment.

Finally, we solve an MAGDM problem to show the feasibility, applicability and efficiency of the proposed MAGDM strategy.

The proposed NS-cross entropy-based MAGDM can be applied in teacher selection, pattern recognition, weaver selection, medical treatment selection options, and other practical problems. In future study, the proposed NS-cross entropy-based MAGDM strategy can be also extended to neutrosophic crisp set environment (Salama & Smarandache, 2015; 2016) and the interval neutrosophic set environment (Wang et al., 2005).

Chapter 3

IN-cross entropy based MAGDM strategy in interval neutrosophic set environment

3.1 Introduction

Wang et al. (2005) introduced interval neutrosophic set (INS) considering truth membership, indeterminate membership and falsity membership as interval number in [0, 1]. Broumi and Smarandache (2013a) defined correlation coefficient of INSs and proved its basic properties. Zhang et al.(2015) defined correlation coefficient for interval neutrosophic numbers (INNs) and applied it MAGDM problems. Zhang et al. (2016) presented an outranking strategy for MADM in INS environment. Recently, Huang et al. (2017) employed VIKOR strategy to solve MAGDM problem with INN. Ye (2014a) defined similarity measure in INS environmentto solve MADM problem. Pramanik and Mondal (2015d) extended the single valued neutrosophic grey relational analysis strategy to interval neutrosophic environment to deal with MADM problem. Aiwu et al. (2015) proposed an MADM strategy based on generalized weighted aggregation operator with INSs. Zhang et al. (2014) proposed an MADM strategy based on two interval neutrosophic number aggregation operators.Ye (2015b) defined an improved cross entropy measure for INSsand employed it to solve MADM problem. Tian et al. (2015) proposed a cross entropy measure with INSs and TOPSIS for solving MADM problems.

The content of this chapter is based on the paper published in**"Neutrosophic Sets and Systems**" 18, 43-57, 2017.

Sahin (2017) defined two cross entropy measures with INSs based on fuzzy cross entropy measure and single valued neutrosophic cross entropy measure for solving MADM problem. Sahin(2017), Ye (2015b), Tian et al. (2015) proposed cross entropy measures under the interval-valued neutrosophic set environment, which are suitable only for single decision maker. So, multiple decision makers cannot participate in their strategies in (Tian et al., 2015; Ye, 2015b; Sahin, 2017).

In this chapter we define IN-cross entropy measure in INS environment and prove its basic properties. The proposed IN-cross entropy measure is straightforward symmetric. We define a new weighted IN-cross entropy measure in the INS environment and prove its basic properties. It is straightforward symmetric. In this chapter, we develop a new MAGDM strategy based on weighted IN cross entropy measure to solve MAGDM problems. Also, we illustrate the proposed strategy by solving a numerical MAGDM problem.

The chapter unfolds as follows: Section 3.2 presents the definition of proposed IN-cross entropy measure, weighted IN-cross entropy measure and their basic properties. Section 3.3 devotes to develop a MAGDM strategy with proposed weighted IN-cross entropy measure. Section 3.4 solves an MAGDM problem to show the feasibility, validity and efficiency of the proposed strategy. Section 3.5 presents conclusion and future scope of research.

3.2 IN-cross-entropy measure

In this section, we define a new interval neutrosophic cross-entropy measure for measuring the deviation of interval neutrosophic variables from an a priori one.

Definition 3.1 IN-cross-entropy measure

Assume that $J_1$and J_2 be any two INSs in U = $\{u_1, u_2, u_3, ..., u_n\}$. Then, the interval neutrosophic cross-entropy measure of $J_1$and J_2 is denoted by CE_{IN} (J_1, J_2) and defined as follows:

$$CE_{IN}(J_1, J_2)$$

$$= \frac{1}{4}\left\{\sum_{i=1}^{n}\left[\frac{2\left|T_{J_1}^{-}(u_i)-T_{J_2}^{-}(u_i)\right|}{\sqrt{1+\left|T_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-T_{J_1}^{-}(u_i))-(1-T_{J_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^{-}(u_i))\right|^2}}\right]+\right.$$

$$\left[\frac{2\left|T_{J_1}^{+}(u_i)-T_{J_2}^{+}(u_i)\right|}{\sqrt{1+\left|T_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-T_{J_1}^{+}(u_i))-(1-T_{J_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^{+}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{J_1}^{-}(u_i)-I_{J_2}^{-}(u_i)\right|}{\sqrt{1+\left|I_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-I_{J_1}^{-}(u_i))-(1-I_{J_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^{-}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{J_1}^{+}(u_i)-I_{J_2}^{+}(u_i)\right|}{\sqrt{1+\left|I_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-I_{J_1}^{+}(u_i))-(1-I_{J_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^{+}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F_{J_1}^{-}(u_i)-F_{J_2}^{-}(u_i)\right|}{\sqrt{1+\left|F_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-F_{J_1}^{-}(u_i))-(1-F_{J_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_2}^{-}(u_i))\right|^2}}\right]+$$

$$\left.\left[\frac{2\left|F_{J_1}^{+}(u_i)-F_{J_2}^{+}(u_i)\right|}{\sqrt{1+\left|F_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-F_{J_1}^{+}(u_i))-(1-F_{J_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_2}^{+}(u_i))\right|^2}}\right]\right\} \tag{3.1}$$

Theorem 3.1

Interval-valued neutrosophic cross entropy $CE_{IN}(J_1, J_2)$ for any two INSs J_1, J_2 of U, satisfies the following properties:

i. $CE_{IN}(J_1, J_2) \geq 0,\ \forall u_i \in U.$

ii. $CE_{IN}(J_1, J_2) = 0$ if and only if $T_{J_1}^{-}(u_i) = T_{J_2}^{-}(u_i)$, $T_{J_1}^{+}(u_i) = T_{J_2}^{+}(u_i)$, $I_{J_1}^{-}(u_i) = I_{J_2}^{-}(u_i)$, $I_{J_1}^{+}(u_i) = I_{J_2}^{+}(u_i)$, $F_{J_1}^{-}(u_i) = F_{J_2}^{-}(u_i)$, $F_{J_1}^{+}(u_i) = F_{J_2}^{+}(u_i)$ for all $\forall u_i \in U$.

iii. $CE_{IN}(J_1, J_2) = CE_{IN}(J_1^c, J_2^c)$, $\forall u_i \in U$.

iv. $CE_{IN}(J_1, J_2) = CE_{IN}(J_2, J_1)$, $\forall u_i \in U$.

Proof: i.

For all values of $u_i \in U$, $\left|T_{J_1}^-(u_i)\right| \geq 0$, $\left|T_{J_2}^-(u_i)\right| \geq 0$, $\left|T_{J_1}^-(u_i) - T_{J_2}^-(u_i)\right| \geq 0$,

$\sqrt{1+\left|T_{J_1}^-(u_i)\right|^2} \geq 0$, $\sqrt{1+\left|T_{J_2}^-(u_i)\right|^2} \geq 0$,

$\left|(1-T_{J_1}^-(u_i))\right| \geq 0$, $\left|(1-T_{J_2}^-(u_i))\right| \geq 0$, $\left|(1-T_{J_1}^-(u_i))-(1-T_{J_2}^-(u_i))\right| \geq 0$,

$\sqrt{1+\left|(1-T_{J_1}^-(u_i))\right|^2} \geq 0$, $\sqrt{1+\left|(1-T_{J_2}^-(u_i))\right|^2} \geq 0$

$$\Rightarrow \left[\frac{2\left|T_{J_1}^-(u_i)-T_{J_2}^-(u_i)\right|}{\sqrt{1+\left|T_{J_1}^-(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^-(u_i)\right|^2}} + \frac{2\left|(1-T_{J_1}^-(u_i))-(1-T_{J_2}^-(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^-(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^-(u_i))\right|^2}}\right] \geq 0$$

and $\left|T_{J_1}^+(u_i)\right| \geq 0$, $\left|T_{J_2}^+(u_i)\right| \geq 0$, $\left|T_{J_1}^+(u_i) - T_{J_2}^+(u_i)\right| \geq 0$, $\sqrt{1+\left|T_{J_1}^+(u_i)\right|^2} \geq 0$,

$\sqrt{1+\left|T_{J_2}^+(u_i)\right|^2} \geq 0$,

$\left|(1-T_{J_1}^+(u_i))\right| \geq 0$, $\left|(1-T_{J_2}^+(u_i))\right| \geq 0$, $\left|(1-T_{J_1}^+(u_i))-(1-T_{J_2}^+(u_i))\right| \geq 0$,

$\sqrt{1+\left|(1-T_{J_1}^+(u_i))\right|^2} \geq 0$, $\sqrt{1+\left|(1-T_{J_2}^+(u_i))\right|^2} \geq 0$

$$\Rightarrow \left[\frac{2\left|T_{J_1}^+(u_i)-T_{J_2}^+(u_i)\right|}{\sqrt{1+\left|T_{J_1}^+(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^+(u_i)\right|^2}} + \frac{2\left|(1-T_{J_1}^+(u_i))-(1-T_{J_2}^+(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^+(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^+(u_i))\right|^2}}\right] \geq 0$$

Similarly, we have

$$\left[\frac{2\left|I_{J_1}^-(u_i)-I_{J_2}^-(u_i)\right|}{\sqrt{1+\left|I_{J_1}^-(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^-(u_i)\right|^2}} + \frac{2\left|(1-I_{J_1}^-(u_i))-(1-I_{J_2}^-(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^-(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^-(u_i))\right|^2}}\right] \geq 0,$$

$$\left[\frac{2\left|I_{J_1}^+(u_i)-I_{J_2}^+(u_i)\right|}{\sqrt{1+\left|I_{J_1}^+(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^+(u_i)\right|^2}} + \frac{2\left|(1-I_{J_1}^+(u_i))-(1-I_{J_2}^+(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^+(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^+(u_i))\right|^2}}\right] \geq 0,$$

$$\left[\frac{2\left|F^-_{J_1}(u_i)-F^-_{J_2}(u_i)\right|}{\sqrt{1+\left|F^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^-_{J_1}(u_i))-(1-F^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^-_{J_2}(u_i))\right|^2}}\right]\geq 0 \text{ and}$$

$$\left[\frac{2\left|F^+_{J_1}(u_i)-F^+_{J_2}(u_i)\right|}{\sqrt{1+\left|F^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^+_{J_1}(u_i))-(1-F^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^+_{J_2}(u_i))\right|^2}}\right]\geq 0$$

Hence, we conclude that $CE_{IN}(J_1, J_2)\geq 0$.□

ii.

$$\left[\frac{2\left|T^-_{J_1}(u_i)-T^-_{J_2}(u_i)\right|}{\sqrt{1+\left|T^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^-_{J_1}(u_i))-(1-T^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{J_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow T^-_{J_1}(u_i)=T^-_{J_2}(u_i)$$

$$\left[\frac{2\left|T^+_{J_1}(u_i)-T^+_{J_2}(u_i)\right|}{\sqrt{1+\left|T^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^+_{J_1}(u_i))-(1-T^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{J_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow T^+_{J_1}(u_i)=T^+_{J_2}(u_i)$$

$$\left[\frac{2\left|I^-_{J_1}(u_i)-I^-_{J_2}(u_i)\right|}{\sqrt{1+\left|I^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^-_{J_1}(u_i))-(1-I^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^-_{J_2}(u_i))\right|^2}}\right]=0,$$

$$\Leftrightarrow I^-_{J_1}(u_i)=I^-_{J_2}(u_i)$$

$$\left[\frac{2\left|I^+_{J_1}(u_i)-I^+_{J_2}(u_i)\right|}{\sqrt{1+\left|I^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^+_{J_1}(u_i))-(1-I^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^+_{J_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow I^+_{J_1}(u_i)=I^+_{J_2}(u_i)$$

$$\left[\frac{2\left|F^-_{J_1}(u_i)-F^-_{J_2}(u_i)\right|}{\sqrt{1+\left|F^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^-_{J_1}(u_i))-(1-F^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^-_{J_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow F^-_{J_1}(u_i)=F^-_{J_2}(u_i)$$

$$\left[\frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow F^{+}_{J_1}(u_i)=F^{+}_{J_2}(u_i)$$

So, $CE_{IN}(J_1, J_2)=0$ if and only if $T^{-}_{J_1}(u_i)=T^{-}_{J_2}(u_i)$, $T^{+}_{J_1}(u_i)=T^{+}_{J_2}(u_i)$, $I^{-}_{J_1}(u_i)=I^{-}_{J_2}(u_i)$,

$I^{+}_{J_1}(u_i)=I^{+}_{J_2}(u_i)$, $F^{-}_{J_1}(u_i)=F^{-}_{J_2}(u_i)$, $F^{+}_{J_1}(u_i)=F^{+}_{J_2}(u_i)$ for all $\forall u_i \in U$.

Hence complete the proof.□.

iii. We have,

$$CE_{IN}\ (J^c_1, J^c_2)$$

$$=\frac{1}{4}\left\{\sum_{i=1}^{n}\left[\frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}}+\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}}\right]+\right.$$

$$\left[\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}+\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}+\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}+\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}+\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}}\right]+$$

$$\left.\left[\frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}}+\frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}}\right]\right\}$$

$$= \frac{1}{4}\left\{\sum_{i=1}^{n}\left[\left[\frac{2\left|T_{J_1}^{-}(u_i)-T_{J_2}^{-}(u_i)\right|}{\sqrt{1+\left|T_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-T_{J_1}^{-}(u_i))-(1-T_{J_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^{-}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T_{J_1}^{+}(u_i)-T_{J_2}^{+}(u_i)\right|}{\sqrt{1+\left|T_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-T_{J_1}^{+}(u_i))-(1-T_{J_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^{+}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{J_1}^{-}(u_i)-I_{J_2}^{-}(u_i)\right|}{\sqrt{1+\left|I_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-I_{J_1}^{-}(u_i))-(1-I_{J_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^{-}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{J_1}^{+}(u_i)-I_{J_2}^{+}(u_i)\right|}{\sqrt{1+\left|I_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-I_{J_1}^{+}(u_i))-(1-I_{J_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^{+}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F_{J_1}^{-}(u_i)-F_{J_2}^{-}(u_i)\right|}{\sqrt{1+\left|F_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-F_{J_1}^{-}(u_i))-(1-F_{J_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_2}^{-}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F_{J_1}^{+}(u_i)-F_{J_2}^{+}(u_i)\right|}{\sqrt{1+\left|F_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-F_{J_1}^{+}(u_i))-(1-F_{J_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_2}^{+}(u_i))\right|^2}}\right]\right]\right\}=$$

CE_{IN} (J_1, J_2).

Hence complete the proof.□

iv. $CE_{IN}(J_1, J_2) =$

$$\frac{1}{4}\left\{\sum_{i=1}^{n}\left[\left[\frac{2\left|T_{J_1}^{-}(u_i)-T_{J_2}^{-}(u_i)\right|}{\sqrt{1+\left|T_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-T_{J_1}^{-}(u_i))-(1-T_{J_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^{-}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T_{J_1}^{+}(u_i)-T_{J_2}^{+}(u_i)\right|}{\sqrt{1+\left|T_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-T_{J_1}^{+}(u_i))-(1-T_{J_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^{+}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{J_1}^{-}(u_i)-I_{J_2}^{-}(u_i)\right|}{\sqrt{1+\left|I_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-I_{J_1}^{-}(u_i))-(1-I_{J_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^{-}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{J_1}^{+}(u_i)-I_{J_2}^{+}(u_i)\right|}{\sqrt{1+\left|I_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-I_{J_1}^{+}(u_i))-(1-I_{J_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^{+}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F_{J_1}^{-}(u_i)-F_{J_2}^{-}(u_i)\right|}{\sqrt{1+\left|F_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-F_{J_1}^{-}(u_i))-(1-F_{J_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_2}^{-}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F_{J_1}^{+}(u_i)-F_{J_2}^{+}(u_i)\right|}{\sqrt{1+\left|F_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-F_{J_1}^{+}(u_i))-(1-F_{J_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_2}^{+}(u_i))\right|^2}}\right]\right]\right\}$$

$$=\frac{1}{4}\left\{\sum_{i=1}^{n}\left[\left[\frac{2\left|T_{J_2}^{-}(u_i)-T_{J_1}^{-}(u_i)\right|}{\sqrt{1+\left|T_{J_2}^{-}(u_i)\right|^2}+\sqrt{1+\left|T_{J_1}^{-}(u_i)\right|^2}}+\frac{2\left|(1-T_{J_2}^{-}(u_i))-(1-T_{J_1}^{-}(u_i))\right|}{\sqrt{1+\left|(1-T_{J_2}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_1}^{-}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T_{J_2}^{+}(u_i)-T_{J_1}^{+}(u_i)\right|}{\sqrt{1+\left|T_{J_2}^{+}(u_i)\right|^2}+\sqrt{1+\left|T_{J_1}^{+}(u_i)\right|^2}}+\frac{2\left|(1-T_{J_2}^{+}(u_i))-(1-T_{J_1}^{+}(u_i))\right|}{\sqrt{1+\left|(1-T_{J_2}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_1}^{+}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{J_2}^{-}(u_i)-I_{J_1}^{-}(u_i)\right|}{\sqrt{1+\left|I_{J_2}^{-}(u_i)\right|^2}+\sqrt{1+\left|I_{J_1}^{-}(u_i)\right|^2}}+\frac{2\left|(1-I_{J_2}^{-}(u_i))-(1-I_{J_1}^{-}(u_i))\right|}{\sqrt{1+\left|(1-I_{J_2}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_1}^{-}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{J_2}^{+}(u_i)-I_{J_1}^{+}(u_i)\right|}{\sqrt{1+\left|I_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-I_{J_2}^{+}(u_i))-(1-I_{J_1}^{+}(u_i))\right|}{\sqrt{1+\left|(1-I_{J_2}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_1}^{+}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F_{J_2}^{-}(u_i)-F_{J_1}^{-}(u_i)\right|}{\sqrt{1+\left|F_{J_2}^{-}(u_i)\right|^2}+\sqrt{1+\left|F_{J_1}^{-}(u_i)\right|^2}}+\frac{2\left|(1-F_{J_2}^{-}(u_i))-(1-F_{J_1}^{-}(u_i))\right|}{\sqrt{1+\left|(1-F_{J_2}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_1}^{-}(u_i))\right|^2}}\right]+$$

$$\left.\left[\frac{2\left|F_{J_2}^{+}(u_i)-F_{J_1}^{+}(u_i)\right|}{\sqrt{1+\left|F_{J_2}^{+}(u_i)\right|^2}+\sqrt{1+\left|F_{J_1}^{+}(u_i)\right|^2}}+\frac{2\left|(1-F_{J_2}^{+}(u_i))-(1-F_{J_1}^{+}(u_i))\right|}{\sqrt{1+\left|(1-F_{J_2}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_1}^{+}(u_i))\right|^2}}\right]\right\}\right\rangle$$

$$=CE_{IN}(J_2, J_1).$$

Hence complete the proof.□

Definition 3.2 Weighted IN-cross-entropy measure

We consider the weight w_i (i = 1, 2, 3,…, n) of u_i (i = 1, 2, 3, …, n) with

$w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$.

Then the weighted cross entropy measure between J_1 and J_2 is defined as follows:

$CE_{IN}^{w}(J_1, J_2) =$

$$\frac{1}{4}\left\langle \sum_{i=1}^{n} w_i \left\{\left[\frac{2\left|T_{J_1}^{-}(u_i)-T_{J_2}^{-}(u_i)\right|}{\sqrt{1+\left|T_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-T_{J_1}^{-}(u_i))-(1-T_{J_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^{-}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T_{J_1}^{+}(u_i)-T_{J_2}^{+}(u_i)\right|}{\sqrt{1+\left|T_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-T_{J_1}^{+}(u_i))-(1-T_{J_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^{+}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{J_1}^{-}(u_i)-I_{J_2}^{-}(u_i)\right|}{\sqrt{1+\left|I_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-I_{J_1}^{-}(u_i))-(1-I_{J_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^{-}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{J_1}^{+}(u_i)-I_{J_2}^{+}(u_i)\right|}{\sqrt{1+\left|I_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-I_{J_1}^{+}(u_i))-(1-I_{J_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^{+}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F_{J_1}^{-}(u_i)-F_{J_2}^{-}(u_i)\right|}{\sqrt{1+\left|F_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-F_{J_1}^{-}(u_i))-(1-F_{J_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_2}^{-}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F_{J_1}^{+}(u_i)-F_{J_2}^{+}(u_i)\right|}{\sqrt{1+\left|F_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-F_{J_1}^{+}(u_i))-(1-F_{J_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_2}^{+}(u_i))\right|^2}}\right]\Bigg\}\Bigg\rangle \quad (3.2)$$

Theorem 3.2

Interval neutrosophic weighted cross-entropy measure $CE_{IN}^{w}(J_1, J_2)$ satisfies the following properties:

i. $CE^{w}_{IN}(J_1, J_2) \geq 0$, $\forall u_i \in U$.

ii. $CE^{w}_{IN}(J_1, J_2) = 0$, if and only if $T_{J_1}^{-}(u_i) = T_{J_2}^{-}(u_i)$, $T_{J_1}^{+}(u_i) = T_{J_2}^{+}(u_i)$, $I_{J_1}^{-}(u_i) = I_{J_2}^{-}(u_i)$, $I_{J_1}^{+}(u_i) = I_{J_2}^{+}(u_i)$, $F_{J_1}^{-}(u_i) = F_{J_2}^{-}(u_i)$, $F_{J_1}^{+}(u_i) = F_{J_2}^{+}(u_i)$ for all $\forall u_i \in U$.

iii. $CE^{w}_{IN}(J_1, J_2) = CE^{w}_{IN}(J_1^{c}, J_2^{c})$, $\forall u_i \in U$.

iv. $CE^{w}_{IN}(J_1, J_2) = CE^{w}_{IN}(J_2, J_1)$, $\forall u_i \in U$.

Proof:

i. For all values of $u_i \in U$, $\left|T_{J_1}^{-}(u_i)\right| \geq 0$, $\left|T_{J_2}^{-}(u_i)\right| \geq 0$, $\left|T_{J_1}^{-}(u_i)-T_{J_2}^{-}(u_i)\right| \geq 0$,

$\sqrt{1+\left|T_{J_1}^{-}(u_i)\right|^2} \geq 0$,

$\sqrt{1+\left|T_{J_2}^{-}(u_i)\right|^2} \geq 0$, $\left|(1-T_{J_1}^{-}(u_i))\right| \geq 0$, $\left|(1-T_{J_2}^{-}(u_i))\right| \geq 0$, $\left|(1-T_{J_1}^{-}(u_i))-(1-T_{J_2}^{-}(u_i))\right| \geq 0$,

$\sqrt{1+\left|(1-T_{J_1}^{-}(u_i))\right|^2} \geq 0$, $\sqrt{1+\left|(1-T_{J_2}^{-}(u_i))\right|^2} \geq 0$

$$\Rightarrow \left[\frac{2\left|T_{J_1}^{-}(u_i)-T_{J_2}^{-}(u_i)\right|}{\sqrt{1+\left|T_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-T_{J_1}^{-}(u_i))-(1-T_{J_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^{-}(u_i))\right|^2}}\right] \geq 0$$

and $\left|T^{+}_{J_1}(u_i)\right| \geq 0$, $\left|T^{+}_{J_2}(u_i)\right| \geq 0$, $\left|T^{+}_{J_1}(u_i) - T^{+}_{J_2}(u_i)\right| \geq 0$, $\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2} \geq 0$, $\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2} \geq 0$,

$\left|(1-T^{+}_{J_1}(u_i))\right| \geq 0$, $\left|(1-T^{+}_{J_2}(u_i))\right| \geq 0$, $\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right| \geq 0$,

$\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2} \geq 0$, $\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2} \geq 0$

$$\Rightarrow \left[\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}\right] \geq 0.$$

Similarly, we have

$$\left[\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}\right] \geq 0 \quad ,$$

$$\left[\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}\right] \geq 0,$$

$$\left[\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}\right] \geq 0 \text{ and}$$

$$\left[\frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}}\right] \geq 0.$$

Since, $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1$, we have, $CE^{w}_{IN}(J_1, J_2) \geq 0$.

Hence complete the proof.□

ii.

$$\left[\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow T^{-}_{J_1}(u_i)=T^{-}_{J_2}(u_i)$$

$$\left[\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow T^{+}_{J_1}(u_i)=T^{+}_{J_2}(u_i)$$

$$\left[\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow I^{-}_{J_1}(u_i)=I^{-}_{J_2}(u_i)$$

$$\left[\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow I^{+}_{J_1}(u_i)=I^{+}_{J_2}(u_i)$$

$$\left[\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow F^{-}_{J_1}(u_i)=F^{-}_{J_2}(u_i)$$

$$\left[\frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow F^{+}_{J_1}(u_i)=F^{+}_{J_2}(u_i)$, for all values of $u_i \in U$.

Since, $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$, $w_i \geq 0$, we have

$CE_{IN}^{w}(J_1, J_2) = 0$ iff $T_{J_1}^{-}(u_i) = T_{J_2}^{-}(u_i)$, $T_{J_1}^{+}(u_i) = T_{J_2}^{+}(u_i)$, $I_{J_1}^{-}(u_i) = I_{J_2}^{-}(u_i)$,

$I_{J_1}^{+}(u_i) = I_{J_2}^{+}(u_i)$, $F_{J_1}^{-}(u_i) = F_{J_2}^{-}(u_i)$, $F_{J_1}^{+}(u_i) = F_{J_2}^{+}(u_i)$ and

$T_{J_1}(u_i) = T_{J_2}(u_i)$, $I_{J_1}(u_i) = I_{J_2}(u_i)$, $F_{J_1}(u_i) = F_{J_2}(u_i)$ for all $u_i \in U$. □

iii. We have, $CE_{IN}^{w}(J_1^c, J_2^c)$

$$= \frac{1}{4}\left\langle \sum_{i=1}^{n} w_i \left\{\left[\frac{2\left|T_{J_1^c}^{-}(u_i) - T_{J_2^c}^{-}(u_i)\right|}{\sqrt{1+\left|T_{J_1^c}^{-}(u_i)\right|^2} + \sqrt{1+\left|T_{J_2^c}^{-}(u_i)\right|^2}} + \frac{2\left|(1-T_{J_1^c}^{-}(u_i)) - (1-T_{J_2^c}^{-}(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1^c}^{-}(u_i))\right|^2} + \sqrt{1+\left|(1-T_{J_2^c}^{-}(u_i))\right|^2}}\right] + \right.\right.$$

$$\left[\frac{2\left|T_{J_1^c}^{+}(u_i) - T_{J_2^c}^{+}(u_i)\right|}{\sqrt{1+\left|T_{J_1^c}^{+}(u_i)\right|^2} + \sqrt{1+\left|T_{J_2^c}^{+}(u_i)\right|^2}} + \frac{2\left|(1-T_{J_1^c}^{+}(u_i)) - (1-T_{J_2^c}^{+}(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1^c}^{+}(u_i))\right|^2} + \sqrt{1+\left|(1-T_{J_2^c}^{+}(u_i))\right|^2}}\right] +$$

$$\left[\frac{2\left|I_{J_1^c}^{-}(u_i) - I_{J_2^c}^{-}(u_i)\right|}{\sqrt{1+\left|I_{J_1^c}^{-}(u_i)\right|^2} + \sqrt{1+\left|I_{J_2^c}^{-}(u_i)\right|^2}} + \frac{2\left|(1-I_{J_1^c}^{-}(u_i)) - (1-I_{J_2^c}^{-}(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1^c}^{-}(u_i))\right|^2} + \sqrt{1+\left|(1-I_{J_2^c}^{-}(u_i))\right|^2}}\right] +$$

$$\left[\frac{2\left|I_{J_1^c}^{+}(u_i) - I_{J_2^c}^{+}(u_i)\right|}{\sqrt{1+\left|I_{J_1^c}^{+}(u_i)\right|^2} + \sqrt{1+\left|I_{J_2^c}^{+}(u_i)\right|^2}} + \frac{2\left|(1-I_{J_1^c}^{+}(u_i)) - (1-I_{J_2^c}^{+}(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1^c}^{+}(u_i))\right|^2} + \sqrt{1+\left|(1-I_{J_2^c}^{+}(u_i))\right|^2}}\right] +$$

$$\left[\frac{2\left|F_{J_1^c}^{-}(u_i) - F_{J_2^c}^{-}(u_i)\right|}{\sqrt{1+\left|F_{J_1^c}^{-}(u_i)\right|^2} + \sqrt{1+\left|F_{J_2^c}^{-}(u_i)\right|^2}} + \frac{2\left|(1-F_{J_1^c}^{-}(u_i)) - (1-F_{J_2^c}^{-}(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1^c}^{-}(u_i))\right|^2} + \sqrt{1+\left|(1-F_{J_2^c}^{-}(u_i))\right|^2}}\right] +$$

$$\left[\frac{2\left|F_{J_1^c}^{+}(u_i) - F_{J_2^c}^{+}(u_i)\right|}{\sqrt{1+\left|F_{J_1^c}^{+}(u_i)\right|^2} + \sqrt{1+\left|F_{J_2^c}^{+}(u_i)\right|^2}} + \frac{2\left|(1-F_{J_1^c}^{+}(u_i)) - (1-F_{J_2^c}^{+}(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1^c}^{+}(u_i))\right|^2} + \sqrt{1+\left|(1-F_{J_2^c}^{+}(u_i))\right|^2}}\right]$$

$$= \frac{1}{4}\left\langle \sum_{i=1}^{n} w_i \left\{ \left[\frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}} + \frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}} \right] + \right.\right.$$

$$\left[\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}} + \frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}} \right] +$$

$$\left[\frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}} + \frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}} \right] +$$

$$\left[\frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}} + \frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}} \right] +$$

$$\left[\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}} + \frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}} \right] +$$

$$\left.\left.\left[\frac{2\left|(1-F^{+}_{Q_1}(u_i))-(1-F^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{Q_2}(u_i))\right|^2}} + \frac{2\left|F^{+}_{Q_1}(u_i)-F^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{Q_2}(u_i)\right|^2}} \right] \right\} \right\rangle$$

$$= \frac{1}{4}\left\langle \sum_{i=1}^{n} w_i \left\{ \left[\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}} + \frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}} \right] + \right.\right.$$

$$\left[\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}} + \frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}} \right] +$$

$$\left[\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}} + \frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}} \right] +$$

$$\left[\frac{2\left|I_{J_1}^{+}(u_i)-I_{J_2}^{+}(u_i)\right|}{\sqrt{1+\left|I_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-I_{J_1}^{+}(u_i))-(1-I_{J_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^{+}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F_{J_1}^{-}(u_i)-F_{J_2}^{-}(u_i)\right|}{\sqrt{1+\left|F_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-F_{J_1}^{-}(u_i))-(1-F_{Q_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_2}^{-}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F_{J_1}^{+}(u_i)-F_{J_2}^{+}(u_i)\right|}{\sqrt{1+\left|F_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-F_{J_1}^{+}(u_i))-(1-F_{J_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_2}^{+}(u_i))\right|^2}}\right]\right\}\right\rangle$$

$$=CE_{IN}^{w}(J_1, J_2), \forall u_i \in U.$$

iv. Since,

$$\left|T_{J_1}^{-}(u_i)-T_{J_2}^{-}(u_i)\right|=\left|T_{J_2}^{-}(u_i)-T_{J_1}^{-}(u_i)\right|, \left|I_{J_1}^{-}(u_i)-I_{J_2}^{-}(u_i)\right|=\left|I_{J_2}^{-}(u_i)-I_{J_1}^{-}(u_i)\right|,$$

$$\left|F_{J_1}^{-}(u_i)-F_{J_2}^{-}(u_i)\right|=\left|F_{J_2}^{-}(u_i)-F_{J_1}^{-}(u_i)\right|,$$

$$\left|(1-T_{J_1}^{-}(u_i))-(1-T_{J_2}^{-}(u_i))\right|=\left|(1-T_{J_2}^{-}(u_i))-(1-T_{J_1}^{-}(u_i))\right|,$$

$$\left|(1-I_{J_1}^{-}(u_i))-(1-I_{J_2}^{-}(u_i))\right|=\left|(1-I_{J_2}^{-}(u_i))-(1-I_{J_1}^{-}(u_i))\right|,$$

$$\left|(1-F_{J_1}^{-}(u_i))-(1-F_{J_2}^{-}(u_i))\right|=\left|(1-F_{J_2}^{-}(u_i))-(1-F_{J_1}^{-}(u_i))\right|.$$

Then, we obtain

$$\sqrt{1+\left|T_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^{-}(u_i)\right|^2}=\sqrt{1+\left|T_{J_2}^{-}(u_i)\right|^2}+\sqrt{1+\left|T_{J_1}^{-}(u_i)\right|^2},$$

$$\sqrt{1+\left|I_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^{-}(u_i)\right|^2}=\sqrt{1+\left|I_{J_2}^{-}(u_i)\right|^2}+\sqrt{1+\left|I_{J_1}^{-}(u_i)\right|^2},$$

$$\sqrt{1+\left|F_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^{-}(u_i)\right|^2}=\sqrt{1+\left|F_{J_2}^{-}(u_i)\right|^2}+\sqrt{1+\left|F_{J_1}^{-}(u_i)\right|^2},$$

$$\sqrt{1+\left|(1-T_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|(1-T_{J_2}^{-}(u_i)\right|^2}=\sqrt{1+\left|(1-T_{J_2}^{-}(u_i)\right|^2}+\sqrt{1+\left|(1-T_{J_1}^{-}(u_i)\right|^2},$$

$$\sqrt{1+\left|(1-I_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|(1-I_{J_2}^{-}(u_i)\right|^2}=\sqrt{1+\left|(1-I_{J_2}^{-}(u_i)\right|^2}+\sqrt{1+\left|(1-I_{J_1}^{-}(u_i)\right|^2},$$

$\sqrt{1+\left|(1-F_{J_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|(1-F_{J_2}^{-}(u_i)\right|^2}=\sqrt{1+\left|(1-F_{J_2}^{-}(u_i)\right|^2}+\sqrt{1+\left|(1-F_{J_1}^{-}(u_i)\right|^2}$,

$\forall u_i \in U$.

Similarly, $\left|T_{J_1}^{+}(u_i)-T_{J_2}^{+}(u_i)\right|=\left|T_{J_2}^{+}(u_i)-T_{J_1}^{+}(u_i)\right|$, $\left|I_{J_1}^{+}(u_i)-I_{J_2}^{+}(u_i)\right|=\left|I_{J_2}^{+}(u_i)-I_{J_1}^{+}(u_i)\right|$,

$\left|F_{J_1}^{+}(u_i)-F_{J_2}^{+}(u_i)\right|=\left|F_{J_2}^{+}(u_i)-F_{J_1}^{+}(u_i)\right|$,

$\left|(1-T_{J_1}^{+}(u_i))-(1-T_{J_2}^{+}(u_i))\right|=\left|(1-T_{J_2}^{+}(u_i))-(1-T_{J_1}^{+}(u_i))\right|$,

$\left|(1-I_{J_1}^{+}(u_i))-(1-I_{J_2}^{+}(u_i))\right|=\left|(1-I_{J_2}^{+}(u_i))-(1-I_{J_1}^{+}(u_i))\right|$,

$\left|(1-F_{J_1}^{+}(u_i))-(1-F_{J_2}^{+}(u_i))\right|=\left|(1-F_{J_2}^{+}(u_i))-(1-F_{J_1}^{+}(u_i))\right|$, then

$\sqrt{1+\left|T_{Q_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2}^{+}(u_i)\right|^2}=\sqrt{1+\left|T_{Q_2}^{+}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_1}^{+}(u_i)\right|^2}$,

$\sqrt{1+\left|I_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^{+}(u_i)\right|^2}=\sqrt{1+\left|I_{J_2}^{+}(u_i)\right|^2}+\sqrt{1+\left|I_{J_1}^{+}(u_i)\right|^2}$,

$\sqrt{1+\left|F_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^{+}(u_i)\right|^2}=\sqrt{1+\left|F_{J_2}^{+}(u_i)\right|^2}+\sqrt{1+\left|F_{J_1}^{+}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-T_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|(1-T_{J_2}^{+}(u_i)\right|^2}=\sqrt{1+\left|(1-T_{J_2}^{+}(u_i)\right|^2}+\sqrt{1+\left|(1-T_{J_1}^{+}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-I_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|(1-I_{J_2}^{+}(u_i)\right|^2}=\sqrt{1+\left|(1-I_{J_2}^{+}(u_i)\right|^2}+\sqrt{1+\left|(1-I_{J_1}^{+}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-F_{J_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|(1-F_{J_2}^{+}(u_i)\right|^2}=\sqrt{1+\left|(1-F_{J_2}^{+}(u_i)\right|^2}+\sqrt{1+\left|(1-F_{J_1}^{+}(u_i)\right|^2}$, $\forall u_i \in U$.

And $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$, $w_i \ge 0$.

So, $CE_{IN}^{w}(J_1, J_2) = CE_{IN}^{w}(J_2, J_1)$. □

3.3 Multi attribute group decision making strategy using IN-cross entropy measure in interval neutrosophic set environment

In this section we develop a novel MAGDM strategy based on proposed IN-cross entropy measure.

Description of the MAGDM problem:

Assume that $A = \{A_1, A_2, A_3, ..., A_m\}$ and $G = \{G_1, G_2, G_3, ..., G_n\}$ be the discrete set of alternatives and attributes respectively. Let $W = \{w_1, w_2, w_3, ..., w_n\}$ be the weight vector of attributes G_j (j = 1, 2, 3, …, n), where $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$. Let $E = \{E_1, E_2, E_3, ..., E_\rho\}$ be the set of decision makers. The weight vector of the decision makers E_k $(k = 1, 2, 3, ..., \rho)$ is $\lambda = \{\lambda_1, \lambda_2, \lambda_3, ..., \lambda_\rho\}$, where $\lambda \geq 0$ and $\sum_{k=1}^{\rho} \lambda_k = 1$.

Now, we describe the steps of the proposed MAGDM strategy.

Step 1. Formulate the decision matrices

For MAGDM with INSs information, the rating values of the alternatives A_i $(i = 1, 2, 3, ..., m)$ on the basis of criterion G_j $(j = 1, 2, 3, ..., n)$ provided by the k-th decision maker can be expressed in terms of INN as $a_{ij}^k = \langle [{}^{-}T_{ij}^k, {}^{+}T_{ij}^k], [{}^{-}I_{ij}^k, {}^{+}I_{ij}^k], [{}^{-}F_{ij}^k, {}^{+}F_{ij}^k] \rangle$ (i = 1, 2, 3, …, m; j = 1, 2, 3, …, n; k = 1, 2, 3, …, ρ) . We present these rating values of alternatives provided by the decision makers in matrix form as follows:

$$M^k = \begin{pmatrix} & G_1 & G_2 & \dots & G_n \\ A_1 & a_{11}^k & a_{12}^k & \cdots & a_{1n}^k \\ A_2 & a_{21}^k & a_{22}^k & & a_{2n}^k \\ . & . & & \dots & . \\ A_m & a_{m1}^k & a_{m2}^k & \cdots & a_{mn}^k \end{pmatrix} \tag{3.3}$$

Step 2. Formulate the weighted aggregated decision matrix

For obtaining one group opinion, we aggregate all individual decision matrices (M^k) to an aggregated decision matrix (M) using interval-valued neutrosophic weighted Aggregation (INNWA) operator (Zhang et al., 2014) as follows:

$$a_{ij} = INNWA_\lambda(a_{ij}^1, a_{ij}^2, a_{ij}^3, \dots, a_{ij}^\rho) =$$

$$(\lambda_1 a_{ij}^1 \oplus \lambda_2 a_{ij}^2 \oplus \lambda_3 a_{ij}^3 \oplus \dots \oplus \lambda_\rho a_{ij}^\rho) =$$

$$< [1 - \prod_{k=1}^{\rho} (1 - {}^{-}T_{ij}^k)^{\lambda_k}, 1 - \prod_{k=1}^{\rho} (1 - {}^{+}T_{ij}^k)^{\lambda_k}], [\prod_{k=1}^{\rho} ({}^{-}I_{ij}^k)^{\lambda_k}, \prod_{k=1}^{\rho} ({}^{+}I_{ij}^k)^{\lambda_k}],$$

$$[\prod_{k=1}^{\rho} ({}^{-}F_{ij}^k)^{\lambda_k}, \prod_{k=1}^{\rho} ({}^{+}F_{ij}^k)^{\lambda_k}] > \tag{3.4}$$

(i = 1, 2, 3, …, m; j = 1, 2, 3, …, n; k = 1, 2, 3, …, ρ).

Therefore, the aggregated decision matrix is defined as follows:

$$M = \begin{pmatrix} & G_1 & G_2 \; \ldots & .G_n \\ A_1 & a_{11} & a_{12} \ldots & a_{1n} \\ A_2 & a_{21} & a_{22} & a_{2n} \\ . & . & \ldots & . \\ A_m & a_{m1} & a_{m2} \ldots & a_{mn} \end{pmatrix} \quad (3.5)$$

Step 3. Formulate priori/ ideal decision matrix

In the MAGDM processes, the priori decision matrix is used to select the best alternative among the set of collected feasible alternatives. In the decision making strategy, we use the following decision matrix as priori decision matrix.

$$P = \begin{pmatrix} & G_1 & G_2 \; \ldots & .G_n \\ A_1 & a_{11}^* & a_{12}^* \ldots & a_{1n}^* \\ A_2 & a_{21}^* & a_{22}^* & a_{2n}^* \\ . & . & \ldots & . \\ A_m & a_{m1}^* & a_{m2}^* \ldots & a_{mn}^* \end{pmatrix} \quad (3.6)$$

Where, $a_{ij}^* = <[1,1],[0,0],[0,0]>$ for benefit type attributes and $a_{ij}^* = <[0,0],[1,1],[1,1]>$ for cost type attributes, (i = 1, 2, 3, …, m; j = 1, 2, 3, …, n)

Step 4. Formulate the weighted IN-cross entropy matrix

Using equation (3.2), we calculate weighted cross entropy value between aggregate matrix and priori matrix. The cross entropy value is presented in matrix form as follows:

$$^{INS}M_{CE}^{w} = \begin{pmatrix} CE_{IN}^{w}\,(A_1) \\ CE_{IN}^{w}\,(A_2) \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ CE_{IN}^{w}\,(A_m) \end{pmatrix} \quad (3.7)$$

Step 5. Rank the priority

Smaller value of the cross entropy reflects that an alternative is closer to the ideal alternative. Therefore, the priority order of all the alternatives is determined according to the increasing order of the cross entropy values $CE_{IN}^{w}\,(A_i)$ (i = 1, 2, 3,…, m). Smallest cross entropy value indicates the best alternative and the greatest cross entropy value indicates the worst alternative.

3.4 Illustrative example

In this section, we provide an illustrative example of MAGDM problems to reflect the validity and efficiency of the proposed strategy in INS environment.

Now, we solve an illustrative example adapted from (He & Liu, 2013) for cultivation and analysis. A venture capital firm intends to make evaluation and selection to five enterprises with the investment potential:

1) Automobile company (A_1)
2) Military manufacturing enterprise (A_2)
3) TV media company (A_3)
4) Food enterprises (A_4)
5) Computer software company (A_5)

On the basis of four attributes namely:

1) Social and political factor (G_1)
2) The environmental factor (G_2)
3) Investment risk factor (G_3)
4) The enterprise growth factor (G_4).

The investment firm makes a panel of three decision makers $E = \{E_1, E_2, E_3\}$ having their weights vector $\lambda = \{0.42, 0.28, 0.30\}$ and weights vector of attributes is $W = \{0.24, 0.25, 0.23, 0.28\}$.

The steps of decision making strategy to rank alternatives are presented below:

Step 1. Formulate the decision matrices

The decision maker presents the rating values of alternative A_i (i = 1, 2, 3, 4, 5) with respect to the attribute G_j (j = 1, 2, 3, 4) in terms of interval neutrosophic numbers provided by the decision makers E_k (k = 1, 2, 3) matrix form as follows:

Decision matrix for decision maker E_1

$$M^1 = \begin{array}{cc} & \begin{array}{cccc} \quad G_1 \quad & \quad G_2 \quad & \quad G_3 \quad & \quad G_4 \quad \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} & \left(\begin{array}{cccc} <[.7,.9],[.3,.4],[.3,.4]> & <[.6,.7],[.3,.4],[.4,.5]> & <[.6,.7],[.2,.3],[.2,.4]> & <[.4,.5],[.3,.4],[.7,.8]> \\ <[.6,.7],[.1,.2],[.2,.3]> & <[.7,.8],[.2,.4],[.2,.3]> & <[.7,.9],[.5,.6],[.4,.5]> & <[.7,.9],[.1,.2],[.1,.3]> \\ <[.6,.8],[.2,.4],[.3,.4]> & <[.5,.7],[.3,.4],[.1,.2]> & <[.8,.9],[.5,.7],[.3,.6]> & <[.6,.7],[.1,.3],[.2,.3]> \\ <[.4,.5],[.7,.8],[.6,.7]> & <[.3,.6],[.2,.3],[.3,.4]> & <[.6,.7],[.1,.2],[.4,.5]> & <[.4,.5],[.3,.4],[.6,7]> \\ <[.7,.8],[.3,.4],[.2,.3]> & <[.4,.5],[.2,.4],[.3,.5]> & <[.5,.6],[.2,.4],[.3,.4]> & <[.7,.9],[.6,.7],[.4,.5]> \end{array}\right) \end{array} \tag{3.8}$$

Decision matrix for decision maker E_2

$$M^2 = \begin{array}{cc} & \begin{array}{cccc} \quad G_1 \quad & \quad G_2 \quad & \quad G_3 \quad & \quad G_4 \quad \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} & \left(\begin{array}{cccc} <[.6,.7],[.1,.2],[.2,.3]> & <[.3,.5],[.2,.4],[.4,.5]> & <[.7,.9],[.3,.4],[.3,.5]> & <[.4,.6],[.4,.5],[.2,.3]> \\ <[.4,.7],[.2,.4],[.3,.4]> & <[.6,.7],[.2,.3],[.3,.4]> & <[.5,.7],[.1,.3],[.3,.4]> & <[.4,.6],[.3,.4],[.2,.3]> \\ <[.3,.6],[.2,.4],[.3,.4]> & <[.4,.5],[.2,.3],[.3,.5]> & <[.8,.9],[.2,.5],[.3,.4]> & <[.5,.6],[.3,.5],[.3,.6]> \\ <[.5,.7],[.3,.5],[.1,.3]> & <[.5,.6],[.1,.3],[.4,.6]> & <[.4,.7],[.1,.4],[.3,.4]> & <[.6,.8],[.3,.5],[.3,4]> \\ <[.6,.9],[.3,.4],[.2,.3]> & <[.3,.6],[.3,.4],[.2,.5]> & <[.6,.8],[.3,.5],[.,4.6]> & <[.3,.5],[.3,.4],[.4,.5]> \end{array}\right) \end{array} \tag{3.9}$$

Decision matrix for decision maker E_3

$$M^3 = \begin{array}{cc} & \begin{array}{cccc} \quad G_1 \quad & \quad G_2 \quad & \quad G_3 \quad & \quad G_4 \quad \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} & \left(\begin{array}{cccc} <[.4,.7],[.1,.2],[.3,.5]> & <[.3,.6],[.2,.4],[.3,.4]> & <[.6,.7],[.2,.4],[.3,.5]> & <[.8,.9],[.2,.4],[.1,.3]> \\ <[.3,.6],[.4,.5],[.4,.5]> & <[.7,.9],[.1,.3],[.3,.4]> & <[.5,.7],[.2,.4],[.2,.3]> & <[.6,.8],[.2,.4],[.3,.5]> \\ <[.7,.8],[.1,.3],[.4,.5]> & <[.8,.9],[.1,.3],[.3,.4]> & <[.6,.8],[.2,.3],[.3,.4]> & <[.6,.7],[.2,.3],[.3,.4]> \\ <[.6,.9],[.2,.3],[.2,.4]> & <[.5,.6],[.1,.3],[.2,.4]> & <[.3,.5],[.1,.2],[.2,.4]> & <[.5,.7],[.2,.3],[.3,5]> \\ <[.7,.8],[.1,.3],[.2,.3]> & <[.5,.6],[.2,.4],[.1,.3]> & <[.4,.6],[.1,.3],[.2,.4]> & <[.5,.7],[.2,.3],[.3,.5]> \end{array}\right) \end{array} \tag{3.10}$$

Step 2. Formulate the weighted aggregated decision matrix

Using equation (3.4), the aggregated decision matrix is presented below:

Aggregated decision matrix

$$M = \begin{array}{cc} & \begin{array}{cccc} \quad G_1 \quad & \quad G_2 \quad & \quad G_3 \quad & \quad G_4 \quad \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} & \left(\begin{array}{cccc} <[.6,.8],[.2,.3],[.3,.4]> & <[.5,.6],[.2,.4],[.4,.4]> & <[.6,.8],[.2,.3],[.2,.4]> & <[.6,.7],[.3,.4],[.3,.4]> \\ <[.5,.7],[.2,.3],[.3,.4]> & <[.7,.8],[.2,.3],[.2,.4]> & <[.6,.8],[.2,.4],[.3,.4]> & <[.6,.8],[.2,.3],[.2,.3]> \\ <[.6,.8],[.2,.4],[.3,.4]> & <[.6,.8],[.2,.3],[.2,.3]> & <[.8,.9],[.3,.5],[.3,.5]> & <[.6,.7],[.2,.3],[.2,.4]> \\ <[.5,.7],[.4,.5],[.3,.5]> & <[.4,.6],[.1,.3],[.3,.4]> & <[.5,.6],[.1,.2],[.3,.4]> & <[.5,.7],[.3,.4],[.4,.5]> \\ <[.7,.8],[.2,.4],[.2,.3]> & <[.4,.6],[.2,.4],[.2,.4]> & <[.5,.7],[.2,.4],[.3,.4]> & <[.6,.8],[.4,.5],[.4,.5]> \end{array}\right) \end{array} \tag{3.11}$$

Step 3. Formulate priori/ ideal decision matrix

Priori/ ideal decision matrix

$$M^1 = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> \\ A_2 & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> \\ A_3 & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> \\ A_4 & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> \\ A_5 & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> & <[1,1],[0,0],[0,0]> \end{pmatrix} \tag{3.12}$$

Step 4. Calculate the weighted IN-cross entropy matrix

Using equation (3.2), we calculate the interval neutrosophic weighted cross entropy values between ideal matrixes (3.12) and weighted aggregated decision matrix (3.11).

$$^{IN}M^{w}_{CE} = \begin{pmatrix} 0.86 \\ 0.77 \\ 0.78 \\ 0.95 \\ 0.90 \end{pmatrix} \tag{3.13}$$

Step 5. Rank the priority

The position of cross entropy values of alternatives arranging in increasing order is 0.77 < 0.78 < 0.86 < 0.90 < 0.95. Since, the smaller valueof cross entropy indicatesthat the alternative is closer to the ideal alternative. Thus the ranking priority of alternatives is $A_2 > A_3 > A_1 > A_5 > A_4$. Hence, military manufacturing enterprise (A_2) is the best alternative for investment.

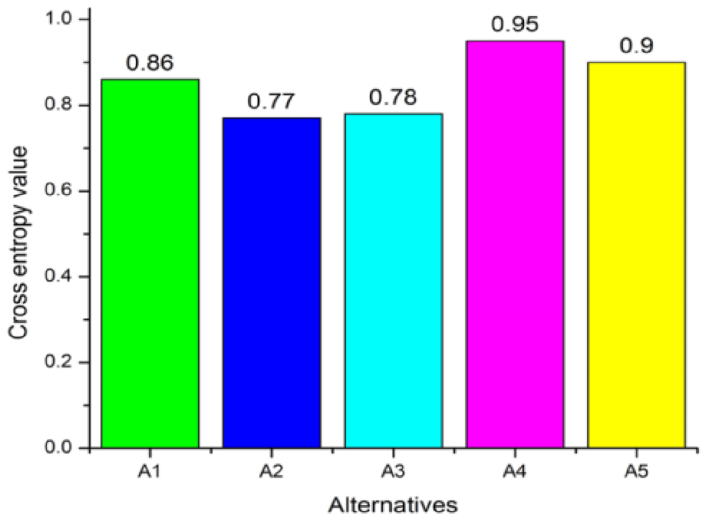

Figure 3.1 Bar diagram of alternatives versus cross entropy values of alternatives

3.5 Conclusion

In this chapter we have defined IN-cross entropy measure in INS environment which is free from the asymmetry and undefined phenomena. We have proved the basic properties of the cross entropy measures. We have also defined weighted IN- cross entropy measure and proved its basic properties. Based on the weighted IN-cross entropy measure, we have proposed a novel MAGDM strategy. Finally, we solve an MAGDM problem to show the feasibility and efficiency of the proposed MAGDM making strategy. The proposed IN-cross entropy based MAGDM strategy can be employed to solve a variety of problems such as logistics center selection, teacher selection, renewable energy selection, fault diagnosis, etc.

Chapter 4

NC-TODIM-based MAGDM in a neutrosophic cubic set environment

4.1 Introduction

TODIM (an acronym in Portuguese for interactive multi-criteria decision making strategy named Tomada de decisaointerativa e multicritévio) is an important MADM strategy, since it considers the decision makers' bounded rationality. Firstly, Gomes and Lima (1992) introduced the TODIM strategy based on prospect theory (Kahneman, 1979). Krohling and Souza (2012) defined the fuzzy TODIM strategy to solve MADM problems. Several researchers applied the TODIM strategy in various fuzzy MADM or MAGDM problems (Liu & Teng, 2014; Tosun & Akyu, 2015; Gomes et al., 2013). Fan et al. (2013) introduced the extended TODIM strategy to deal with the hybrid MADM problems. Krohling et al. (2013) extended the TODIM strategy from fuzzy environment to intuitionistic fuzzy environment. Wang (2015) introduced TODIM strategy into multi-valued neutrosophic set environment. Zhang et al. (2016) proposed the TODIM strategy for MAGDM problems with neutrosophic number (NN) environment. Ji et al. (2016) proposed the TODIM strategy under a multi-valued neutrosophic environment and employed it to solve personal selection problems. Xu et al. (2017) developed the TODIM strategy in a single valued neutrosophic setting and extended it into interval neutrosophic setting. Neutrosophic TODIM studied by Xu et al., (2017) is capable of dealing with only single-valued neutrosophic information or interval neutrosophic information. Neutrosophic cubic set (NCS)is capable of expressing of the interval neutrosophic information and neutrosophic information in the process of MADGM.

The content of this chapter is based on the paper published in **"Information"** 2017, 8(4), 149; doi:10.3390/info8040149.

In the NCS environment, the TODIM strategy is yet to appear. Motivated by these, we initiate the study of TODIM in the NCS environment, which we call NC-TODIM. To apply NCSs to MADGM problems, we introduce some basic operations of neutrosophic cubic (NC) numbers and the score, and accuracy functions of NC- numbers, and the ranking strategy of NC numbers.

In this Chapter, we develop a TODIM strategy (for short, NC-TODIM strategy) for MAGDM in the NCS environment. The proposed NC-TODIM strategy is employed to solve an illustrative numerical example of an MAGDM.

The remainder of the chapter is divided into six sections that are organized as follows: Section 4.2 presents comparison strategy for any two NC-numbers. Section 4.3 is devoted to present the proposed NC-TODIM strategy. Section 4.4 presents an illustrative numerical example of MAGDM in the NCS environment. Section 4.5 is devoted to analyzing the ranking order with different values of attenuation factors of losses. Section 4.6 presents a comparative analysis between the developed strategy and other existing strategies in the NCS environment. Section 4.7 presents the conclusion and the future scope of research.

4.2 Comparison strategy of two NC-numbers

Definition 4.1 Score function. Let $©_1$ be a NC-number in a non-empty set G. Then, a score function of $©_1$, denoted by $Sc(©_1)$ is defined as:

$$Sc(©_1)=\frac{1}{2}[(\frac{2+a_1+a_2-2b_1-2b_2-c_1-c_2}{4})+(\frac{1+a-2b-c}{2})] \tag{4.1}$$

where, $©_1$= <[a_1, a_2], [b_1, b_2], [c_1, c_2], (a, b, c) > and $Sc(©_1) \in [-1, 1]$.

Proposition 4.1

Score function of two NC-numbers lies between −1 to 1.

Proof. Using the definition of INS and NS, we have all a_1, a_2, b_1, b_2, c_1, c_2, a, b, and c $\in [0,1]$.

Since,

$$0\le a_1 \le 1, 0\le a_2 \le 1$$

$$\Rightarrow 0\le a_1+a_2 \le 2$$

$$\Rightarrow 2\le 2+a_1+a_2 \le 4 \tag{4.2}$$

$0 \le b_1 \le 1 \Rightarrow 0 \le 2b_1 \le 2$, and $0 \le b_2 \le 1$

$\Rightarrow 0 \le 2b_2 \le 2$

$\Rightarrow -2 \le -2b_1 \le 0$

$\Rightarrow -2 \le -2b_2 \le 0$

$$\Rightarrow -4 \le -2b_1 - 2b_2 \le 0 \tag{4.3}$$

$0 \le c_1 \le 1 \Rightarrow -1 \le -c_1 \le 0$

$0 \le c_2 \le 1 \Rightarrow -1 \le -c_2 \le 0$

$$\Rightarrow -2 \le -c_1 - c_2 \le 0 \tag{4.4}$$

Adding Equation (4.2), Equation (4.3) and Equation (4.4), we obtain

$\Rightarrow -4 \le 2 + a_1 + a_2 - 2b_1 - 2b_2 - c_1 - c_2 \le 4$,

$$\Rightarrow -1 \le \frac{2 + a_1 + a_2 - 2b_1 - 2b_2 - c_1 - c_2}{4} \le 1 \tag{4.5}$$

Again,

$$0 \le a \le 1 \Rightarrow 1 \le 1 + a \le 2 \tag{4.6}$$

$0 \le b \le 1 \Rightarrow 0 \le 2b \le 2$,

$0 \le c \le 1$,

$\Rightarrow 0 \le 2b + c \le 3$,

$$\Rightarrow -3 \le -2b - c \le 0 \tag{4.7}$$

Adding Equation (4.6) and Equation (4.7), we obtain

$-2 \le 1 + a - 2b - c \le 2$,

$$\Rightarrow -1 \le \frac{1 + a - 2b - c}{2} \le 1 \tag{4.8}$$

Adding Equation (4.5) and Equation (4.8) and dividing by 2, we obtain

$$-1 \le \frac{1}{2}[(\frac{2 + a_1 + a_2 - 2b_1 - 2b_2 - c_1 - c_2}{4}) + (\frac{1 + a - 2b - c}{2})] \le 1$$

$Sc(©_1) \in [-1, 1]$.

Hence the proof is complete. □

Example 4.1 Let $©_1$ and $©_2$ be two NC-numbers in G, presented as follows:

$©_1$ = <[0.39, 0.47], [0.17, 0.43], [0.18, 0.36], (0.6, 0.3, 0.4)>and

$©_2$ = <[0.56, 0.70], [0.27, 0.42], [0.15, 0.26], (0.7, 0.3, 0.6)>.

Then, by using Definition (4.1), we obtain $Sc(©_1)$ = −0.01 and $Sc(©_2)$ = 0.07, In this case, we can say that $©_2 > ©_1$.

Definition 4.2 Accuracy function

Let $©_1$ be an NC-number in a non-empty set G, an accuracy function of $©_1$ is defined as:

$$Ac(©_1) = ½[½(a_1 + a_2 − b_2(1 − a_2) − b_1(1 − a_1) − c_2(1 − b_1) − c_1(1 − b_2) + a − b(1 − a) − c(1 − b) \quad (4.9)$$

Here, $Ac(©_1) \in [-1, 1]$.

When the value of $Ac(©_1)$ increases, we say that the degree of accuracy of the NC-number $©_1$increases.

Proposition 4.2

Accuracy function of two NC-numbers lies between −1 to 1.

Proof. The values of accuracy function depend upon

$$\{\frac{1}{2}(a_1 + a_2 - b_2(1-a_2) - b_1(1-a_1) - c_2(1-b_1) - c_1(1-b_2)) \text{ and} \{a - b(1-a) - c(1-b)\}$$

The values of

$$\{\frac{1}{2}(a_1 + a_2 - b_2(1-a_2) - b_1(1-a_1) - c_2(1-b_1) - c_1(1-b_2))\} \text{ and}$$

$\{a - b(1-a) - c(1-b)\}$ lie between −1 to 1 from (Şahin, 2014).

Thus, $-1 \le Ac(©_1) \le 1$.

Hence the proof is completed.

Example 4.2 Let $©_1$ and $©_2$ be two NC-numbers in G presented as follows:

$©_1$ = <[0.41, 0.52], [0.10, 0.18], [0.06, 0.17], (0.48, 0.11, 0.11)> and

$©_2$ = <[0.40, 0.51], [0.10, 0.20], [0.10, 0.19], (0.50, 0.11, 0.11).

Then, by applying Definition 4.2, we obtain $Ac(©_1)$ = 0.14 and $Ac(©_2)$ = 0.30.

With respect to the score function Sc and the accuracy function Ac, a strategy for comparing NC-numbers can be defined as follows:

Comparison strategy of two NC-numbers

Let $©_1$ and $©_2$ be any two NC-numbers. Then we define comparison strategy as follows:

i. If $Sc(©_1) > Sc(©_2)$, then $©_1 > ©_2$ (4.10) ii.If $Sc(©_1) = Sc(©_2)$ and $Ac(©_1) > Ac(©_2)$, then $©_1 > ©_2$ (4.11)

iii.If $Sc(©_1) = Sc(©_2)$ and $Ac(©_1) = Ac(©_2)$, then $©_1 = ©_2$ (4.12)

Example 4.3 Let $©_1$ and $©_2$ be two NC-numbers in G, presented as follows:

$©_1$ = <[0.23, 0.29], [0.37, 0.46], [0.34, 0.42], (0.26, 0.26, 0.26)>

and $©_2$ = <[0.25, 0.31], [0.35, 0.44], [0.35, 0.44], (0.28, 0.28, 0.28)>.

Then, applying Definition 4.1, we obtain $Sc(©_1)$ = 0.13 and $Sc(©_2)$ = 0.13. Applying Definition 4.2, we obtain $Ac(©_1)$ = −0.20 and $Ac(©_2)$ = −0.18. In this case, we say that alternative $©_2 > ©_1$. (Score values and Accuracy values taking correct up to two decimal places).

Definition 4.3 Let $©_1$ and $©_2$ be any two NC-numbers, then the distance between them is defined by

$\partial(©_1, ©_2) =$

$$\frac{1}{9}[|a_1 - d_1| + |a_2 - d_2| + |b_1 - e_1| + |b_2 - e_2| + |c_1 - f_1| + |c_2 - f_2| + |a - d| + |b - e| + |c - f|] \tag{4.13}$$

where, $©_1$= <[a_1, a_2], [b_1, b_2], [c_1, c_2], (a, b, c)> and $©_2$= <[d_1, d_2], [e_1, e_2], [f_1, f_2], (d, e, f)>.

Example 4.4 Let $©_1$ and $©_2$ be two NC-numbers in G presented as follows:

$©_1$ = <[0.66, 0.75], [0.25, 0.32], [0.17, 0.34], (0.53, 0.17, 0.22)>

and $©_2$ = < [0.35, 0.55], [0.12, 0.25], [0.12, 0.20], (0.60, 0.23, 0.43)>

Then, we obtain $\partial(©_1, ©_2)$ = 0.12.

Definition 4.4 Let $©_{ij}$ = $\{<[t_{ij}^-, t_{ij}^+], [i_{ij}^-, i_{ij}^+], [f_{ij}^-, f_{ij}^+], (t, i, f)>\}$ be a neutrosophic cubic value, which is used to evaluate i-th alternative with respect to j-th criterion. The normalized form of $©_{ij}$ is defined as follows:

$$©_{ij}^{\otimes} = \{<[\frac{t_{ij}^{-}}{(\sum_{i=1}^{m}(t_{ij}^{-})^2+(t_{ij}^{+})^2)^{\frac{1}{2}}}, \frac{t_{ij}^{+}}{(\sum_{i=1}^{m}(t_{ij}^{-})^2+(t_{ij}^{+})^2)^{\frac{1}{2}}}], [\frac{i_{ij}^{-}}{(\sum_{i=1}^{m}(i_{ij}^{-})^2+(i_{ij}^{+})^2)^{\frac{1}{2}}}, \frac{i_{ij}^{+}}{(\sum_{i=1}^{m}(i_{ij}^{-})^2+(i_{ij}^{+})^2)^{\frac{1}{2}}}],$$

$$[\frac{f_{ij}^{-}}{(\sum_{i=1}^{m}(f_{ij}^{-})^2+(f_{ij}^{+})^2)^{\frac{1}{2}}}, \frac{f_{ij}^{+}}{(\sum_{i=1}^{m}(f_{ij}^{-})^2+(f_{ij}^{+})^2)^{\frac{1}{2}}}],$$

$$[\frac{t_{ij}}{(\sum_{i=1}^{m}(t_{ij})^2+(i_{ij})^2+(f_{ij})^2)^{\frac{1}{2}}}, \frac{i_{ij}}{(\sum_{i=1}^{m}(t_{ij})^2+(i_{ij})^2+(f_{ij})^2)^{\frac{1}{2}}}, \frac{f_{ij}}{(\sum_{i=1}^{m}(t_{ij})^2+(i_{ij})^2+(f_{ij})^2)^{\frac{1}{2}}}]>\}. \quad (4.14)$$

A conceptual model of the evolution of the neutrosophic cubic set is shown in Figure 4.1.

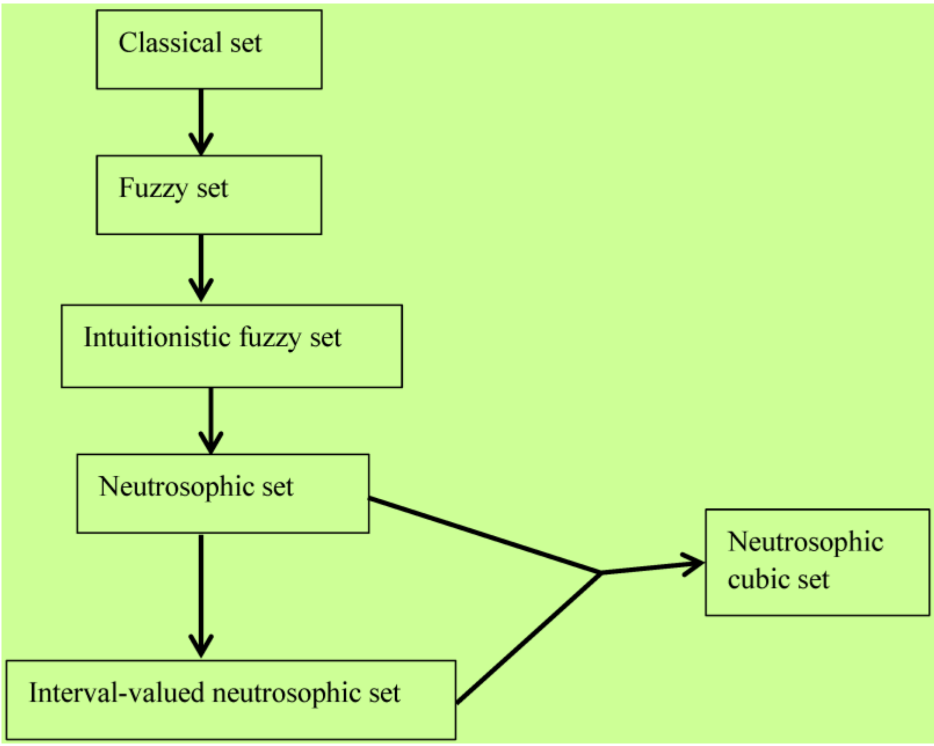

Figure 4.1 Evolution of the neutrosophic cubic set.

4.3 NC-TODIM based MAGDM in a NCS environment

Assume that A = $\{A_1, A_2, \ldots, A_m\}$ ($m \geq 2$) and C = $\{C_1, C_2, \ldots, C_n\}$ ($n \geq 2$) are the discrete set of alternatives and attributes respectively. W = $\{W_1, W_2, \ldots, W_n\}$ is the weight vector of attributes C_j ($j = 1, 2, \ldots, n$), where $W_j > 0$ and $\sum_{j=1}^{n} W_j = 1$. Let E = $\{E_1, E_2,$

…,E_r} be the set of decision makers and $\gamma=\{\gamma_1,\gamma_2,...,\gamma_r\}$ be the weight vector of decision makers, where $\gamma_k > 0$ and $\sum_{k=1}^{r}\gamma_k=1$.

NC-TODIM Strategy

Now, we describe the NC-TODIM strategy to solve the MAGDM problems with NC-numbers. The NC-TODIM strategy consists of the following steps:

Step 1. Formulate the decision matrix

Assume that $M^k = (©_{ij}^k)_{m\times n}$ be the decision matrix, where $©_{ij}^k = <\tilde{G}_{ij}^k, R_{ij}^k>$ is the rating value provided by the k-th ($E_{k)}$ decision maker for alternative A_i, with respect to attribute C_j. The matrix form of M^k is presented as:

$$M^k = \begin{pmatrix} & C_1 & C_2 & ... & C_n \\ A_1 & ©_{11}^k & ©_{12}^k & ... & ©_{1n}^k \\ A_2 & ©_{21}^k & ©_{22}^k & & ©_{2n}^k \\ . & . & . & . & . \\ A_m & ©_{m1}^k & ©_{m2}^k & ... & ©_{mnj}^k \end{pmatrix} \quad (4.15)$$

Step 2. Normalize the decision matrix

The MAGDM problem generally consists of cost criteria and benefit criteria. So, the decision matrix needs to be normalized. For cost criterion C_j, we use the Definition 4.4 to normalize the decision matrix (Equation (4.15)) provided by the decision makers. For benefit criterion C_j we don't need to normalize the decision matrix. When C_j is a cost criterion, the normalized form of decision matrix (see Equation (4.15)) is presented below.

$$M^{\otimes k} = \begin{pmatrix} & C_1 & C_2 & ... & C_n \\ A_1 & ©_{11}^{\otimes k} & ©_{12}^{\otimes k} & ... & ©_{1n}^{\otimes k} \\ A_2 & ©_{21}^{\otimes k} & ©_{22}^{\otimes k} & ... & ©_{2n}^{\otimes k} \\ . & . & . & ... & . \\ A_m & ©_{m1}^{\otimes k} & ©_{m2}^{\otimes k} & ... & ©_{mnj}^{\otimes k} \end{pmatrix} \quad (4.16)$$

Here $©_{ij}^{\otimes k}$ is the normalized form of the NC-number.

Step 3. Determine the relative weight of each criterion

The relative weight W_{ch} of each criterion is obtained by the following equation.

$$W_{ch} = \frac{W_C}{W_h} \tag{4.17}$$

where, W_h = max {W_1, W_2, …, W_n}.

Step 4. Calculate score values

Using Equation (4.1), calculate the score value $Sc(©_{ij}^{\otimes k})$ (i = 1, 2, …, m; j = 1, 2, …, n) of $©_{ij}^{\otimes k}$ for the cost criterion C_j. Using Equation (4.1), calculate the score value $Sc(©_{ij}^{k})$ (i = 1, 2, …, m; j = 1, 2, …, n) of $©_{ij}^{k}$ for the benefit criterion C_j.

Step 5. Calculate accuracy values

Using Equation (4.9), calculate the accuracy value $Ac(©_{ij}^{\otimes k})$ (I = 1, 2, …, m; j= 1, 2, …, n) of $©_{ij}^{\otimes k}$ for the cost criterion C_j. Using Equation (4.9), calculate the accuracy value $Ac(©_{ij}^{k})$ (I = 1, 2, …, m; j = 1, 2, …, n) of $©_{ij}^{k}$ for the benefit criterion C_j.

Step 6. Formulate the dominance matrix

Calculate the dominance of each alternative A_i over each alternative A_j with respect to the criteria C_1, C_2, …, C_n of the k-th decision maker E_k by the following Equation (4.18) and Equation (4.19).

(For cost criteria)

$$\left.\begin{aligned} \Psi_c^k(A_i, A_j) &= \sqrt{\left(\frac{W_{Ch}}{\sum_{c=1}^{n} W_{ch}}\partial(©_{ic}^{\otimes k}, ©_{jc}^{\otimes k}\right)}\;, \text{ if } ©_{ic}^{\otimes k} > ©_{jc}^{\otimes k} \\ &= 0 \qquad\qquad\qquad\qquad\qquad , \text{if } ©_{ic}^{\otimes k} = ©_{jc}^{\otimes k} \\ &= -\frac{1}{\alpha}\sqrt{\left(\frac{\sum_{c=1}^{n} W_{ch}}{W_{Ch}}\partial(©_{ic}^{\otimes k}, ©_{jc}^{\otimes k}\right)}\;, \text{ if } ©_{ic}^{\otimes k} < ©_{jc}^{\otimes k} \end{aligned}\right\} \tag{4.18}$$

(For benefit criteria)

$$\left.\begin{array}{l} \Psi_c^k(A_i, A_j) = \sqrt{\left(\frac{W_{Ch}}{\sum_{c=1}^{n} W_{ch}} \partial(©_{ic}^k, ©_{jc}^k)\right)}, \text{if } ©_{ic}^k > ©_{jc}^k \\ = 0 \qquad\qquad\qquad\qquad , \text{if } ©_{ic}^k = ©_{jc}^k \\ = -\frac{1}{\alpha}\sqrt{\left(\frac{\sum_{c=1}^{n} W_{ch}}{W_{Ch}} \partial(©_{ic}^k, , ©_{jc}^k)\right)}, \text{if } ©_{ic}^k < ©_{jc}^k \end{array}\right\} \tag{4.19}$$

where, parameter α represents the attenuation factor of losses and α must be positive.

Step 7. Formulatethe individual overall dominance matrix

Using Equation (4.20), calculate the individual total dominance matrix of each alternative A_i over each alternative A_j under the criterion C_j.

$$\varphi^k = (A_i, A_j) = \sum_{c=1}^{n} \Psi_c^k(A_i, A_j) \tag{4.20}$$

Step 8. Aggregate the dominance matrix

Using Equation (4.21), calculate the collective overall dominance of alternative A_i over each alternative A_j.

$$\varphi(A_i, A_j) = \sum_{k=1}^{m} \gamma_k \lambda^k(A_i, A_j) \tag{4.21}$$

Step 9. Calculate global values

We present the global value of each alternative as follows:

$$\Omega_i = \frac{\sum_{j=1}^{n} \varphi(A_i, A_j) - \min_{1\le i\le m}\left(\sum_{j=1}^{n} \varphi(A_i, A_j)\right)}{\max_{1\le i\le m}\left(\sum_{j=1}^{n} \varphi(A_i, A_j)\right) - \min_{1\le i\le m}\left(\sum_{j=1}^{n} \varphi(A_i, A_j)\right)} \tag{4.22}$$

Step 10. Rank the priority

Sorting the values of Ω_i provides the rank of each alternative. A set of alternatives can be preference-ranked according to the descending order of Ω_i. The highest global value corresponds to the best alternative.

A conceptual model of the NC-TODIM strategy is shown in Figure 4.2

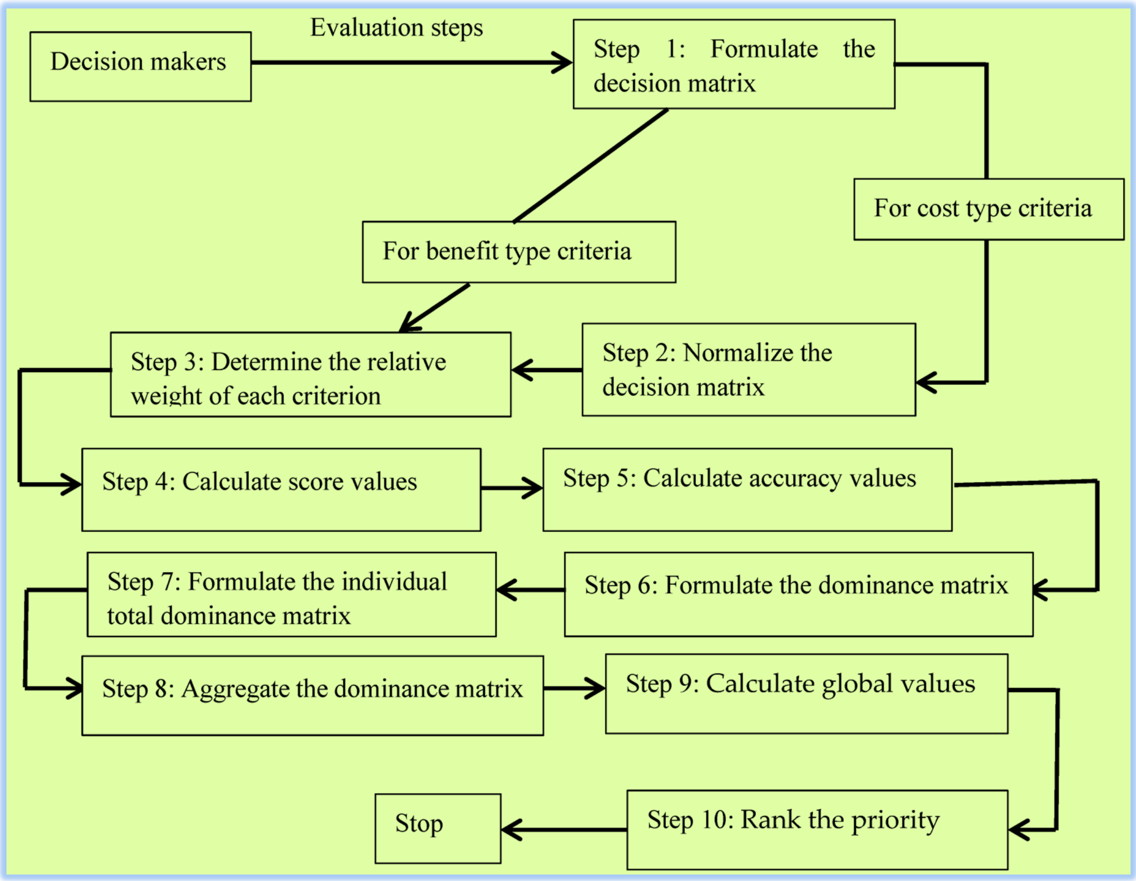

Figure 4.2 A flow chart of the proposed neutrosophic cubic set (NC)-TODIM strategy.

4.4 Illustrative example

In this section, an MAGDM problem is adapted from the study (He & Liu, 2013) in the NCS environment. An investment company wants to select the best alternative among the set of feasible alternatives. The feasible alternatives are

1. Car company (A_1)
2. Food company (A_2)
3. Computer company (A_3)
4. Arms company (A_4).

The best alternative is selected based on the following criteria:

1. Risk analysis (C_1)
2. Growth analysis (C_2)
3. Environmental impact analysis (C_3).

An investment company forms a panel of three decision makers $\{E_1, E_2, E_3\}$ who evaluate four alternatives in decision making process. The weight vector of attributes and decision makers are considered as $W = (0.4, 0.35, 0.25)^T$, $\gamma = (0.32, 0.33, 0.35)^T$ respectively.

The proposed strategy is presented using the following steps:

Step 1. Formulate the decision matrix

Formulate the decision matrices $M^k (k = 1, 2, 3)$ using the rating values of alternatives with respect to three criteria provided by the three decision makers in terms of NC-numbers. Assume that the NC-number $©_{ij}^k = <\tilde{G}_{ij}^k, R_{ij}^k>$ presents the rating value provided by the decision maker E_k for alternative A_i with respect to attribute C_j. Using these rating values $©_{ij}^k$ (k = 1, 2, 3; i = 1, 2, 3, 4; j = 1, 2, 3), three decision matrices $M^k = (©_{ij}^k)_{4\times3}$ (k = 1, 2, 3) are constructed (see Equations (4.23)–(4.25)).

Decision matrix for E_1

$M^1 =$

$$\begin{array}{cc} & \begin{array}{ccc} \qquad\qquad C_1 \qquad\qquad & \qquad\qquad C_2 \qquad\qquad & \qquad\qquad C_3 \qquad\qquad \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \left(\begin{array}{ccc} <[.41,.52],[.10,.18],[.06,.17],(.48,.11,.11)> & <[.40,.51],[.10,.20],[.10,.19],(.50,.11,.11)> & <[.22,.27],[.41,.52],[.41,.52],(.31,.31,.31)> \\ <[.35,.46],[.18,.27],[.17,.34],(.43,.16,.21)> & <[.22,.28],[.40,.50],[.39,.48],(.28,.28,.28)> & <[.38,.49],[.10,.21],[.10,.21],(.57,.12,.12)> \\ <[.23,.29],[.36,.45],[.34,.42],(.26,.26,.26)> & <[.34,.45],[.20,.30],[.19,.39],(.44,.16,.22)> & <[.22,.27],[.41,.52],[.41,.52],(.31,.31,.31)> \\ <[.17,.23],[.45,.55],[.42,.59],(.21,.32,.37)> & <[.22,.28],[.40,.50],[.39,.48],(.28,.28,.28)> & <[.38,.49],[.10,.21],[.10,.21],(.57,.12,.12)> \end{array}\right) \end{array} \tag{4.23}$$

Decision matrix for E_2

$M^2 =$

$$\begin{array}{cc} & \begin{array}{ccc} \qquad\qquad C_1 \qquad\qquad & \qquad\qquad C_2 \qquad\qquad & \qquad\qquad C_3 \qquad\qquad \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \left(\begin{array}{ccc} <[.17, .23], [.46, .55], [.42, .59], (.21, .32, .37)> & <[.25, .31], [.35, .44], [.35, .44], (.28, .28, .28)> & <[.34, .43], [.13, .27], [.13, .27], (.49, .11, .11)> \\ <[.23, .29], [.37, .46], [.34, .42], (.26, .26, .26)> & <[.25, .31], [.35, .44], [.35, .44], (.28, .28, .28)> & <[.34, .43], [.13, .27], [.13, .27], (.49, .11, .11)> \\ <[.41, .52], [.10, .18], [.10, .17], (.48, .11,.11)> & <[.44, .57], [.10, .17], [.10, .17], (.51, .11,.11)> & <[.19, .24], [.53, .67], [.53, .67], (.27, .27, .27)> \\ <[.35, .46], [.20, .28], [.17, .34], (.42, .16, .21)> & <[.25, .31], [.35, .44], [.35, .44], (.28, .28, .28)> & <[.34, .43], [.13, .27], [.13, .27], (.49, .11, .11)> \end{array}\right) \end{array} \tag{4.24}$$

Decision matrix for E_3

$$M^3 = \begin{array}{cc}
 & \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\
\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} &
\begin{pmatrix}
\langle [.22, .27], [.42, .52], [.42, .52], (.28, .28, .28)\rangle & \langle [.22, .28], [.40, .50], [.39, .48], (.28, .28, .28)\rangle & \langle [.41, .52], [.10, .18], [.10, .17], (.48, .11, .11)\rangle \\
\langle [.22, .27], [.42, .52], [.42, .52], (.28, .28, .28)\rangle & \langle [.40, .51], [.10, .20], [.10, .19], (.50, .11, .11)\rangle & \langle [.23, .29], [.36, .45], [.34, .42], (.26, .26, .26)\rangle \\
\langle [.38, .49], [.10, .21], [.10, .21], (.50, .11, .11)\rangle & \langle [.34, .45], [.20, .30], [.19, .39], (.44, .16, .22)\rangle & \langle [.38, .49], [.10, .21], [.10, .21], (.50, .11, .11)\rangle \\
\langle [.38, .49], [.10, .21], [.10, .21], (.50, .11, .11)\rangle & \langle [.22, .28], [.40, .50], [.39, .48], (.28, .28, .28)\rangle & \langle [.17, .23], [.45, .54], [.42, .59], (.21, .32, .37)\rangle
\end{pmatrix}
\end{array} \tag{4.25}$$

Step 2. Normalize the decision matrix

Since all the criteria are benefit type, we do not need to normalize the decision matrix.

Step 3. Determine the relative weight of each criterion

Using Equation (4.17), we obtain the relative weight vector W_{ch} of criteria as follows:

$W_{ch} = (1, 0.875, 0.625)^T$.

Step 4. Calculate score values

The score values of each alternative relative to each criterion obtained by Equation (4.1) are presented in the Tables (4.1)–(4.3).

Table 4.1 Score values for M1

	C_1	**C_2**	**C_3**
A_1	0.56	0.54	0.06
A_2	0.40	0.09	0.54
A_3	0.50	0.38	0.06
A_4	−0.03	0.09	0.54

Table 4.2 Score values for M^2

	C_1	**C_2**	**C_3**
A_1	−0.03	0.13	0.49
A_2	0.13	0.13	0.49
A_3	0.56	0.60	−0.04
A_4	0.39	0.13	0.49

Table 4.3 Score values for M^3

	C_1	**C_2**	**C_3**
A_1	0.07	0.09	0.56
A_2	0.07	0.52	0.13
A_3	0.51	0.37	0.39
A_4	0.51	0.09	−0.03

Step 5. Calculate accuracy values

The accuracy values of each alternative relative to each criterion are presented in Tables 4.4–4.6.

Table 4.4 Accuracy values for M^1

	C_1	C_2	C_3
A_1	0.14	0.30	−0.24
A_2	0.12	−0.23	0.32
A_3	−0.20	0.09	−0.24
A_4	−0.38	−0.23	0.32

Table 4.5 Accuracy values for M^2

	C_1	C_2	C_3
A_1	−0.38	−0.18	0.21
A_2	−0.20	−0.18	0.21
A_3	0.14	0.36	−0.21
A_4	0.12	−0.18	0.21

Table 4.6 Accuracy values for M^3

	C_1	C_2	C_3
A_1	−0.24	−0.23	0.41
A_2	−0.24	0.30	−0.20
A_3	0.26	0.09	0.12
A_4	0.26	−0.23	−0.38

Step 6. Formulate the dominance matrix

Using Equation (4.19), we construct dominance matrix for α = 1. The dominance matrices are represented in matrix form (See Equations (4.26)–(4.34)).

The dominance matrix Ψ_1^1

$$\Psi_1^1 = \begin{pmatrix} & A_1 & A_2 & A_3 & A_4 \\ A_1 & 0 & 0.18 & 0.30 & 0.35 \\ A_2 & -0.46 & 0 & -0.58 & 0.30 \\ A_3 & -0.74 & 0.23 & 0 & 0.19 \\ A_4 & -0.88 & -0.74 & -0.47 & 0 \end{pmatrix} \tag{4.26}$$

The dominance matrix Ψ_2^1

$$\Psi_2^1 = \begin{pmatrix} & A_1 & A_2 & A_3 & A_4 \\ A_1 & 0 & 0.29 & 0.18 & 0.28 \\ A_2 & -0.82 & 0 & -0.69 & 0 \\ A_3 & -0.51 & 0.24 & 0 & 0.29 \\ A_4 & -0.81 & 0 & -0.65 & 0 \end{pmatrix} \tag{4.27}$$

The dominance matrix Ψ_3^1

$$\Psi_3^1 = \begin{array}{cc} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0 & -1 & 0 & -1 \\ 0.25 & 0 & 0.26 & 0 \\ 0 & -1 & 0 & -1 \\ 0.25 & 0 & 0.26 & 0 \end{pmatrix} \end{array} \tag{4.28}$$

The dominance matrix Ψ_1^2

$$\Psi_1^2 = \begin{array}{cc} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0 & -0.46 & -0.88 & -0.74 \\ 0.18 & 0 & -0.75 & -0.58 \\ 0.35 & 0.09 & 0 & 0.04 \\ 0.30 & 0.23 & 0.19 & 0 \end{pmatrix} \end{array} \tag{4.29}$$

The dominance matrix Ψ_2^2

$$\Psi_2^2 = \begin{array}{cc} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0 & 0 & -0.84 & 0 \\ 0 & 0 & -0.84 & 0 \\ 0.29 & 0.29 & 0 & 0.29 \\ 0 & 0 & -0.84 & 0 \end{pmatrix} \end{array} \tag{4.30}$$

The dominance matrix Ψ_3^2

$$\Psi_3^2 = \begin{array}{cc} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0 & 0 & 0.26 & 0 \\ 0 & 0 & 0.26 & 0 \\ -1 & -1 & 0 & -1 \\ 0 & 0 & 0.26 & 0 \end{pmatrix} \end{array} \tag{4.31}$$

The dominance matrix Ψ_1^3

$$\Psi_1^3 = \begin{array}{cc} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0 & 0 & -0.78 & -0.78 \\ 0 & 0 & -0.78 & -0.78 \\ 0.31 & 0.31 & 0 & 0 \\ 0.31 & 0.31 & 0 & 0 \end{pmatrix} \end{array} \tag{4.32}$$

The dominance matrix Ψ_2^3

$$\Psi_2^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} 0 & -0.83 & -0.65 & 0 \\ 0.29 & 0 & 0.18 & 0.29 \\ 0.23 & -0.51 & 0 & 0.23 \\ 0 & -0.83 & -0.65 & 0 \end{pmatrix} \end{array} \tag{4.33}$$

The dominance matrix Ψ_3^3

$$\Psi_3^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} 0 & -0.94 & -0.59 & -1.1 \\ 0.23 & 0 & -0.73 & 0.15 \\ -0.59 & 0.18 & 0 & 0.23 \\ -1.1 & -0.58 & -0.94 & 0 \end{pmatrix} \end{array} \tag{4.34}$$

Step 7. Formulate the individual overall dominance matrix

The individual overall dominance matrix is calculated by the Equation (4.20) and the dominance matrices are represented in matrix form (see Equations (4.35)–(4.37)).

First decision maker's overall dominance matrix φ^1

$$\varphi^1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} 0 & -0.53 & 0.47 & -0.37 \\ -1 & 0 & -1 & 0.30 \\ -1.3 & -0.53 & 0 & -0.52 \\ -1.5 & -0.74 & -0.86 & 0 \end{pmatrix} \end{array} \tag{4.35}$$

Second decision maker's overall dominance matrix φ^2

$$\varphi^2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} 0 & -0.46 & -1.5 & -0.74 \\ 0.18 & 0 & -1.3 & -0.58 \\ -0.36 & -0.62 & 0 & -0.67 \\ 0.30 & 0.23 & -0.39 & 0 \end{pmatrix} \end{array} \tag{4.36}$$

Third decision maker's overall dominance matrix φ^3

$$\varphi^3 = \begin{pmatrix} & A_1 & A_2 & A_3 & A_4 \\ A_1 & 0 & -1.8 & -2 & -1.9 \\ A_2 & 0.52 & 0 & -1.3 & -0.34 \\ A_3 & -0.05 & -0.02 & 0 & 0.46 \\ A_4 & -0.79 & .-1.1 & -1.6 & 0 \end{pmatrix} \quad (4.37)$$

Step 8. Aggregate the dominance matrix

Using Equation (4.21), the aggregate dominance matrix φ is constructed (see Equation (4.38)) as follows:

$$\varphi = \begin{pmatrix} & A_1 & A_2 & A_3 & A_4 \\ A_1 & 0 & -0.94 & -1.1 & -0.53 \\ A_2 & -0.10 & 0 & -1.23 & -0.22 \\ A_3 & -0.54 & -0.38 & 0 & -0.23 \\ A_4 & -0.64 & .-0.55 & -0.96 & 0 \end{pmatrix} \quad (4.38)$$

Step 9. Calculate global values

Using Equation (4.22), we calculate the values of Ω_i (i = 1, 2, 3, 4) and represented in Table 4.7.

Table 4.7 Global values of alternatives

A_i	A_1	A_2	A_3	A_4
Ω_i	0.49	0.61	1	0

Step 10. Rank the priority

Since $\Omega_3 > \Omega_2 > \Omega_1 > \Omega_4$, alternatives are then preference ranked as follows: $A_3 > A_2 > A_1 > A_4$.

Hence A_3 is the best alternative.

From the illustrative example, we see that the proposed NC-TODIM strategy is more suitable for real scientific and engineering applications because it can handle hybrid information consisting of INS and SVNS information simultaneously to cope with indeterminate and inconsistent information. Thus, NC-TODIM extends the existing decision-making strategies and provides a sophisticated mathematical tool for decision makers.

4.5 Rank of alternatives with different values of α

Table 4.8 shows that the ranking order of alternatives depends on the values of the attenuation factor, which reflects the importance of the attenuation factor in the NC-TODIM strategy.

Table 4.8 Global values and ranking of alternatives for different values of α

Values of α	Global Values of Alternative (Ω_i)	Rank Order of A_i
0.5	$\Omega_1 = 0$, $\Omega_2 = 0.89$, $\Omega_3 = 1$, $\Omega_4 = 0.46$ $\Omega_3 > \Omega_2 > \Omega_4 > \Omega_1$	$A_3 > A_2 > A_4 > A_1$
1	$\Omega_1 = 0.49$, $\Omega_2 = 0.61$, $\Omega_3 = 1$, $\Omega_4 = 0$ $\Omega_3 > \Omega_2 > \Omega_1 > \Omega_4$	$A_3 > A_2 > A_1 > A_4$
1.5	$\Omega_1 = 0$, $\Omega_2 = 0.72$, $\Omega_3 = 1$, $\Omega_4 = 0.44$ $\Omega_3 > \Omega_2 > \Omega_4 > \Omega_1$	$A_3 > A_2 > A_4 > A_1$
2	$\Omega_1 = 0$, $\Omega_2 = 1$, $\Omega_3 = 0.81$, $\Omega_4 = 0.38$ $\Omega_2 > \Omega_3 > \Omega_4 > \Omega_1$	$A_2 > A_3 > A_4 > A_1$
3	$\Omega_1 = 0$, $\Omega_2 = 0.56$, $\Omega_3 = 1$, $\Omega_4 = 0.45$ $\Omega_3 > \Omega_2 > \Omega_4 > \Omega_1$	$A_3 > A_2 > A_4 > A_1$

4.6 Analysis on influence of the parameter α to ranking order

The impact of parameter α on ranking order is examined by comparing the ranking orders taken with varying the different values of α. When α = 0.5, 1, 1.5, 2, 3, ranking order are presented in Table 8. We draw Figures 3 and 4 to compare the ranking order for different values of α. When α = 0.5, α = 1.5 and α = 3, the ranking order is unchanged and A_3 is the best alternative, while A_1 is the worst alternative. When α = 1, the ranking order is changed and A_3 is the best alternative and A_4 is the worst alternative. For α = 2, the ranking order is changed and A_2 is the best alternative and A_1 is the worst alternative. From Table 4.8, we see that A_3 is the best alternative in four cases and A_1 is the worst for four cases.

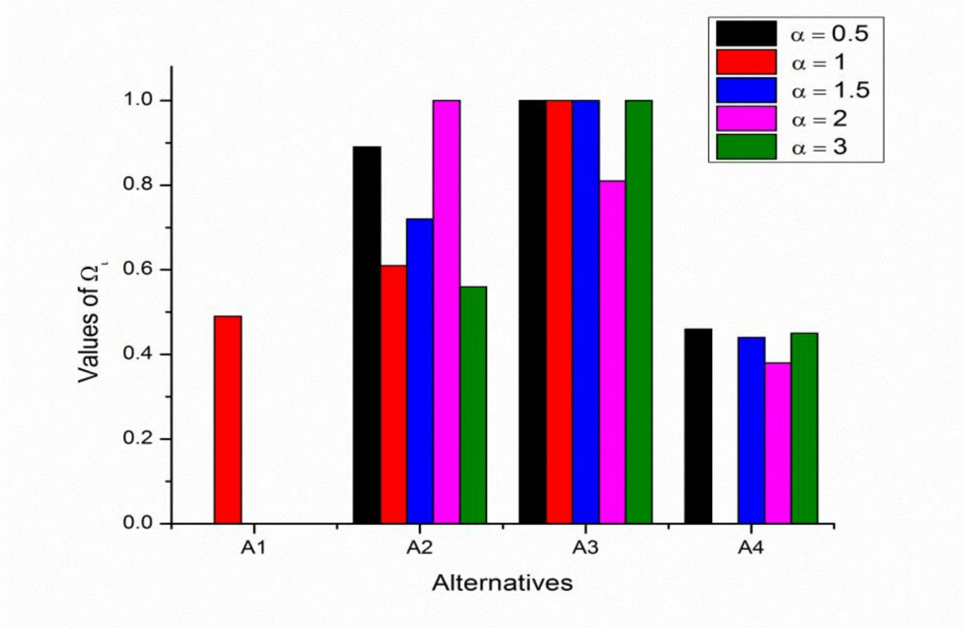

Figure 4.3 Global values of the alternatives for different values of attenuation factor α = 0.5, 1, 1.5, 2, 3.

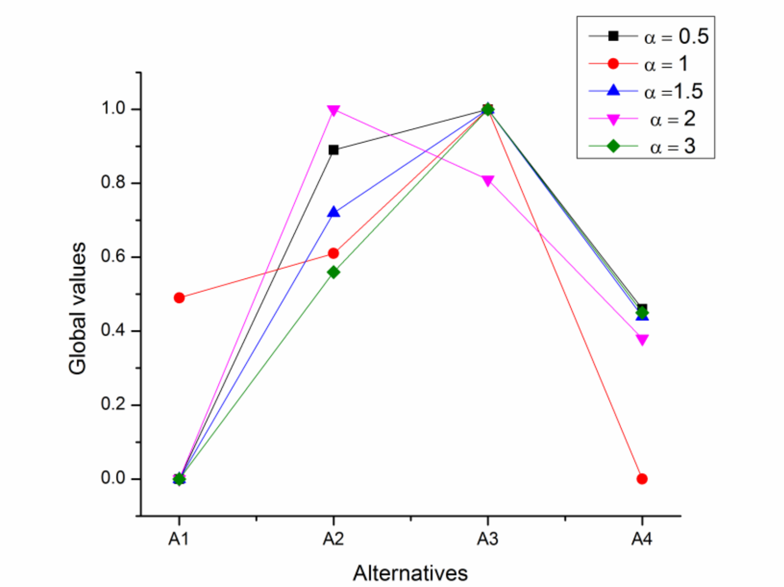

Figure 4.4 Ranking of the alternatives for α = 0.5, 1, 1.5, 2, 3.

4.7 Comparative analysis and discussion

On comparing with the existing neutrosophic decision making strategies (Pramanik, Biswas et al., 2017; Sahin, and Küçük, 2015; Ye, 2015a; Biswas et al. 2016a,; Sahin, and Liu, 2016, 2017; Sahin, 2017; Xu et al., 2017; Liu, and Wang, 2014, 2016; Liu, and Tang, 2016;Liu, 2016; Liu et al., 2014) we see that the decision information used in the proposed NC-TODIM strategy is NC numbers, which comprises of interval neutrosophic information and single-valued neutrosophic information simultaneously; whereas the decision information in the existing literature is either SVNSs or INSs. Since NC numbers comprises of much more information, the NC numbers based on the TODIM strategy proposed in this chapter is more elegant, typical and more general in applications, while the existing neutrosophic decision-making strategies cannot deal with the NC number decision-making problem.

The first decision making paper in NCS environment was studied by Banerjee et al. (2017). On comparison with existing GRA-based NCS decision making strategies (Banerjee et al., 2017), we observe that the proposed NC-TODIM strategy uses the score, and accuracy functions, while the decision making-strategy in (Banerjee et al., 2017) uses Hamming distances for weighted grey relational coefficients and standard (ideal) grey relational coefficients, and ranks the alternatives based on the relative closeness coefficients. Hence, the proposed NC-TODIM strategy is relatively simple in the decision making process.

The decision making strategy proposed by Lu, and Ye (2017) cannot deal with group decision makers while the proposed NC-TODIM strategy is more sophisticated as it can deal with single as well as group decision makers in the NCS environment.

On comparison with extended TOPSIS (Pramanik, Dey et al., 2017) with neutrosophic cubic information, we observe that nine components involve. Therefore, calculation of a weighted decision matrix, a neutrosophic cubic positive ideal solution (NCPIS), and a neutrosophic cubic negative ideal solution, the distance measures of alternatives from NCPIS and NCNIS (NCNIS,) and entropy weight, and use of an aggregation operator are lengthy, time consuming, and hence expensive. The proposed NC-TODIM strategy is free from different kinds of typical aggregation operators. The calculations required for the proposed strategy are relatively straightforward and time-saving. Therefore, the final ranking obtained by the proposed strategy is more conclusive than those produced by the other strategies, and it is evident that the proposed astrategy is accurate and reliable.

On comparison with the strategy proposed by Zhan et al. (2017), we see that they employ score, accuracy, and certainty functions, and a weighted average operator and weighted geometric operator of NCSs for decision making problem involving only a single decision maker. This reflects that the strategy introduced by Zhan et al. (2017) is only applicable for decision making problems involving single decision maker.

However, our proposed NC-TODIM strategy is more general as it is capable of dealing with group decision-making problems.

A comparative study is conducted with the existing strategy (Pramanik, Dalapati, Alam, Roy et al., 2017) for group decision making under a NCS environment (See Table 4.9). Since the philosophy oftwo strategies are different, the obtained results (ranking order) are different. At a glance, it cannot be said which strategy is superior to the other. However, on comparison with similarity measure-based strategy studied in (Pramanik, Dalapati, Alam, Roy et al., 2017), we observed that ideal solutions are needed for ranking of alternatives but in a real world ideal solution, this is an imaginary case, which means that an indeterminacy arises automatically, whereas in our proposed NC-TODIM strategy we can calculate the rank of the alternatives based on global values of alternatives. So, the proposed NC-TODIM strategy is relatively easy to implement and apply for solving MAGDM problems.

Table 4.9 Ranking order of alternatives using three different decision making strategies in the neutrosophic cubic set (NCS) environment.

Proposed NC-TODIM Strategy	**Similarity Measure** (Pramanik, Dalapati, Alam, Roy et al., 2017)
$\Omega_1 = 0$, $\Omega_2 = 0.89$, $\Omega_3 = 1$, $\Omega_4 = 0.46$	$\rho_1 = 0.20$, $\rho_2 = 0.80$, $\rho_3 = 0.22$, $\rho_4 = 0.19$
Ranking order: $A_3 > A_2 > A_4 > A_1$	Ranking order: $A_2 > A_3 > A_1 > A_4$

4.8 Conclusion

In this study, we proposed a score function and an accuracy function, and established their properties. We developed a NC-TODIM strategy, which is capable for tackling MAGDM problems affected by uncertainty and indeterminacy represented by NC numbers. The standard TODIM, in its original formulation, is only applicable to a crisp environment. Existing neutrosophic TODIM strategies deal with single valued neutrosophic information or interval neutrosophic information. Therefore, proposed NC-TODIM strategy demonstrates the advantages of presenting and manipulating MAGDM problems with NCSs comprising of the hybrid information of INSs and NSs. Furthermore, NC-TODIM strategy that considers the risk preferences of decisionmakers is significant to solve MAGDM problems. The proposed NC-TODIM strategy is verified to be applicable, feasible, and effective by solving an illustrative example regarding the selection problem of investment alternatives. In addition, we investigate the influence of attenuation factor of losses α on ranking the order of alternatives.

The contribution of this study can be concluded as follows. First, this study utilized NCSs to present the interval neutrosophic information and neutrosophic information in the MAGDM process. Second, the NC-TODIM strategy established in this chapter is simpler and easier than the existing strategy proposed by Pramanik, Dalapati, Alam, Roy et al. (2017) for group decision making with neutrosophic cubic information based on similarity measure and demonstrates the main advantage of its simple and easy group decision making process. Third, TODIM strategy has been extended to the NCS environment.Fourth, we defined the NC number. Fifth, we defined the score and accuracy functions and proved their basic properties. Sixth, we developed the ranking of NC numbers using score and accuracy functions. Therefore, two functions namely, score function, accuracy function, and proofs of their basic properties, ranking of NC numbers, and NC-TODIM strategy for MAGDM are the main contributions of the chapter. Several directions for future research are generated from this study. First, this study employs the NC-TODIM strategy to deal with MAGDM. In addition to MAGDM, MAGDM problems in a variety of other fields can be solved using the NC-TODIM strategy, including logistics center selection, personnel selection, teacher selection, renewable energy selection, medical diagnosis, image processing, fault diagnosis, etc. Second, this study considers the risk preferences of decision makers i.e., the essence of TODIM, while the interrelationship between criteria are ignored. In future research, the NC-TODIM strategy will be improved to address this deficiency. Third, the proposed strategy can only deal with crisp weights of attributes and decision makers, rather than NCS, which reflects its main limitation. This limitation will be effectively addressed in our future research. Fourth, in our illustrative example, three criteria are considered as an example. However, in real world group decision making problems, many other criteria should be included. A comprehensive framework for MAGDM problem comprising of all relevant criteria should be designed based on prior studies and the proposed NC-TODIM strategy in future research. Finally, we conclude that the developed NC-TODIM strategy offers a novel and effective strategy for decision makers under the NCS environment, and will open up a new avenue of research into the neutrosophic hybrid environment.

Chapter 5

Neutrosophic cubic MAGDM method based on similarity measure

5.1 Introduction

Similarity measure is a vital topic in fuzzy set theory, Chen and Hsiao (1995) presented comparisons of similarity measures of fuzzy sets. Pramanik and Mondal (2015a) studied weighted fuzzy similarity measure based on tangent function for medical diagnosis. Hwang and Yang (2013) constructed a new similarity measure between intuitionistic fuzzy sets based on lower, upper and middle fuzzy sets. Mondal and Pramanik (2015a) developed tangent similarity measures in intuitionistic fuzzy environment to deal with medical diagnosis. Ren and Wang (2015) proposed similarity measures in interval- valued intuitionistic fuzzy environment and applied it to MADM problems. Baccour et al. (2013) presented survey of similarity measures for intuitionistic fuzzy sets. Baroumi and Smarandache (2013b) discussed several similarity measures of neutrosophic sets. Majumdar and Samanta (2014) introduced some measures of similarity and entropy of single valued neutrosophic sets. Aydogdu (2015a) proposed similarity and entropy measure of single valued neutrosophic sets. Mondal and Pramanik (2015c) extended the concept of intuitionistic tangent similarity measure to neutrosophic environment. Biswas et al. (2015) studied cosine similarity measure with trapezoidal fuzzy neutrosophic numbersto deal with MADM problems.

The content of this chapter is based on the paper published in**“Neutrosophic Sets and Systems**” 16, 44-56, 2017.

Aydogdu (2015b) also defined entropy and similarity measures of interval neutrosophic sets.Ye (2014a) proposed a similarity measures under interval neutrosophic domain using Hamming distance and Euclidean distance. Pramanik and Mondal (2015b) proposed cosine similarity measure of rough neutrosophic set and applied it to medical diagnosis problems. Pramanik and Mondal (2015c) developed cotangent similarity measure of rough neutrosophic sets to deal with medical diagnosis.

In neutrosophic cubic set environment, similarity measure is yet to appear. To fill the research gap, we define similarity measures in neutrosophic cubic set environment and develop an MAGDM strategy in neutrosophic cubic set setting. The decision makers' weights and criteria (attributes) weights are described by neutrosophic cubic numbers using linguistic variables. The ranking of alternatives is presented in descending order. Finally, anillustrative numerical example MAGDMproblem in neutrosophic cubic set environment is solved to show the effectiveness of the proposed strategy.

Rest of the chapter is presented as follows. Section 5.2devotes to define similarity measure for neutrosophic cubic sets and prove their basic properties. Section 5.3 presents a MAGDM strategy based on similarity measure in neutrosophic cubic set environment. Section 5.4 presents a numerical example for a MAGDM problem. Finally, section 5.5 presents conclusion and future scope of research.

5.2 Similarity measure of NCS

We define similarity measure for neutrosophic cubic set.

Definition 5.1

Let Q_1 and Q_2 be two NCSs in G. Similarity measure for $Q_1$and Q_2 is defined as a mapping

SM: NCS (G) × NCS (G) $\rightarrow$ [0, 1] that satisfies the following properties:

5.1 $0 \leq SM(Q_1, Q_2) \leq 1$

5.2 $SM(Q_1, Q_2) = 1$ iff $Q_1 = Q_2$

5.3 $SM(Q_1, Q_2) = SM(Q_2, Q_1)$

5.4 If $Q_1 \subseteq Q_2 \subseteq Q_3$ then $SM(Q_1, Q_3) \leq SM(Q_1, Q_2)$ and $SM(Q_1, Q_3) \leq SM(Q_2, Q_3)$ for all $Q_1, Q_2, Q_3 \in$ NCS (G).

Similarity measure for two NCSs Q_1 and Q_2 expressed as

$$SM(Q_1, Q_2) = \frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{D_i}{9}\right),$$

where $D_i = \Big(\left| t^{-}_{\tilde{G}_1}(g_i) - t^{-}_{\tilde{G}_2}(g_i) \right| + \left| t^{+}_{\tilde{G}_1}(g_i) - t^{+}_{\tilde{G}_2}(g_i) \right| + \left| i^{-}_{\tilde{G}_1}(g_i) - i^{-}_{\tilde{G}_2}(g_i) \right| + \left| i^{+}_{\tilde{G}_1}(g_i) - i^{+}_{\tilde{G}_2}(g_i) \right| + \left| f^{-}_{\tilde{G}_1}(g_i) - f^{-}_{\tilde{G}_2}(g_i) \right| + \left| f^{+}_{\tilde{G}_1}(g_i) - f^{+}_{\tilde{G}_2}(g_i) \right| + \left| t_{R_1}(g_i) - t_{R_2}(g_i) \right| + \left| i_{R_1}(g_i) - i_{R_2}(g_i) \right| + \left| f_{R_1}(g_i) - f_{R_2}(g_i) \right| \Big)$.

We now prove that the similarity measure satisfies the four stated properties:

Property 5.1

$0 \leq SM(Q_1, Q_2) \leq 1$

Proof: If D_i has extreme value i.e. $D_i = 0$ or 9, then $SM(Q_1, Q_2) = 1$ or 0 (5.1)

If D_i lies between 0 and 9 i.e $0<D_i<9$, then $0<\frac{D_i}{9}<1$

$\Rightarrow 0> -\frac{D_i}{9} > -1$

Adding 1 each part of the above inequality, we obtain

$0< 1-\frac{D_i}{9}<1$

$\frac{1}{n}\sum_{i=1}^{n} 0 < \frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{D_i}{9}\right) < \frac{1}{n}\sum_{i=1}^{n} 1 = 1 \Rightarrow 0 < \frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{D_i}{9}\right) < 1$

$\Rightarrow 0<SM(Q_1, Q_2)<1$ (5.2)

Combining (5.1) and (5.2), we get $0 \leq SM(Q_1, Q_2) \leq 1$

Property 5.2

$SM(Q_1, Q_2) = 1$ iff $Q_1 = Q_2$

Proof:

If $Q_1 = Q_2$, then $D_i = 0$ by the definition of equality.

$SM(Q_1, Q_2) = \frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{D_i}{9}\right) = 1.\square$

Property 5.3

SM (Q_1, Q_2) = SM (Q_2, Q_1)

Proof: SM (Q_1, Q_2) = $\frac{1}{n}\sum_{i=1}^{n}(1-\frac{D_i}{9})$,

where $D_i(Q_1, Q_2) = (\left| t^{-}_{\tilde{G}_1}(g_i) - t^{-}_{\tilde{G}_2}(g_i)\right| + \left| t^{+}_{\tilde{G}_1}(g_i) - t^{+}_{\tilde{G}_2}(g_i)\right| + \left| i^{-}_{\tilde{G}_1}(g_i) - i^{-}_{\tilde{G}_2}(g_i)\right| + \left| i^{+}_{\tilde{G}_1}(g_i) - i^{+}_{\tilde{G}_2}(g_i)\right| + \left| f^{-}_{\tilde{G}_1}(g_i) - f^{-}_{\tilde{G}_2}(g_i)\right| + \left| f^{+}_{\tilde{G}_1}(g_i) - f^{+}_{\tilde{G}_2}(g_i)\right| + \left| t_{R_1}(g_i) - t_{R_2}(g_i)\right| + \left| i_{R_1}(g_i) - i_{R_2}(g_i)\right| + \left| f_{R_1}(g_i) - f_{R_2}(g_i)\right|)$

since, $\left| t^{-}_{\tilde{G}_1}(g_i) - t^{-}_{\tilde{G}_2}(g_i)\right| = \left| t^{-}_{\tilde{G}_2}(g_i) - t^{-}_{\tilde{G}_1}(g_i)\right|$, $\left| t^{+}_{\tilde{G}_1}(g_i) - t^{+}_{\tilde{G}_2}(g_i)\right| = \left| t^{+}_{\tilde{G}_2}(g_i) - t^{+}_{\tilde{G}_1}(g_i)\right|$, $\left| i^{-}_{\tilde{G}_1}(g_i) - i^{-}_{\tilde{G}_2}(g_i)\right| = \left| i^{-}_{\tilde{G}_2}(g_i) - i^{-}_{\tilde{G}_1}(g_i)\right|$, $\left| i^{+}_{\tilde{G}_1}(g_i) - i^{+}_{\tilde{G}_2}(g_i)\right| = \left| i^{+}_{\tilde{G}_2}(g_i) - i^{+}_{\tilde{G}_1}(g_i)\right|$, $\left| f^{-}_{\tilde{G}_1}(g_i) - f^{-}_{\tilde{G}_2}(g_i)\right| = \left| f^{-}_{\tilde{G}_2}(g_i) - f^{-}_{\tilde{G}_1}(g_i)\right|$, $\left| f^{+}_{\tilde{G}_1}(g_i) - f^{+}_{\tilde{G}_2}(g_i)\right| = \left| f^{+}_{\tilde{G}_2}(g_i) - f^{+}_{\tilde{G}_1}(g_i)\right|$, $\left| t_{R_1}(g_i) - t_{R_2}(g_i)\right| = \left| t_{R_2}(g_i) - t_{R_1}(g_i)\right|$, $\left| i_{R_1}(g_i) - i_{R_2}(g_i)\right| = \left| i_{R_2}(g_i) - i_{R_1}(g_i)\right|$, $\left| f_{R_1}(g_i) - f_{R_2}(g_i)\right| = \left| f_{R_2}(g_i) - f_{R_1}(g_i)\right|$.

$\Rightarrow D_i(Q_1, Q_2) = D_i(Q_2, Q_1)$

Therefore, SM (Q_1, Q_2) = SM (Q_2, Q_1).☐

Property 5.4

If $Q_1 \subseteq Q_2 \subseteq Q_3$, then SM (Q_1, Q_3) $\leq$ SM (Q_1, Q_2) and SM (Q_1, Q_3) $\leq$ SM (Q_2, Q_3) for all $Q_1, Q_2, Q_3 \in$ NCS (G).

Proof:

Let $Q_1 \subseteq Q_2 \subseteq Q_3$ then,

$$t^{-}_{\tilde{G}_1}(g_i) \leq t^{-}_{\tilde{G}_2}(g_i) \leq t^{-}_{\tilde{G}_3}(g_i),\ t^{+}_{\tilde{G}_1}(g_i) \leq t^{+}_{\tilde{G}_2}(g_i) \leq t^{+}_{\tilde{G}_3}(g_i),\ i^{-}_{\tilde{G}_1}(g_i) \leq i^{-}_{\tilde{G}_2}(g_i) \leq i^{-}_{\tilde{G}_3}(g_i)$$
$$i^{+}_{\tilde{G}_1}(g_i) \leq i^{+}_{\tilde{G}_2}(g_i), \leq i^{+}_{\tilde{G}_3}(g_i),\ f^{-}_{\tilde{G}_1}(g_i) \geq f^{-}_{\tilde{G}_2}(g_i) \geq f^{-}_{\tilde{G}_3}(g_i), f^{+}_{\tilde{G}_1}(g_i) \geq f^{+}_{\tilde{G}_2}(g_i) \geq f^{+}_{\tilde{G}_3}(g_i)$$
$$t_{R_1}(g_i) \leq t_{R_2}(g_i) \leq t_{R_3}(g_i), i_{R_1}(g_i) \leq i_{R_2}(g_i) \leq i_{R_3}(g_i), f_{R_1}(g_i) \geq f_{R_2}(g_i) \geq f_{R_3}(g_i) \tag{5.3}$$

Now $D_i(Q_1, Q_2) = (\left| t^{-}_{\tilde{G}_1}(g_i) - t^{-}_{\tilde{G}_2}(g_i)\right| + \left| t^{+}_{\tilde{G}_1}(g_i) - t^{+}_{\tilde{G}_2}(g_i)\right| + \left| i^{-}_{\tilde{G}_1}(g_i) - i^{-}_{\tilde{G}_2}(g_i)\right| + \left| i^{+}_{\tilde{G}_1}(g_i) - i^{+}_{\tilde{G}_2}(g_i)\right| + \left| f^{-}_{\tilde{G}_1}(g_i) - f^{-}_{\tilde{G}_2}(g_i)\right| + \left| f^{+}_{\tilde{G}_1}(g_i) - f^{+}_{\tilde{G}_2}(g_i)\right| + \left| t_{R_1}(g_i) - t_{R_2}(g_i)\right| + \left| i_{R_1}(g_i) - i_{R_2}(g_i)\right| + \left| f_{R_1}(g_i) - f_{R_2}(g_i)\right|)$

And $D_i(Q_1, Q_3) = (\,|\, t^{-}_{\tilde{G}_1}(g_i) - t^{-}_{\tilde{G}_3}(g_i) \,|+|\, t^{+}_{\tilde{G}_1}(g_i) - t^{+}_{\tilde{G}_3}(g_i) \,|+|\, i^{-}_{\tilde{G}_1}(g_i) - i^{-}_{\tilde{G}_3}(g_i) \,|+|\, i^{+}_{\tilde{G}_1}(g_i) - i^{+}_{\tilde{G}_3}(g_i) \,|+|\, f^{-}_{\tilde{G}_1}(g_i) - f^{-}_{\tilde{G}_3}(g_i) \,|+|\, f^{+}_{\tilde{G}_1}(g_i) - f^{+}_{\tilde{G}_3}(g_i) \,|+|\, t_{R_1}(g_i) - t_{R_2}(g_i) \,|+|\, i_{R_1}(g_i) - i_{R_3}(g_i) \,|+|\, f_{R_1}(g_i) - f_{R_3}(g_i) \,|\,)$

From (5.3), we conclude that

$D_i(Q_1, Q_3) \geq D_i(Q_1, Q_2)$

$\Rightarrow \dfrac{D_i(Q_1,Q_3)}{9} \geq \dfrac{D_i(Q_1,Q_2)}{9}$

$\Rightarrow -\dfrac{D_i(Q_1,Q_3)}{9} \leq -\dfrac{D_i(Q_1,Q_2)}{9}$

$\Rightarrow [1-\dfrac{D_i(Q_1,Q_3)}{9}] \leq [1-\dfrac{D_i(Q_1,Q_2)}{9}]$

$\Rightarrow \dfrac{1}{n}\sum_{i=1}^{n}[1-\dfrac{D_i(Q_1,Q_3)}{9}] \leq \dfrac{1}{n}\sum_{i=1}^{n}[1-\dfrac{D_i(Q_1,Q_3)}{9}]$

$\Rightarrow SM(Q_1, Q_3) \leq SM(Q_1, Q_2)$

Similarly, we can shows that $SM(Q_1, Q_3) \leq SM(Q_2, Q_3)$.

This completes the proof. ☐

5.3 MAGDM strategy based on similarity measure in NCS environment

In this section we propose a new MAGDM strategy based on similarity measure in NCS environment. Assume that $\alpha = \{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n\}$ be a set of n alternatives with criteria $\beta = \{\beta_1, \beta_2, \beta_3, ..., \beta_m\}$ and $E = \{E_1, E_2, E_3, ..., E_r\}$ be the r Experts/decision makers. Let $\Psi = \{\Psi_1, \Psi_2, \Psi_3, ..., \Psi_r\}$ be the weight vector of decision makers, where $\Psi_k > 0$ and $\sum_{k=1}^{r} \Psi_k = 1$. Proposed MAGDM strategy is presented using the following steps:

Step1. Formation of ideal NCS decision matrix

Ideal NCS decision matrix is an important matrix for similarity measure of MAGDM. Here we construct an ideal NCS matrix in the form

$$M=\begin{pmatrix} & \beta_1 & \beta_2 & \dots\dots & \beta_m \\ \alpha_1 & Q_{11} & Q_{12} & \dots & Q_{1m} \\ \alpha_2 & Q_{21} & Q_{22} & & Q_{2m} \\ . & . & \dots & . & \\ \alpha_n & Q_{n1} & Q_{n2} & \dots & .Q_{nm} \end{pmatrix} \tag{5.4}$$

Where, $Q_{ij} = < [1,1],[0,0],[0,0],(1,0,0) >$ for benefit type attributes and $Q_{ij} = < [0,0],[1,1],[1,1],(0,1,1) >$ for cost type attributes, (i = 1, 2, 3, …,n; j = 1, 2, 3, …, m)

Step 2. Construction of NCS decision matrix

Since r decision makers are involved in the decision making process, the k-th (k = 1, 2, 3,…, r) decision maker provides the evaluation information of the alternative α_i (i= 1, 2, 3,…, n) with respect to criterion β_j (j= 1, 2, 3,…, m) in terms of the neutrosophic cubic numbers (NCNs). The k-th decision matrix denoted by M^k (See Equation (5.5)) is constructed as follows:

$$M^k=< Q^k_{ij} >=\begin{pmatrix} & \beta_1 & \beta_2 & \dots\dots & \beta_m \\ \alpha_1 & Q^k_{11} & Q^k_{12} & \dots & Q^k_{1m} \\ \alpha_2 & Q^k_{21} & Q^k_{22} & & Q^k_{2m} \\ . & . & \dots & . & \\ \alpha_n & Q^k_{n1} & Q^k_{n2} & \dots & .Q^k_{nm} \end{pmatrix} \tag{5.5}$$

where k = 1, 2, 3,…, r. i = 1, 2, 3,…, n. j = 1, 2, 3,…, m.

Step 3. Determination of attribute weight

Every decision maker provides their own opinion regarding to the attribute weight in terms of linguistic variables that can be converted into NCNs. Let $w_k(\beta_j)$ be the attribute weight for the attribute β_j given by the k-th decision maker in term of NCS. We convert $w_k(\beta)$ into fuzzy number as follows:

$$w^F_k(\beta_j) = \begin{Bmatrix} (1-\sqrt{\dfrac{V_{kj}}{9}}), \text{if } \beta_j \in \beta \\ 0 \qquad\qquad \text{otherwise} \end{Bmatrix} \tag{5.6}$$

where $V_{kj} = \sqrt{\left\{\begin{array}{c}(1-t_k^-(\beta_j))^2+(1-t_k^+(\beta_j))^2+(i_k^-(\beta_j))^2+(i_k^+(\beta_j))^2 \\ +(f_k^-(\beta_j))^2+(f_k^+(\beta_j))^2+(1-t_k(\beta_j))^2 \\ +(i_k(\beta_j))^2+(f_k(\beta_j))^2\end{array}\right\}}$

Then aggregate weight for the criterion β_j can be determined as:

$$W_j = \frac{(1-\prod_{k=1}^{r}(1-w_k^F(\beta_j))}{\sum_{k=1}^{r}(1-\prod_{k=1}^{r}(1-w_k^F(\beta_j))} \tag{5.7}$$

Here, $\sum_{k=1}^{r} W_j = 1$.

Step 4. Calculation of weighted similarity measure

We now calculate weighted similarity measure between ideal matrix M and M^k as follows:

$$S^w\left(M, M^k\right) = \left\langle \lambda_i^k \right\rangle$$

$$= \left(\lambda_1^k, \lambda_2^k, ..., \lambda_n^k\right)^T = \left(\frac{1}{m}\sum_{j=1}^{m}(1-\frac{D_{ij}^k}{9})\, W_j\right)_{i=1}^{n} \tag{5.8}$$

Here, k =1, 2, 3,…, r.

Step 5. Ranking of alternatives

In order to rank alternatives, we propose the formula (see Equation (5.9)):

$$\rho_i = \sum_{k=1}^{r} \psi_k \lambda_i^k \tag{5.9}$$

We arrange alternatives according to the descending order values of ρ_i. The highest value of ρ_i (i= 1, 2, 3,…, n) reflects the best alternative.

5.4 Numerical example

We solve an MAGDM problem adapted from (He & Liu, 2013) to demonstrate the applicability and effectiveness of the proposed method. Assume that an investment company wants to invest a sum of money in the best option. The investment company forms a decision making committee comprising of three experts/decision makers {E_1, E_2, E_3}with weight vector $\Psi = \{0.25, 0.4, 0.35\}$ to make a panel of four alternatives to invest money. The alternatives are Car company (α_1), Food company (α_2), Computer company (α_3) and Arms company (α_4). Decision makers take decision based on the criteria namely, risk analysis (β_1), growth analysis (β_2), environment impact (β_3) and criterion weights are provided by the decision makers in terms of linguistic variables that can be converted into NCNs (See Table 5.1).

Table 5.1 Linguistic term for rating of attribute/ criterion

Linguistic terms	**NCN**
Very important (VI)	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>
Important (I)	<[.6, .8], [.2, .3], [.2, .4], (.8, .3, .4)>
Medium (M)	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>
Unimportant (UI)	<[.3, .4], [.5, .6], [.5, .7], (.4, .6, .7)>
Very unimportant (VUI)	<[.1, .2], [.6, .8], [.7, .9], (.2, .8, .9)>

Step 1. Formation of ideal NCS decision matrix

We construct ideal NCS decision matrix (See Equation (5.10).

$$M=\left(\begin{array}{cccc} & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> \\ \alpha_2 & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> \\ \alpha_3 & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> \\ \alpha_4 & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> \end{array}\right) \quad (5.10)$$

Step 2. Construction of NCS decision matrix

The NCS decision matrices are constructed for four alternatives with respect to the three criteria.

Decision matrix for E_1 in NCS form

$M^1 =$

$$\begin{array}{cc} & \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \end{array} \\ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{array} & \begin{pmatrix} <[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)> & <[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)> & <[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)> \\ <[.6, .8], [.2, .3], [.2, .4], (.8, .3, .4)> & <[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)> & <[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)> \\ <[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)> & <[.6, .8], [.2, .3], [.2, .4], (.8, .3, .4)> & <[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)> \\ <[.3, .4], [.5, .6], [.5, .7], (.4, .6, .7)> & <[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)> & <[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)> \end{pmatrix} \end{array}$$

Decision matrix for E_2 in NCS form

$M^2 =$

$$\begin{array}{cc} & \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \end{array} \\ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{array} & \begin{pmatrix} <[.3, .4], [.5, .6], [.5, .7], (.4, .6, .7)> & <[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)> & <[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)> \\ <[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)> & <[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)> & <[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)> \\ <[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)> & <[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)> & <[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)> \\ <[.6, .8], [.2, .3], [.2, .4], (.8, .3, .4)> & <[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)> & <[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)> \end{pmatrix} \end{array}$$

Decision matrix for E_3 in NCS form

$M^3 =$

$$\begin{array}{cc} & \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \end{array} \\ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{array} & \begin{pmatrix} <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> \\ <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> \\ <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)> & <[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)> \\ <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.3,.4],[.5,.6],[.5,.7],(.4,.6,.7)> \end{pmatrix} \end{array}$$

Step 3. Determination of attribute weight

The linguistic terms shown in Table 5.1 are used to evaluate each attribute. The importance of each attribute for every decision maker is rated with linguistic terms shown in Table 5.2 Linguistic terms are converted into NCN (See Table 5.3.).

Table 5.2 Attribute rating in linguistic variables

	β_1	β_2	β_3
E_1	VI	M	I
E_2	VI	VI	M
E_3	M	VI	M

Table 5.3 Attribute rating in NCS

	β_1	β_2	β_3
E_1	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>	<[.6, .8], [.2, .3], [.2, .4], (.8, .3, .4)>
E_2	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>
E_3	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>

Using Equation (5.6) and Equation (5.7), we obtain the attribute weights as follows:

$$w_1 = 0.36, w_2 = 0.37, w_3 = 0.27. \tag{5.11}$$

Step 4. Calculation of weighted similarity measures

We now calculate weighted similarity measures using the Equation (5.8).

$$S^w(M, M^1) = \begin{pmatrix} 0.25 \\ 0.22 \\ 0.19 \\ 0.24 \end{pmatrix}, S^w(M, M^2) = \begin{pmatrix} 0.18 \\ 0.20 \\ 0.25 \\ 0.22 \end{pmatrix}, S^w(M, M^3) = \begin{pmatrix} 0.20 \\ 0.21 \\ 0.25 \\ 0.20 \end{pmatrix} \tag{5.12}$$

Step 5. Ranking of alternatives

Using Equation (5.9), we rank the alternatives according to the descending value of ρ_i (i = 1, 2, 3, 4).

We obtain $\rho_1 = 0.202, \rho_2 = 0.206, \rho_3 = 0.232, \rho_4 = 0.216$, Therefore the ranking order is $\rho_3 > \rho_4 > \rho_2 > \rho_1 \Rightarrow \alpha_3 > \alpha_4 > \alpha_2 > \alpha_1$.

Hence Computer company (α_3) is the best alternative for money investment.

5.5 Conclusion

In this chapter we have defined similarity measure between neutrosophic cubic sets and proved its basic properties. We have developed a new multi attribute group decision making strategy based on the proposed similarity measure. We also provide an illustrative example for multi attribute group decision making to show its applicability and effectiveness. We have employed linguistic variables to present criterion weights and presented conversion of linguistic variables into neutrosophic cubic numbers. We have also proposed a conversion formula for neutrosophic cubic number into fuzzy number. The proposed strategy can be applied to other MAGDM problems in neutrosophic cubic set environment. We also hope that the proposed strategy will open up a new direction of research work in neutrosophic cubic set environment.

Chapter 6

NC-VIKOR based MAGDM strategy in neutrosophic cubic set environment

6.1 Introduction

Opricovic (1998) proposed the VIKOR strategy for an MAGDM problem with conflicting attributes (Opricovic & Tzeng, 2004, 2007). Bausys and Zavadskas (2015) extended the VIKOR strategy to interval neutrosophic set (INS) environment and applied it to solve MADM problem. Further, Hung et al. (2017) proposed VIKOR method for interval neutrosophic MAGDM. Pouresmaeil et al. (2017) proposed an MAGDM strategy based on TOPSIS and VIKOR in single valued neutrosophic set (SVNS) environment. Hu et al. (2017) proposed interval neutrosophic projection based VIKOR strategy and applied it for doctor selection. Selvakumari et al. (2017) proposed VIKOR Method for decision making problem using octagonal neutrosophic soft matrix.VIKOR strategy in neutrosophic cubic set (NCS) environment is yet to appear in the literature.To fill up the research gap, we propose a new NC-VIKOR strategy to deal with MAGDM problems in NCS environment. We also introduce a neutrosophic cubic number aggregation operator and prove its basic properties.We solve an MAGDM problem based on proposed NC-VIKOR strategy.

The remainder of the Chapter is organized as follows: Section 6.2 develops a novel MAGDM strategy based on NC-VIKOR to solve the MADGM problems with NCS environment.

Section 6.3, solves an illustrative numerical example using the proposed NC-VIKOR in NCS environment. Then, Section 6.4 presents the sensitivity analysis. The conclusion of the whole chapter and further direction of research are presented in Section 6.5.

The content of this chapter is based on the paper published in **“Neutrosophic Sets and Systems”** 20, 95-108, 2018.

6.2 VIKOR strategy for solving MAGDM problem in NCS environment

Assume that $\Phi = \{\Phi_1, \Phi_2, \Phi_3, ..., \Phi_r\}$ be a set of r alternatives and $\Psi = \{\Psi_1, \Psi_2, \Psi_3, ..., \Psi_s\}$ be a set of s attributes. Assume that $W = \{w_1, w_2, w_3, ..., w_s\}$ be the weight vector of the attributes, where $w_k \geq 0$ and $\sum_{k=1}^{s} w_k = 1$. Assume that $E = \{E_1, E_2, E_3, ..., E_M\}$ be the set of M decision makers and $\zeta = \{\zeta_1, \zeta_2, \zeta_3, ..., \zeta_M\}$ be the set of weight vector of decision makers, where $\zeta_p \geq 0$ and $\sum_{p=1}^{M} \zeta_p = 1$.

The proposed MAGDM strategy consists of the following steps:

Step 1. Construction of the decision matrix

Let $DM^p = (a_{ij}^p)_{r\times s}$ (p = 1, 2, 3, …,t) be the p-th decision matrix, where information about the alternative Φ_i provided by the decision maker or expert E_p with respect to attribute Ψ_j (j = 1, 2, 3, …, s). The p-th decision matrix denoted by DM^p (See Equation (6.1)) is constructed as follows:

$$DM^p = \begin{pmatrix} & \Psi_1 & \Psi_2 & ... & \Psi_s \\ \Phi_1 & a_{11}^p & a_{12}^p & ... & a_{1s}^p \\ \Phi_2 & a_{21}^p & a_{22}^p & ... & a_{2s}^p \\ . & . & . & & . \\ \Phi_r & a_{r1}^p & a_{r2}^p & ... & a_{rs}^p \end{pmatrix} \tag{6.1}$$

Here p = 1, 2, 3,…, M; i = 1, 2, 3,…, r; j = 1, 2, 3,…, s.

Step 2. Normalization of the decision matrix

We use Equation (1.2) for normalizing the cost type attributes and benefit type attributes. After normalization, the normalized decision matrix (Equation (6.1)) is represented as follows (see Equation (6.2)):

$$DM^p = \begin{pmatrix} & \Psi_1 & \Psi_2 & \dots & \Psi_s \\ \Phi_1 & a^{*p}_{11} & a^{*p}_{12} & \dots & a^{*p}_{1s} \\ \Phi_2 & a^{*p}_{21} & a^{*p}_{22} & & a^{*p}_{2s} \\ . & . & \dots & . & \\ \Phi_r & a^{*p}_{r1} & a^{*p}_{r2} & \dots & .a^{*p}_{rs} \end{pmatrix} \quad (6.2)$$

Here, p = 1, 2, 3,…, M; i = 1, 2, 3,…, r; j = 1, 2, 3,…, s.

Step 3. Aggregated decision matrix

For obtaining group decision, we aggregate all the individual decision matrices ($DM^p, p=1,2,...,M$) to an aggregated decision matrix (DM) using the neutrosophic cubic numbers weighted aggregation (NCNWA) operator as follows:

$$a_{ij} = NCNWA_\zeta(a^1_{ij}, a^2_{ij},...,a^M_{ij}) = (\zeta_1 a^1_{ij} \oplus \zeta_2 a^2_{ij} \oplus \zeta_3 a^3_{ij} \oplus ... \oplus \zeta_M a^M_{ij}) =$$

$$<\left([\sum_{p=1}^{M} \zeta_p t_{ij}^{-(p)}, \sum_{p=1}^{M} \zeta_p t_{ij}^{+(p)}], [\sum_{p=1}^{M} \zeta_p i_{ij}^{-(p)}, \sum_{p=1}^{M} \zeta_p i_{ij}^{+(p)}], \right.$$

$$\left. [\sum_{p=1}^{M} \zeta_p f_{ij}^{-(p)}, \sum_{p=1}^{M} \zeta_p f_{ij}^{+(p)}], (\sum_{p=1}^{M} \zeta_p t_{ij}^{(p)}, \sum_{p=1}^{M} \zeta_p i_{ij}^{(p)}, \sum_{p=1}^{M} \zeta_p f_{ij}^{(p)}] \right)> \quad (6.3)$$

The NCNWA operator satisfies the following properties:

6.1 Idempotency

6.2 Monotonicity

6.3 Boundedness

Property 6.1 Idempotency

If all $a^1_{ij}, a^2_{ij},... \quad ,a^M_{ij} = a$ are equal, then $a_{ij} = NCNWA_\zeta(a^1_{ij}, a^2_{ij},... \quad ,a^M_{ij}) = a$

Proof: Since, $a^1_{ij} = a^2_{ij} = ... \quad = a^M_{ij} = a$,using the Equation (6.3), we obtain

$$a_{ij} = NCNWA_\zeta(a^1_{ij} \ a^2_{ij}... \quad a^M_{ij}) = (\zeta_1 a^1_{ij} \oplus \zeta_2 a^2_{ij} \oplus \zeta_3 a^3_{ij} \oplus ... \oplus \zeta_M a^M_{ij}) =$$

$$(\zeta_1 a \oplus \zeta_2 a \oplus \zeta_3 a \oplus ... \oplus \zeta_M a) = <\left([t^- \sum_{p=1}^{M} \zeta_p, t^+ \sum_{p=1}^{M} \zeta_p], [i^- \sum_{p=1}^{M} \zeta_p, i^+ \sum_{p=1}^{M} \zeta_p], \right.$$

$$\left. [f^- \sum_{p=1}^{M} \zeta_p, f^+ \sum_{p=1}^{M} \zeta_p], (t \sum_{p=1}^{M} \zeta_p, i \sum_{p=1}^{M} \zeta_p, f \sum_{p=1}^{M} \zeta_p] \right)> = <\left([t^-, t^+], [i^-, i^+], [f^-, f^+], (t, i, f]\right)> = a.$$

Property 6.2 Monotonicity

Assume that $\{a_{ij}^{1}, a_{ij}^{2},..., a_{ij}^{M}\}$ and $\{a_{ij}^{*1}, a_{ij}^{*2},..., a_{ij}^{*M}\}$ be any two set of collections of M neutrosophic cubic numbers with the condition $a_{ij}^{p} \le a_{ij}^{*p}$ (p = 1, 2,..., M), then

$NCNWA_{\zeta}(a_{ij}^{1}, a_{ij}^{2},..., a_{ij}^{M}) \le NCNWA_{\zeta}(a_{ij}^{*1}, a_{ij}^{*2},..., a_{ij}^{*M})$.

Proof:

From the given condition $t_{ij}^{-(p)} \le t_{ij}^{-*(p)}$, we have $\zeta_p t_{ij}^{-(p)} \le \zeta_p t_{ij}^{-*(p)}$

$\Rightarrow \sum_{p=1}^{M} \zeta_p t_{ij}^{-(p)} \le \sum_{p=1}^{M} \zeta_p t_{ij}^{-*(p)}$.

From the given condition $t_{ij}^{+(p)} \le t_{ij}^{+*(p)}$, we have $\zeta_p t_{ij}^{+(p)} \le \zeta_p t_{ij}^{+*(p)}$

$\Rightarrow \sum_{p=1}^{M} \zeta_p t_{ij}^{+(p)} \le \sum_{p=1}^{M} \zeta_p t_{ij}^{+*(p)}$.

From the given condition $i_{ij}^{-(p)} \ge i_{ij}^{-*(p)}$, we have $\zeta_p i_{ij}^{-(p)} \ge \zeta_p i_{ij}^{-*(p)}$

$\Rightarrow \sum_{p=1}^{M} \zeta_p i_{ij}^{-(p)} \ge \sum_{p=1}^{M} \zeta_p i_{ij}^{-*(p)}$.

From the given condition, $i_{ij}^{+(p)} \ge i_{ij}^{+*(p)}$, we have $\zeta_p i_{ij}^{+(p)} \ge \zeta_p i_{ij}^{+*(p)}$

$\Rightarrow \sum_{p=1}^{M} \zeta_p i_{ij}^{+(p)} \ge \sum_{p=1}^{M} \zeta_p i_{ij}^{+*(p)}$.

From the given condition, $f_{ij}^{-(p)} \ge f_{ij}^{-*(p)}$, we have $\zeta_p f_{ij}^{-(p)} \ge \zeta_p f_{ij}^{-*(p)}$

$\Rightarrow \sum_{p=1}^{M} \zeta_p f_{ij}^{-(p)} \ge \sum_{p=1}^{M} \zeta_p f_{ij}^{-*(p)}$.

From the given condition, $f_{ij}^{+(p)} \ge f_{ij}^{+*(p)}$, we have $\zeta_p f_{ij}^{+(p)} \ge \zeta_p f_{ij}^{+*(p)}$

$\Rightarrow \sum_{p=1}^{M} \zeta_p f_{ij}^{+(p)} \ge \sum_{p=1}^{M} \zeta_p f_{ij}^{+*(p)}$.

From the given condition $t_{ij}^{(p)} \le t_{ij}^{*(p)}$, we have $\zeta_p t_{ij}^{(p)} \le \zeta_p t_{ij}^{*(p)}$

$\Rightarrow \sum_{p=1}^{M} \zeta_p t_{ij}^{(p)} \le \sum_{p=1}^{M} \zeta_p t_{ij}^{*(p)}$.

From the given condition $i_{ij}^{(p)} \ge i_{ij}^{*(p)}$, we have $\zeta_p i_{ij}^{(p)} \ge \zeta_p i_{ij}^{*(p)}$

$$\Rightarrow \sum_{p=1}^{M} \zeta_p i_{ij}^{(p)} \leq \sum_{p=1}^{M} \zeta_p i_{ij}^{*(p)}.$$

From the given condition $t_{ij}^{(p)} \leq t_{ij}^{*(p)}$, we have $\zeta_p f_{ij}^{(p)} \geq \zeta_p f_{ij}^{*(p)}$

$$\Rightarrow \sum_{p=1}^{M} \zeta_p f_{ij}^{(p)} \leq \sum_{p=1}^{M} \zeta_p f_{ij}^{*(p)}$$

From the above relations, we obtain

$$NCNWA_{\zeta}(a_{ij}^{1}, a_{ij}^{2}, ..., a_{ij}^{M}) \leq NCNWA_{\zeta}(a_{ij}^{*1}, a_{ij}^{*2}, ..., a_{ij}^{*M}).$$

Property6.3 Boundedness

Let $\{a_{ij}^{1}, a_{ij}^{2}, ..., a_{ij}^{M}\}$ be any collection of M neutrosophic cubic numbers.

If $a^{+} = <[\max_p\{t_{ij}^{-(p)}\}, [\max_p\{t_{ij}^{+(p)}\}], [\min_p\{i_{ij}^{-(p)}\}, \min_p\{i_{ij}^{+(p)}\}], [\min_p\{f_{ij}^{-(p)}\}, \min_p\{f_{ij}^{+(p)}\}],$

$(\max_p\{t_{ij}^{p}\}, \min_p\{i_{ij}^{p}\}, \min_p\{f_{ij}^{p}\}) >$

$a^{-} = <[\min_p\{t_{ij}^{-(p)}\}, [\min_p\{t_{ij}^{+(p)}\}], [\max_p\{i_{ij}^{-(p)}\}, \max_p\{i_{ij}^{+(p)}\}], [\max_p\{f_{ij}^{-(p)}\}, \max_p\{f_{ij}^{+(p)}\}],$

$(\min_p\{t_{ij}^{p}\}, \max_p\{i_{ij}^{p}\}, \max_p\{f_{ij}^{p}\}) >.$

Then, $a^{-} \leq NCNWA_{\zeta}(a_{ij}^{1}\ a_{ij}^{2} ...\ a_{ij}^{M}) \leq a^{+}$.

Proof:

From Property 6.1 and Property 6.2, we obtain

$$NCNWA_{\zeta}(a_{ij}^{1}, a_{ij}^{2}, ..., a_{ij}^{M}) \geq NCNWA_{\zeta}(a^{-}, a^{-}, ..., a^{-}) = a^{-} \text{ and}$$

$$NCNWA_{\zeta}(a_{ij}^{1}, a_{ij}^{2}, ..., a_{ij}^{M}) \leq NCNWA_{\zeta}(a^{+}, a^{+}, ..., a^{+}) = a^{+}.$$

So, we have $a^{-} \leq NCNWA_{\zeta}(a_{ij}^{1}, a_{ij}^{2}, ..., a_{ij}^{M}) \leq a^{+}$.

Therefore, the aggregated decision matrix is defined as follows:

$$DM = \begin{pmatrix} & \Psi_1 & \Psi_2 & ... & \Psi_s \\ \Phi_1 & a_{11} & a_{12} & ... & a_{1s} \\ \Phi_2 & a_{21} & a_{22} & ... & a_{2s} \\ . & . & . & . & . \\ \Phi_r & a_{r1} & a_{r2} & ... & a_{rs} \end{pmatrix} \quad (6.4)$$

Here, i = 1, 2, 3, …, r; j = 1, 2, 3, …, s; p =1, 2, …., M.

Step 4. Define the positive ideal solution and negative ideal solution

$$a_{ij}^{+} = <[\max_i t_{ij}^{-}, \max_i t_{ij}^{+}], [\min_i i_{ij}^{-}, \min_i i_{ij}^{+}], [\min_i f_{ij}^{-}, \min_i i_{ij}^{+}], (\max_i t_{ij}, \min_i f_{ij}, \min_i f_{ij}) > \tag{6.5}$$

$$a_{ij}^{-} = <[\min_i t_{ij}^{-}, \min_i t_{ij}^{+}], [\max_i i_{ij}^{-}, \max_i i_{ij}^{+}], [\max_i f_{ij}^{-}, \max_i i_{ij}^{+}], (\min_i t_{ij}, \max_i f_{ij}, \max_i f_{ij}) > \tag{6.6}$$

Step 5. Compute Γ_i and Z_i

Γ_i and Z_i represents the average and worst group scores for the alternative A_i respectively with the relations

$$\Gamma_i = \sum_{j=1}^{s} \frac{w_j \times D(a_{ij}^{+}, a_{ij}^{*})}{D(a_{ij}^{+}, a_{ij}^{-})} \tag{6.7}$$

$$Z_i = \max_j \left\{ \frac{w_j \times D(a_{ij}^{+}, a_{ij}^{*})}{D(a_{ij}^{+}, a_{ij}^{-})} \right\} \tag{6.8}$$

Here, $w_j$is the weight of Ψ_j.

The smaller value of Γ_i corresponds to the better average and the smaller value of Z_i represents worse group scores for alternative A_i.

Step 6. Calculate the values of φ_i (i = 1, 2, 3, …, r)

$$\varphi_i = \gamma \frac{(\Gamma_i - \Gamma^{-})}{(\Gamma^{+} - \Gamma^{-})} + (1-\gamma) \frac{(Z_i - Z^{-})}{(Z^{+} - Z^{-})} \tag{6.9}$$

Here, $\Gamma_i^{-} = \min_i \Gamma_i$, $\Gamma_i^{+} = \max_i \Gamma_i$, $Z_i^{-} = \min_i Z_i$, $Z_i^{+} = \max_i Z_i$ (6.10)

and γ depicts the decision making mechanism coefficient. If $\gamma > 0.5$, it is for "the maximum group utility"; If $\gamma < 0.5$, it is " the minimum regret"; and it is both if $\gamma = 0.5$.

Step 7. Rank the priority of alternatives

Rank the alternatives by φ_i, Γ_i and Z_i according to the rule of traditional VIKOR strategy.

Step 8. Determine the compromise solution

Obtain alternative Φ^1 as a compromise solution, which is ranked as the best by the measure φ (Minimum) if the following two conditions are satisfied:

Condition 1. Acceptable stability: $\varphi(\Phi^2)-\varphi(\Phi^1)\geq\frac{1}{(r-1)}$, where Φ^1,Φ^2 are the alternatives with first and second position in the ranking list by φ; r is the number of alternatives.

Condition 2. Acceptable stability in decision making: Alternative Φ^1 must also be the best ranked by Γ or/and Z. This compromise solution is stable within whole decision making process.

If one of the conditions is not satisfied, then a set of compromise solutions is proposed as follows:

◊ Alternatives Φ^1 and Φ^2 are compromise solutions if only condition 2 is not satisfied, or

◊ Φ^1, Φ^2, Φ^3,…, Φ^r are compromise solutions if condition 1 is not satisfied and Φ^r is decided by constraint $\varphi(\Phi^r)-\varphi(\Phi^1)\leq\frac{1}{(r-1)}$ for maximum r.

6.3 Illustrative example

To demonstrate the feasibility, applicability and effectiveness of the proposed strategy, we solve an MAGDM problem adapted from (He & Liu, 2013). We assume that an investment company wants to invest a sum of money in the best option. The investment company forms a decision making board involving of three members (E_1, E_2, E_3) who evaluate the four alternatives to invest money. The alternatives are Car company (Φ_1), Food company (Φ_2), Computer company (Φ_3) and Arms company (Φ_4). Decision makers take decision to evaluate alternatives based on the attributes namely, risk factor (Ψ_1), growth factor (Ψ_2), environment impact (Ψ_3). We consider three criteria as benefit type based on Pramanik, Dalapati, Alam et al., 2017. Assume that the weight vector of attributes is $W=(0.36, 0.37, 0.27)^T$ and weight vector of decision makers or experts is $\zeta=(0.26, 0.40, 0.34)^T$. Now, we apply the proposed MAGDM strategy using the following steps.

Step 1. Construction of the decision matrix

We construct the decision matrices as follows:

Decision matrix for DM^1 in neutrosophic cubic number form:

$$\begin{array}{cc} & \begin{array}{ccc} \Psi_1 & \Psi_2 & \Psi_3 \end{array} \\ \begin{array}{c} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{array} & \begin{pmatrix} \langle[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)\rangle & \langle[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)\rangle & \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle \\ \langle[.6, .8], [.2, .3], [.2, .4], (.8, .3, .4)\rangle & \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle & \langle[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)\rangle \\ \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle & \langle[.6, .8], [.2, .3], [.2, .4], (.8, .3, .4)\rangle & \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle \\ \langle[.3, .4], [.5, .6], [.5, .7], (.4, .6, .7)\rangle & \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle & \langle[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)\rangle \end{pmatrix} \end{array} \tag{6.11}$$

Decision matrix for DM^2 in neutrosophic cubic number form:

$$\begin{array}{cc} & \begin{array}{ccc} \Psi_1 & \Psi_2 & \Psi_3 \end{array} \\ \begin{array}{c} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{array} & \begin{pmatrix} \langle[.3, .4], [.5, .6], [.5, .7], (.4, .6, .7)\rangle & \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle & \langle[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)\rangle \\ \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle & \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle & \langle[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)\rangle \\ \langle[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)\rangle & \langle[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)\rangle & \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle \\ \langle[.6, .8], [.2, .3], [.2, .4], (.8, .3, .4)\rangle & \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle & \langle[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)\rangle \end{pmatrix} \end{array} \tag{6.12}$$

Decision matrix for DM^3 in neutrosophic cubic number form:

$$\begin{array}{cc} & \begin{array}{ccc} \Psi_1 & \Psi_2 & \Psi_3 \end{array} \\ \begin{array}{c} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{array} & \begin{pmatrix} \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle & \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle & \langle[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)\rangle \\ \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle & \langle[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)\rangle & \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle \\ \langle[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)\rangle & \langle[.6, .8], [.2, .3], [.2, .4], (.8, .3, .4)\rangle & \langle[.6, .8], [.2, .3], [.2, .4], (.8, .3, .4)\rangle \\ \langle[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)\rangle & \langle[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)\rangle & \langle[.3, .4], [.5, .6], [.5, .7], (.4, .6, .7)\rangle \end{pmatrix} \end{array} \tag{6.13}$$

Step 2. Normalization of the decision matrix

Since all the criteria are considered as benefit type, we do not need to normalize the decision matrices (DM^1, DM^2, DM^3).

Step 3. Aggregated decision matrix

Using Equation (6.3), the aggregated decision matrix of (6.11, 6.12, 6.13) is presented below:

$$\begin{array}{cc} & \begin{array}{ccc} \Psi_1 & \Psi_2 & \Psi_3 \end{array} \\ \begin{array}{c} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{array} & \begin{pmatrix} \langle[.44, .56], [.36, .46], [.36, .51], (.56, .46,.50)\rangle & \langle[.48, .60], [.32, .42], [.32, .42], (.60, .42,.42)\rangle & \langle[.62, .80], [.18, .28], [.18, .28], (.80, .28, .28)\rangle \\ \langle[.45, .58], [.35, .45], [.35, .47], (.58, .45, .47)\rangle & \langle[.50, .64], [.30, .40], [.30, .40], (.64, .40, .40)\rangle & \langle[.60, .76], [.20, .30], [.20, .30], (.76, .30,.30)\rangle \\ \langle[.62, .80], [.18, .28], [.18, .28], (.80, .28, .28)\rangle & \langle[.64, .84], [.16, .26], [.16, .32], (.84, .26, .32)\rangle & \langle[.47, .60], [.33, .43], [.33, .47], (.60, .43, .47)\rangle \\ \langle[.56, .73], [.24, .34], [.24, .41], (.73, .34, .41)\rangle & \langle[.40, .50], [.40, .50], [.40, .50], (.50, .50, .50)\rangle & \langle[.56, .73], [.24, .34], [.24, .37], (.73, .34,.37)\rangle \end{pmatrix} \end{array} \tag{6.14}$$

Step 4. Define the positive ideal solution and negative ideal solution

The positive ideal solution a_{ij}^{+} =

Ψ_1	Ψ_2	Ψ_3
<[.62, .80], [.18, .28], [.18, .28], (.8 0, .28,.28)>	<[.64, .84], [.16, .26], [. 16, .32], (.84, .26,.32)>	<[.62, .80], [. 18, .28], [.18, .28], (.80, .28, .28)>

and the negative ideal solution a_{ij}^{-} =

Ψ_1	Ψ_2	Ψ_3
<[.44, .56], [.36, .46], [.36, .51], (.56, .46,.50)>	<[.40, .50], [.40, .50], [.40, .50], (.50, .50,.50)>	<[.47, .60], [.33, .43], [.33, .43], (.60, .43, .47)>

Step 5. Compute Γ_i and Z_i

Using Equation (6.9) and Equation (6.10), we obtain

$$\Gamma_1 = \left(\frac{0.36\times 0.2}{0.37}\right)+\left(\frac{0.37\times 0.16}{0.25}\right)+\left(\frac{0.27\times 0}{0.16}\right)=0.43,$$

$$\Gamma_2 = \left(\frac{0.36\times 0.18}{0.37}\right)+\left(\frac{0.37\times 0.14}{0.25}\right)+\left(\frac{0.27\times 0.02}{0.16}\right)=0.42,$$

$$\Gamma_3 = \left(\frac{0.36\times 0}{0.37}\right)+\left(\frac{0.37\times 0}{0.25}\right)+\left(\frac{0.27\times 0.19}{0.16}\right)=0.32,$$

$$\Gamma_4 = \left(\frac{0.36\times 0.08}{0.37}\right)+\left(\frac{0.37\times 0.25}{0.25}\right)+\left(\frac{0.27\times 0.07}{0.16}\right)=0.57.$$

And $Z_1 = \max\left\{\left(\frac{0.36\times 0.2}{0.37}\right),\left(\frac{0.37\times 0.16}{0.25}\right),\left(\frac{0.27\times 0}{0.16}\right)\right\}=0.24,$

$$Z_2 = \max\left\{\left(\frac{0.36\times 0.18}{0.37}\right),\left(\frac{0.37\times 0.14}{0.25}\right),\left(\frac{0.27\times 0.02}{0.16}\right)\right\}=0.21,$$

$$Z_3 = \max\left\{\left(\frac{0.36\times 0}{0.37}\right),\left(\frac{0.37\times 0}{0.25}\right),\left(\frac{0.27\times 0.19}{0.16}\right)\right\}=0.32,$$

$$Z_4 = \max\left\{\left(\frac{0.36\times 0.08}{0.37}\right),\left(\frac{0.37\times 0.25}{0.25}\right),\left(\frac{0.27\times 0.07}{0.16}\right)\right\}=0.37.$$

Step 6. Calculate the values of φ_i

Using Equations (6.11), (6.12) and $\gamma = 0.5$, we obtain

$$\varphi_1 = 0.5\times\frac{(0.43-0.32)}{0.25}+0.5\times\frac{(0.24-0.21)}{0.16}=0.31,$$

$$\varphi_2 = 0.5\times\frac{(0.42-0.32)}{0.25}+0.5\times\frac{(0.21-0.21)}{0.16}=0.2,$$

$\varphi_3 = 0.5 \times \frac{(0.32 - 0.32)}{0.25} + 0.5 \times \frac{(0.32 - 0.21)}{0.16} = 0.34,$

$\varphi_4 = 0.5 \times \frac{(0.57 - 0.32)}{0.25} + 0.5 \times \frac{(0.37 - 0.21)}{0.16} = 1.$

Step 7. Rank the priority of alternatives

The preference ranking order of the alternatives is presentedin Table 6.1

Table 6.1 Preference ranking order and compromise solution based on Γ, Z and φ

	Φ_1	Φ_2	Φ_3	Φ_4	Ranking	Compromise solution
Γ	0.43	0.42	0.32	0.57	$\Phi_2 \succ \Phi_1 \succ \Phi_3 \succ \Phi_4$	Φ_2
Z	0.24	0.21	0.32	0.37	$\Phi_2 \succ \Phi_1 \succ \Phi_3 \succ \Phi_4$	Φ_2
$\varphi(\gamma = 0.5)$	0.31	0.20	0.34	1	$\Phi_2 \succ \Phi_1 \succ \Phi_3 \succ \Phi_4$	Φ_2

Step 8: Determine the compromise solution

The preference ranking order based on φ in decreasing order and alternative with best position is Φ_2 with $\varphi(\Phi_2)$ = 0.20, and second best position Φ_1 with $\varphi(\Phi_1)$ = 0.31. Therefore, $\varphi(\Phi_1) - \varphi(\Phi_2) = 0.11 < 0.333$ (since, r = 4; 1/(r-1) = 0.333), which does not satisfy the condition 1

($\varphi(\Phi^2) - \varphi(\Phi^1) \geq \frac{1}{(r-1)}$**),** but alternative Φ_2 is the best ranked by Γ, Z, which satisfies the condition 2.

Therefore, we obtain the compromise solution as follows:

$\varphi(\Phi_1) - \varphi(\Phi_2) = 0.11 < 0.333,$

$\varphi(\Phi_3) - \varphi(\Phi_2) = 0.14 < 0.333,$

$\varphi(\Phi_4) - \varphi(\Phi_2) = 0.80 > 0.333.$

So Φ_1, Φ_2, Φ_3 are compromise solutions.

6.4 The influence of parameter γ

Table 6.1shows how the ranking order of alternatives (Φ_i) changes with the change of the value of γ

Table 6.2 Values of φ_i (i = 1, 2, 3, 4) and ranking of alternatives for different values of γ .

Values of γ	Values of φ_i	Preference order
$\gamma = 0.1$	$\varphi_1 = 0.22$, $\varphi_2 = \mathbf{0.04}$, $\varphi_3 = 0.62$, $\varphi_4 = 1$	$\Phi_2 \succ \Phi_1 \succ \Phi_3 \succ \Phi_4$
$\gamma = 0.2$	$\varphi_1 = 0.24$, $\varphi_2 = \mathbf{0.08}$, $\varphi_3 = 0.55$, $\varphi_4 = 1$	$\Phi_2 \succ \Phi_1 \succ \Phi_3 \succ \Phi_4$
$\gamma = 0.3$	$\varphi_1 = 0.26$, $\varphi_2 = \mathbf{0.12}$, $\varphi_3 = 0.48$, $\varphi_4 = 1$	$\Phi_2 \succ \Phi_1 \succ \Phi_3 \succ \Phi_4$
$\gamma = 0.4$	$\varphi_1 = 0.29$, $\varphi_2 = \mathbf{0.16}$, $\varphi_3 = 0.41$, $\varphi_4 = 1$	$\Phi_2 \succ \Phi_1 \succ \Phi_3 \succ \Phi_4$
$\gamma = 0.5$	$\varphi_1 = 0.31$, $\varphi_2 = \mathbf{0.2}$, $\varphi_3 = 0.34$, $\varphi_4 = 1$	$\Phi_2 \succ \Phi_1 \succ \Phi_3 \succ \Phi_4$
$\gamma = 0.6$	$\varphi_1 = 0.34$, $\varphi_2 = \mathbf{0.24}$, $\varphi_3 = 0.28$, $\varphi_4 = 1$	$\Phi_2 \succ \Phi_3 \succ \Phi_1 \succ \Phi_4$
$\gamma = 0.7$	$\varphi_1 = 0.36$, $\varphi_2 = 0.28$, $\varphi_3 = \mathbf{0.21}$, $\varphi_4 = 1$	$\Phi_3 \succ \Phi_2 \succ \Phi_1 \succ \Phi_4$
$\gamma = 0.8$	$\varphi_1 = 0.39$, $\varphi_2 = 0.32$, $\varphi_3 = \mathbf{0.14}$, $\varphi_4 = 1$	$\Phi_3 \succ \Phi_2 \succ \Phi_1 \succ \Phi_4$
$\gamma = 0.9$	$\varphi_1 = 0.42$, $\varphi_2 = 0.36$, $\varphi_3 = \mathbf{0.07}$, $\varphi_4 = 1$	$\Phi_3 \succ \Phi_2 \succ \Phi_1 \succ \Phi_4$

Figure 6.1 represents the graphical representation of alternatives (A_i) versus φ_i (i = 1, 2, 3, 4) for different values of γ .

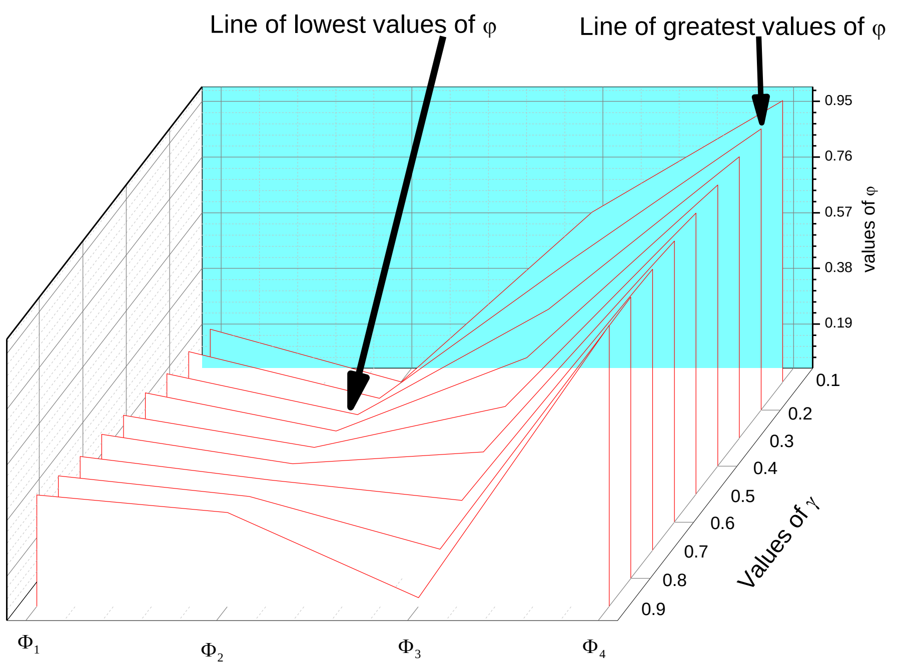

Figure 6.1 Graphical representation of ranking of alternatives for different values of γ .

6.5 Conclusion

In this chapter, we have extended the traditional VIKOR strategy to NC-VIKOR in neutrosophic cubic set environment. We introduced neutrosophic cubic number weighted aggregation (NCNWA) operator and proved its three basic properties. We developed a novel NC-VIKOR based MAGDM strategy in neutrosophic cubic set environment. Finally, we solve a MAGDM problem to show the feasibility, applicability and efficiency of the proposed MAGDM strategy. We have presented a sensitivity analysis to show the impact of different values of the decision making mechanism coefficient on ranking order of the alternatives. The proposed NC-VIKOR based MAGDM strategy can be employed to solve a variety of problems such as logistics center selection (Pramanik, Dalapati et al., 2016, 2018), teacher selection (Pramanik & Mukhopadhyaya, 2011), renewable energy selection (San Cristóbal, 2011), fault diagnosis (Ye, 2016), brick selection (Mondal, & Pramanik, 2014a, 2015d), weaver selection (Dey et al., 2015), etc.

Chapter 7

NC-cross entropy based MADM strategy in neutrosophic cubic set environment

7.1 Introduction

Ye (2013) defined cross entropy for SVNSs and employed it solve to MADM problems. To remove the drawbacks of cross entropy (2013), Ye (2015b) proposed improved cross entropy for SVNSs. In the same study, Ye (2015b) also proposed new cross entropy for INSs. Tian et al. (2015) proposed a cross entropy for interval neutrosophic set (INS) environments and employed it to MADM problems. Sahin (2017) proposed an interval neutrosophic cross entropy measure based on fuzzy cross entropy and single valued neutrosophic cross entropy measures and applied it to MADM problems. Recently, Pramanik, Dalapati, Alam, Smarandache et al. (2018) proposed a novel cross entropy, namely, NS-cross entropy in SVNS environment and proved its basic properties. In the same research, Pramanik, Dalapati, Alam, Smarandacheet al. (2018) also proposed weighted NS-cross entropy and employed it to MAGDM problem. Furthermore, Dalapati et al. (2017) extended NS-cross entropy in INS environments and employed it for solving MADM problems. Pramanik, Dey et al. (2018) developed two new MADM strategies based on cross entropy measures in bipolar neutrosophic set (BNS) and interval BNS environment.

Cross entropy measure is yet to appear in NCS environment. Since MADM strategy based on cross entropy is not studied in the literature, we move to propose a comprehensive NC-cross entropy-based strategy for tackling MADM in the NCS environment. This study develops a novel NC-cross entropy-based MADM strategy.

The content of this chapter is based on the paper published in **"Mathematics"**, 6, 67, 2018.doi:10.3390/math6050067.

The remainder of the Chapter is presented as follows: Section proposes an NC-cross entropy measure and weighted NC-cross entropy measure and establishes their basic properties. Section 7.3devotes to develop MADM strategy using NC-cross entropy. Section 7.4 provides an illustrative numerical example to show the applicability and validity of the proposed strategy in NCS environments. Section 7.5 presents briefly the contribution of the chapter. Section 7.6 offers conclusion and the future scope of research.

7.2 NC-Cross-entropy measure in NCS environment

Definition 7.1 NC-cross entropy measure

Let Q_1 and Q_2 be any two NCSs in $U = \{u_1, u_2, u_3, ..., u_n\}$. Then, neutrosophic cubic cross-entropy measure of $Q_1$and Q_2 is denoted by $CE_{NC}(Q_1, Q_2)$ and defined as follows:

$$CE_{NC}(Q_1, Q_2)$$

$$= \frac{1}{8}\left\{\sum_{i=1}^{n}\left(\left[\frac{2\left|T^-_{Q_1}(u_i) - T^-_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^-_{Q_1}(u_i)\right|^2} + \sqrt{1+\left|T^-_{Q_2}(u_i)\right|^2}} + \frac{2\left|(1-T^-_{Q_1}(u_i)) - (1-T^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{Q_1}(u_i))\right|^2} + \sqrt{1+\left|(1-T^-_{Q_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T^+_{Q_1}(u_i) - T^+_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^+_{Q_1}(u_i)\right|^2} + \sqrt{1+\left|T^+_{Q_2}(u_i)\right|^2}} + \frac{2\left|(1-T^+_{Q_1}(u_i)) - (1-T^+_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{Q_1}(u_i))\right|^2} + \sqrt{1+\left|(1-T^+_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^-_{Q_1}(u_i) - I^-_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^-_{Q_1}(u_i)\right|^2} + \sqrt{1+\left|I^-_{Q_2}(u_i)\right|^2}} + \frac{2\left|(1-I^-_{Q_1}(u_i)) - (1-I^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^-_{Q_1}(u_i))\right|^2} + \sqrt{1+\left|(1-I^-_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^+_{Q_1}(u_i) - I^+_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^+_{Q_1}(u_i)\right|^2} + \sqrt{1+\left|I^+_{Q_2}(u_i)\right|^2}} + \frac{2\left|(1-I^+_{Q_1}(u_i)) - (1-I^+_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^+_{Q_1}(u_i))\right|^2} + \sqrt{1+\left|(1-I^+_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^-_{Q_1}(u_i) - F^-_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^-_{Q_1}(u_i)\right|^2} + \sqrt{1+\left|F^-_{Q_2}(u_i)\right|^2}} + \frac{2\left|(1-F^-_{Q_1}(u_i)) - (1-F^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{Q_1}(u_i))\right|^2} + \sqrt{1+\left|(1-F^-_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{+}_{Q_1}(u_i)-F^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{Q_1}(u_i))-(1-F^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|T_{Q_1}(u_i)-T_{Q_2}(u_i)\right|}{\sqrt{1+\left|T_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{Q_1}(u_i))-(1-T_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{Q_1}(u_i)-I_{Q_2}(u_i)\right|}{\sqrt{1+\left|I_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2}(u)\right|^2}}+\frac{2\left|(1-I_{Q_1}(u_i))-(1-I_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left.\left[\frac{2\left|F_{Q_1}(u_i)-F_{Q_2}(u_i)\right|}{\sqrt{1+\left|F_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{Q_1}(u_i))-(1-F_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2}(u_i))\right|^2}}\right]\right)\right)\right\} \quad (7.1)$$

Theorem 7.1 Let Q_1, Q_2 be any two NCSs in U. The NC-cross entropy measure $CE_{NC}(Q_1, Q_2)$ satisfies the following properties:

i. $CE_{NC}(Q_1, Q_2) \geq 0, \forall u_i \in U$.

ii. $CE_{NC}(Q_1, Q_2) = 0$ iff $T^{-}_{Q_1}(u_i) = T^{-}_{Q_2}(u_i)$, $T^{+}_{Q_1}(u_i) = T^{+}_{Q_2}(u_i)$, $I^{-}_{Q_1}(u_i) = I^{-}_{Q_2}(u_i)$, $I^{+}_{Q_1}(u_i) = I^{+}_{Q_2}(u_i)$, $F^{-}_{Q_1}(u_i) = F^{-}_{Q_2}(u_i)$, $F^{+}_{Q_1}(u_i) = F^{+}_{Q_2}(u_i)$ and $T_{Q_1}(u_i) = T_{Q_2}(u_i)$, $I_{Q_1}(u_i) = I_{Q_2}(u_i)$, $F_{Q_1}(u_i) = F_{Q_2}(u_i)$, $\forall u_i \in U$.

iii. $CE_{NC}(Q_1, Q_2) = CE_{NC}(Q_1^c, Q_2^c)$, $\forall u_i \in U$.

iv. $CE_{NC}(Q_1, Q_2) = CE_{NC}(Q_2, Q_1)$, $\forall u_i \in U$.

Proof of Theorem 7.1

i. For all values of $u_i \in U$, $\left|T_{Q_1}(u_i)\right| \geq 0$, $\left|T_{Q_2}(u_i)\right| \geq 0$, $\left|T_{Q_1}(u_i)-T_{Q_2}(u_i)\right| \geq 0$, $\sqrt{1+\left|T_{Q_1}(u_i)\right|^2} \geq 0$, $\sqrt{1+\left|T_{Q_2}(u_i)\right|^2} \geq 0$, $\left|(1-T_{Q_1}(u_i))\right| \geq 0$, $\left|(1-T_{Q_2}(u_i))\right| \geq 0$, $\left|(1-T_{Q_1}(u_i))-(1-T_{Q_2}(u_i))\right| \geq 0$, $\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2} \geq 0$, $\sqrt{1+\left|(1-T_{Q_2}(u_i))\right|^2} \geq 0$.

Then,

$$\left[\frac{2\left|T_{Q_1}(u_i)-T_{Q_2}(u_i)\right|}{\sqrt{1+\left|T_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{Q_1}(u_i))-(1-T_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2}(u_i))\right|^2}}\right]\geq 0 \quad (7.2)$$

Similarly,

$$\left[\frac{2\left|I_{Q_1}(u_i)-I_{Q_2}(u_i)\right|}{\sqrt{1+\left|I_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2}(u)\right|^2}}+\frac{2\left|(1-I_{Q_1}(u_i))-(1-I_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2}(u_i))\right|^2}}\right]\geq 0 \quad (7.3)$$

and

$$\left[\frac{2\left|F_{Q_1}(u_i)-F_{Q_2}(u_i)\right|}{\sqrt{1+\left|F_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{Q_1}(u_i))-(1-F_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2}(u_i))\right|^2}}\right]\geq 0 \quad (7.4)$$

Again,

For all values of $u_i \in U$, $\left|T^-_{Q_1}(u_i)\right| \geq 0$, $\left|T^-_{Q_2}(u_i)\right| \geq 0$, $\left|T^-_{Q_1}(u_i)-T^-_{Q_2}(u_i)\right| \geq 0$, $\sqrt{1+\left|T^-_{Q_1}(u_i)\right|^2} \geq 0$, $\sqrt{1+\left|T^-_{Q_2}(u_i)\right|^2} \geq 0$, $\left|(1-T^-_{Q_1}(u_i))\right| \geq 0$, $\left|(1-T^-_{Q_2}(u_i))\right| \geq 0$, $\left|(1-T^-_{Q_1}(u_i))-(1-T^-_{Q_2}(u_i))\right| \geq 0$, $\sqrt{1+\left|(1-T^-_{Q_1}(u_i))\right|^2} \geq 0$, $\sqrt{1+\left|(1-T^-_{Q_2}(u_i))\right|^2} \geq 0$

$$\Rightarrow\left[\frac{2\left|T^-_{Q_1}(u_i)-T^-_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T^-_{Q_1}(u_i))-(1-T^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{Q_2}(u_i))\right|^2}}\right]\geq 0 \quad (7.5)$$

and $\left|T^+_{Q_1}(u_i)\right| \geq 0$, $\left|T^+_{Q_2}(u_i)\right| \geq 0$, $\left|T^+_{Q_1}(u_i)-T^+_{Q_2}(u_i)\right| \geq 0$, $\sqrt{1+\left|T^+_{Q_1}(u_i)\right|^2} \geq 0$, $\sqrt{1+\left|T^+_{Q_2}(u_i)\right|^2} \geq 0$, $\left|(1-T^+_{Q_1}(u_i))\right| \geq 0$, $\left|(1-T^+_{Q_2}(u_i))\right| \geq 0$, $\left|(1-T^+_{Q_1}(u_i))-(1-T^+_{Q_2}(u_i))\right| \geq 0$, $\sqrt{1+\left|(1-T^+_{Q_1}(u_i))\right|^2} \geq 0$, $\sqrt{1+\left|(1-T^+_{Q_2}(u_i))\right|^2} \geq 0$

$$\Rightarrow\left[\frac{2\left|T^+_{Q_1}(u_i)-T^+_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T^+_{Q_1}(u_i))-(1-T^+_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{Q_2}(u_i))\right|^2}}\right]\geq 0 \quad (7.6)$$

Similarly, we can show that

$$\left[\frac{2\left|I^{-}_{Q_1}(u_i)-I^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{Q_1}(u_i))-(1-I^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{Q_2}(u_i))\right|^2}}\right]\geq 0 \qquad (7.7)$$

$$\left[\frac{2\left|I^{+}_{Q_1}(u_i)-I^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{Q_1}(u_i))-(1-I^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{Q_2}(u_i))\right|^2}}\right]\geq 0 \qquad (7.8)$$

$$\left[\frac{2\left|F^{-}_{Q_1}(u_i)-F^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{Q_1}(u_i))-(1-F^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{Q_2}(u_i))\right|^2}}\right]\geq 0 \qquad (7.9)$$

and

$$\left[\frac{2\left|F^{+}_{Q_1}(u_i)-F^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{Q_1}(u_i))-(1-F^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{Q_2}(u_i))\right|^2}}\right]\geq 0 \qquad (7.10)$$

Adding Equation (7.2) to Equation (7.10), we obtain $CE_{NC}(Q_1, Q_2)\geq 0$.

ii. $$\left[\frac{2\left|T_{Q_1}(u_i)-T_{Q_2}(u_i)\right|}{\sqrt{1+\left|T_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{Q_1}(u_i))-(1-T_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2}(u_i))\right|^2}}\right]=0,$$

$$\Leftrightarrow T_{Q_1}(u_i)=T_{Q_2}(u_i) \qquad (7.11)$$

$$\left[\frac{2\left|I_{Q_1}(u_i)-I_{Q_2}(u_i)\right|}{\sqrt{1+\left|I_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2}(u)\right|^2}}+\frac{2\left|(1-I_{Q_1}(u_i))-(1-I_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2}(u_i))\right|^2}}\right]=0,$$

$$\Leftrightarrow I_{Q_1}(u_i)=I_{Q_2}(u_i) \qquad (7.12)$$

$$\left[\frac{2\left|F_{Q_1}(u_i)-F_{Q_2}(u_i)\right|}{\sqrt{1+\left|F_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{Q_1}(u_i))-(1-F_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2}(u_i))\right|^2}}\right]=0,$$

$\Leftrightarrow F_{Q_1}(u_i)=F_{Q_2}(u_i)$, For all values of $u_i \in U$. (7.13)

Again,

$$\left[\frac{2\left|T^{-}_{Q_1}(u_i)-T^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{-}_{Q_1}(u_i))-(1-T^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{Q_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow T^{-}_{Q_1}(u_i)=T^{-}_{Q_2}(u_i) \tag{7.14}$$

$$\left[\frac{2\left|T^{+}_{Q_1}(u_i)-T^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{+}_{Q_1}(u_i))-(1-T^{+}_{JQ2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{Q_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow T^{+}_{Q_1}(u_i)=T^{+}_{Q_2}(u_i) \tag{7.15}$$

$$\left[\frac{2\left|I^{-}_{Q_1}(u_i)-I^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{Q_1}(u_i))-(1-I^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{Q_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow I^{-}_{Q_1}(u_i)=I^{-}_{Q_2}(u_i) \tag{7.16}$$

$$\left[\frac{2\left|I^{+}_{Q_1}(u_i)-I^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{Q_1}(u_i))-(1-I^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{Q_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow I^{+}_{Q_1}(u_i)=I^{+}_{Q_2}(u_i) \tag{7.17}$$

$$\left[\frac{2\left|F^{-}_{Q_1}(u_i)-F^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{Q_1}(u_i))-(1-F^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{Q_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow F^{-}_{Q_1}(u_i)=F^{-}_{Q_2}(u_i) \tag{7.18}$$

$$\left[\frac{2\left|F^{+}_{Q_1}(u_i)-F^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{Q_1}(u_i))-(1-F^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{Q_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow F^{+}_{Q_1}(u_i)=F^{+}_{Q_2}(u_i)\text{, for all values of } u_i\in U. \tag{7.19}$$

From, Equation (7.11) to Equation (7.19), we obtain $CE_{NC}(Q_1, Q_2)=0$ iff

$T^{-}_{Q_1}(u_i)=T^{-}_{Q_2}(u_i)$, $T^{+}_{Q_1}(u_i)=T^{+}_{Q_2}(u_i)$, $I^{-}_{Q_1}(u_i)=I^{-}_{Q_2}(u_i)$, $I^{+}_{Q_1}(u_i)=I^{+}_{Q_2}(u_i)$,

$F^{-}_{Q_1}(u_i)=F^{-}_{Q_2}(u_i)$, $F^{+}_{Q_1}(u_i)=F^{+}_{Q_2}(u_i)$ and

$T_{Q_1}(u_i)=T_{Q_2}(u_i), I_{Q_1}(u_i)=I_{Q_2}(u_i), F_{Q_1}(u_i)=F_{Q_2}(u_i)$ for all $\forall u_i\in U$.

iii.

We have, $CE_{NC}(Q_1^c, Q_2^c)$

$$= \frac{1}{8}\left\{\sum_{i=1}^{n}\left(\left(\left[\frac{2\left|T^-_{Q_1^c}(u_i)-T^-_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|T^-_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|T^-_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-T^-_{Q_1^c}(u_i))-(1-T^-_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{Q_2^c}(u_i))\right|^2}}\right]+\right.\right.\right.$$

$$\left[\frac{2\left|T^+_{Q_1^c}(u_i)-T^+_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|T^+_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|T^+_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-T^+_{Q_1^c}(u_i))-(1-T^+_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^-_{Q_1^c}(u_i)-I^-_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|I^-_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|I^-_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-I^-_{Q_1^c}(u_i))-(1-I^-_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-I^-_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-I^-_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^+_{Q_1^c}(u_i)-I^+_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|I^+_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|I^+_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-I^+_{Q_1^c}(u_i))-(1-I^+_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-I^+_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-I^+_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^-_{Q_1^c}(u_i)-F^-_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|F^-_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|F^-_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-F^-_{Q_1^c}(u_i))-(1-F^-_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-F^-_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^+_{Q_1^c}(u_i)-F^+_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|F^+_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|F^+_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-F^+_{Q_1^c}(u_i))-(1-F^+_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-F^+_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-F^+_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|T_{Q_1^c}(u_i)-T_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|T_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-T_{Q_1^c}(u_i))-(1-T_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{Q_1^c}(u_i)-I_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|I_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2^c}(u)\right|^2}}+\frac{2\left|(1-I_{Q_1^c}(u_i))-(1-I_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left.\left[\frac{2\left|F_{Q_1^c}(u_i)-F_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|F_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-F_{Q_1^c}(u_i))-(1-F_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2^c}(u_i))\right|^2}}\right]\right)\right)\right\}$$

$$=\frac{1}{8}\left\{\sum_{i=1}^{n}\left(\left(\left[\frac{2\left|(1-T^-_{Q_1}(u_i))-(1-T^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{Q_2}(u_i))\right|^2}}+\frac{2\left|T^-_{Q_1}(u_i)-T^-_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{Q_2}(u_i)\right|^2}}\right]+\right.\right.\right.$$

$$\left[\frac{2\left|(1-T^+_{Q_1}(u_i))-(1-T^+_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{Q_2}(u_i))\right|^2}}+\frac{2\left|T^+_{Q_1}(u_i)-T^+_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{Q_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-I^-_{Q_1}(u_i))-(1-I^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^-_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^-_{Q_2}(u_i))\right|^2}}+\frac{2\left|I^-_{Q_1}(u_i)-I^-_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{Q_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-I^+_{Q_1}(u_i))-(1-I^+_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^+_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^+_{Q_2}(u_i))\right|^2}}+\frac{2\left|I^+_{Q_1}(u_i)-I^+_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^+_{Q_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-F^-_{Q_1}(u_i))-(1-F^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^-_{Q_2}(u_i))\right|^2}}+\frac{2\left|F^-_{Q_1}(u_i)-F^-_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{Q_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-F^+_{Q_1}(u_i))-(1-F^+_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^+_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^+_{Q_2}(u_i))\right|^2}}+\frac{2\left|F^+_{Q_1}(u_i)-F^+_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{Q_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-T_{Q_1}(u_i))-(1-T_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2}(u_i))\right|^2}}+\frac{2\left|T_{Q_1}(u_i)-T_{Q_2}(u_i)\right|}{\sqrt{1+\left|T_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-I_{Q_1}(u_i))-(1-I_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2}(u_i))\right|^2}}+\frac{2\left|I_{Q_1}(u_i)-I_{Q_2}(u_i)\right|}{\sqrt{1+\left|I_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2}(u)\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|(1-F_{Q_1}(u_i))-(1-F_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2}(u_i))\right|^2}}+\frac{2\left|F_{Q_1}(u_i)-F_{Q_2}(u_i)\right|}{\sqrt{1+\left|F_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2}(u_i)\right|^2}}\right]\right)\right\}$$

$$=\frac{1}{8}\left\{\sum_{i=1}^{n}\left(\left[\frac{2\left|T^-_{Q_1}(u_i)-T^-_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T^-_{Q_1}(u_i))-(1-T^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{Q_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T^+_{Q_1}(u_i)-T^+_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T^+_{Q_1}(u_i))-(1-T^+_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^-_{Q_1}(u_i)-I^-_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-I^-_{Q_1}(u_i))-(1-I^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^-_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^-_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^+_{Q_1}(u_i)-I^+_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^+_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-I^+_{Q_1}(u_i))-(1-I^+_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^+_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^+_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^-_{Q_1}(u_i)-F^-_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F^-_{Q_1}(u_i))-(1-F^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^-_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^+_{Q_1}(u_i)-F^+_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F^+_{Q_1}(u_i))-(1-F^+_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^+_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^+_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|T_{Q_1}(u_i)-T_{Q_2}(u_i)\right|}{\sqrt{1+\left|T_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{Q_1}(u_i))-(1-T_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{Q_1}(u_i)-I_{Q_2}(u_i)\right|}{\sqrt{1+\left|I_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2}(u)\right|^2}}+\frac{2\left|(1-I_{Q_1}(u_i))-(1-I_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left.\left[\frac{2\left|F_{Q_1}(u_i)-F_{Q_2}(u_i)\right|}{\sqrt{1+\left|F_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{Q_1}(u_i))-(1-F_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2}(u_i))\right|^2}}\right]\right)\right)\right\}=CE_{NC}(Q_1, Q_2).$$

iv.

Since, $\forall u_i \in U$, for a single valued part, we obtain:

$$\left|T_{Q_1}(u_i)-T_{Q_2}(u_i)\right|=\left|T_{Q_2}(u_i)-T_{Q_1}(u_i)\right|, \left|I_{Q_1}(u_i)-I_{Q_2}(u_i)\right|=\left|I_{Q_2}(u_i)-I_{Q_1}(u_i)\right|,$$

$$\left|F_{Q_1}(u_i)-F_{Q_2}(u_i)\right|=\left|F_{Q_2}(u_i)-F_{Q_1}(u_i)\right|,$$

$$\left|(1-T_{Q_1}(u_i))-(1-T_{Q_2}(u_i))\right|=\left|(1-T_{Q_2}(u_i))-(1-T_{Q_1}(u_i))\right|,$$

$$\left|(1-I_{Q_1}(u_i))-(1-I_{Q_2}(u_i))\right|=\left|(1-I_{Q_2}(u_i))-(1-I_{Q_1}(u_i))\right|,$$

$$\left|(1-F_{Q_1}(u_i))-(1-F_{Q_2}(u_i))\right|=\left|(1-F_{Q_2}(u_i))-(1-F_{Q_1}(u_i))\right|.$$

Then,

$$\sqrt{1+\left|T_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|T_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|I_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|I_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|F_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|F_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2}(u_i))\right|^2}=\sqrt{1+\left|(-T_{Q_2}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2},$$

$$\sqrt{1+\left|(1-I_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2}(u_i))\right|^2}=\sqrt{1+\left|(1-I_{Q_2}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_1}(u_i))\right|^2},$$

$$\sqrt{1+\left|(1-F_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2}(u_i))\right|^2}=\sqrt{1+\left|(1-F_{Q_2}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_1}(u_i))\right|^2},$$

$\forall u_i \in U.$

For the interval neutrosophic part, we obtain

$$\left|T^-_{Q_1}(u_i)-T^-_{Q_2}(u_i)\right|=\left|T^-_{Q_2}(u_i)-T^-_{Q_1}(u_i)\right|, \left|I^-_{Q_1}(u_i)-I^-_{Q_2}(u_i)\right|=\left|I^-_{Q_2}(u_i)-I^-_{Q_1}(u_i)\right|,$$

$$\left|F^-_{Q_1}(u_i)-F^-_{Q_2}(u_i)\right|=\left|F^-_{Q_2}(u_i)-F^-_{Q_1}(u_i)\right|,$$

$$\left|(1-T^-_{Q_1}(u_i))-(1-T^-_{Q_2}(u_i))\right|=\left|(1-T^-_{Q_2}(u_i))-(1-T^-_{Q_1}(u_i))\right|,$$

$\left|(1-I^-_{Q_1}(u_i))-(1-I^-_{Q_2}(u_i))\right|=\left|(1-I^-_{Q_2}(u_i))-(1-I^-_{Q_1}(u_i))\right|$,

$\left|(1-F^-_{Q_1}(u_i))-(1-F^-_{Q_2}(u_i))\right|=\left|(1-F^-_{Q_2}(u_i))-(1-F^-_{Q_1}(u_i))\right|$.

Then, we obtain $\sqrt{1+\left|T^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|T^-_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|T^-_{Q_1}(u_i)\right|^2}$,

$\sqrt{1+\left|I^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|I^-_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|I^-_{Q_1}(u_i)\right|^2}$,

$\sqrt{1+\left|F^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|F^-_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|F^-_{Q_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-T^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|(1-T^-_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|(1-T^-_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|(1-T^-_{Q_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-I^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|(1-I^-_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|(1-I^-_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|(1-I^-_{Q_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-F^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|(1-F^-_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|(1-F^-_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|(1-F^-_{Q_1}(u_i)\right|^2}$,

$\forall u_i \in U$.

Similarly, $\left|T^+_{Q_1}(u_i)-T^+_{Q_2}(u_i)\right|=\left|T^+_{Q_2}(u_i)-T^+_{Q_1}(u_i)\right|$, $\left|I^+_{Q_1}(u_i)-I^+_{Q_2}(u_i)\right|=\left|I^+_{Q_2}(u_i)-I^+_{Q_1}(u_i)\right|$,

$\left|F^+_{Q_1}(u_i)-F^+_{Q_2}(u_i)\right|=\left|F^+_{Q_2}(u_i)-F^+_{Q_1}(u_i)\right|$,

$\left|(1-T^+_{Q_1}(u_i))-(1-T^+_{Q_2}(u_i))\right|=\left|(1-T^+_{Q_2}(u_i))-(1-T^+_{Q_1}(u_i))\right|$,

$\left|(1-I^+_{Q_1}(u_i))-(1-I^+_{Q_2}(u_i))\right|=\left|(1-I^+_{Q_2}(u_i))-(1-I^+_{Q_1}(u_i))\right|$,

$\left|(1-F^+_{Q_1}(u_i))-(1-F^+_{Q_2}(u_i))\right|=\left|(1-F^+_{Q_2}(u_i))-(1-F^+_{Q_1}(u_i))\right|$, then

$\sqrt{1+\left|T^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|T^+_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|T^+_{Q_1}(u_i)\right|^2}$,

$\sqrt{1+\left|I^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^+_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|I^+_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|I^+_{Q_1}(u_i)\right|^2}$,

$\sqrt{1+\left|F^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|F^+_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|F^+_{Q_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-T^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|(1-T^+_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|(1-T^+_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|(1-T^+_{Q_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-I^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|(1-I^+_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|(1-I^+_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|(1-I^+_{Q_1}(u_i)\right|^2}$,

$$\sqrt{1+\left|(1-F^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|(1-F^{+}_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|(1-F^{+}_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|(1-F^{+}_{Q_1}(u_i)\right|^2},\ \forall u_i \in U.$$

Thus, $CE_{NC}\ (Q_1, Q_2) = CE_{NC}(Q_2, Q_1)$.□

Definition 7.2 Weighted NC-cross-entropy measure

We consider the weight w_i (i = 1, 2, 3,…, n) of u_i (i = 1, 2, 3, …, n) with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, a neutrosophic cubic weighted cross entropy measure between Q_1 and Q_2 can be defined as $CE^{w}_{NC}\ (Q_1, Q_2)$

$$= \frac{1}{8}\left\langle \sum_{i=1}^{n} w_i \left\{ \left[\frac{2\left|T^{-}_{Q_1}(u_i)-T^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{Q_2}(u_i)\right|^2}} + \frac{2\left|(1-T^{-}_{Q_1}(u_i))-(1-T^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{Q_2}(u_i))\right|^2}} \right] + \right.\right.$$

$$\left[\frac{2\left|T^{+}_{Q_1}(u_i)-T^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{Q_2}(u_i)\right|^2}} + \frac{2\left|(1-T^{+}_{Q_1}(u_i))-(1-T^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{Q_2}(u_i))\right|^2}} \right] +$$

$$\left[\frac{2\left|I^{-}_{Q_1}(u_i)-I^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{Q_2}(u_i)\right|^2}} + \frac{2\left|(1-I^{-}_{Q_1}(u_i))-(1-I^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{Q_2}(u_i))\right|^2}} \right] +$$

$$\left[\frac{2\left|I^{+}_{Q_1}(u_i)-I^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{Q_2}(u_i)\right|^2}} + \frac{2\left|(1-I^{+}_{Q_1}(u_i))-(1-I^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{Q_2}(u_i))\right|^2}} \right] +$$

$$\left[\frac{2\left|F^{-}_{Q_1}(u_i)-F^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{Q_2}(u_i)\right|^2}} + \frac{2\left|(1-F^{-}_{Q_1}(u_i))-(1-F^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{Q_2}(u_i))\right|^2}} \right] +$$

$$\left[\frac{2\left|F^{+}_{Q_1}(u_i)-F^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{Q_2}(u_i)\right|^2}} + \frac{2\left|(1-F^{+}_{Q_1}(u_i))-(1-F^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{Q_2}(u_i))\right|^2}} \right] +$$

$$\left[\frac{2\left|T_{Q_1}(u_i)-T_{Q_2}(u_i)\right|}{\sqrt{1+\left|T_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2}(u_i)\right|^2}} + \frac{2\left|(1-T_{Q_1}(u_i))-(1-T_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2}(u_i))\right|^2}} \right] +$$

$$\left[\frac{2\left|I_{Q_1}(u_i)-I_{Q_2}(u_i)\right|}{\sqrt{1+\left|I_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2}(u)\right|^2}}+\frac{2\left|(1-I_{Q_1}(u_i))-(1-I_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F_{Q_1}(u_i)-F_{Q_2}(u_i)\right|}{\sqrt{1+\left|F_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{Q_1}(u_i))-(1-F_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2}(u_i))\right|^2}}\right]\right\}\right\rangle \quad (7.20)$$

Theorem 7.2 Let Q_1, Q_2 be any two NCSs in U. Then, weighted NC-cross entropy measure $CE^w_{NC}(Q_1, Q_2)$ satisfies the following properties:

i. $CE^w_{NC}(Q_1, Q_2) \geq 0$, $\forall u_i \in U$.

ii. $CE^w_{NC}(Q_1, Q_2) = 0$ iff $T^-_{Q_1}(u_i) = T^-_{Q_2}(u_i)$, $T^+_{Q_1}(u_i) = T^+_{Q_2}(u_i)$, $I^-_{Q_1}(u_i) = I^-_{Q_2}(u_i)$, $I^+_{Q_1}(u_i) = I^+_{Q_2}(u_i)$, $F^-_{Q_1}(u_i) = F^-_{Q_2}(u_i)$, $F^+_{Q_1}(u_i) = F^+_{Q_2}(u_i)$ and $T_{Q_1}(u_i) = T_{Q_2}(u_i)$, $I_{Q_1}(u_i) = I_{Q_2}(u_i)$, $F_{Q_1}(u_i) = F_{Q_2}(u_i)$, $\forall u_i \in U$.

iii. $CE^w_{NC}(Q_1, Q_2) = CE^w_{NC}(Q^c_1, Q^c_2)$, $\forall u_i \in U$.

iv. $CE^w_{NC}(Q_1, Q_2) = CE^w_{NC}(Q_2, Q_1)$, $\forall u_i \in U$.

Proof of Theorem 7.2

i.

For all values of $u_i \in U$, $\left|T_{Q_1}(u_i)\right| \geq 0$, $\left|T_{Q_2}(u_i)\right| \geq 0$, $\left|T_{Q_1}(u_i) - T_{Q_2}(u_i)\right| \geq 0$, $\sqrt{1+\left|T_{Q_1}(u_i)\right|^2} \geq 0$, $\sqrt{1+\left|T_{Q_2}(u_i)\right|^2} \geq 0$, $\left|(1-T_{Q_1}(u_i))\right| \geq 0$, $\left|(1-T_{Q_2}(u_i))\right| \geq 0$, $\left|(1-T_{Q_1}(u_i))-(1-T_{Q_2}(u_i))\right| \geq 0$, $\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2} \geq 0$, $\sqrt{1+\left|(1-T_{Q_2}(u_i))\right|^2} \geq 0$.

Then,

$$\left[\frac{2\left|T_{Q_1}(u_i)-T_{Q_2}(u_i)\right|}{\sqrt{1+\left|T_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{Q_1}(u_i))-(1-T_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2}(u_i))\right|^2}}\right] \geq 0 \quad (7.21)$$

Similarly,

$$\left[\frac{2\left|I_{Q_1}(u_i)-I_{Q_2}(u_i)\right|}{\sqrt{1+\left|I_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2}(u)\right|^2}}+\frac{2\left|(1-I_{Q_1}(u_i))-(1-I_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2}(u_i))\right|^2}}\right]\geq 0 \qquad (7.22)$$

and

$$\left[\frac{2\left|F_{Q_1}(u_i)-F_{Q_2}(u_i)\right|}{\sqrt{1+\left|F_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{Q_1}(u_i))-(1-F_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2}(u_i))\right|^2}}\right]\geq 0 \qquad (7.23)$$

Again, for all values of $u_i \in U$, $\left|T^-_{Q_1}(u_i)\right|\geq 0$, $\left|T^-_{Q_2}(u_i)\right|\geq 0$, $\left|T^-_{Q_1}(u_i)-T^-_{Q_2}(u_i)\right|\geq 0$,

$\sqrt{1+\left|T^-_{Q_1}(u_i)\right|^2}\geq 0$, $\sqrt{1+\left|T^-_{Q_2}(u_i)\right|^2}\geq 0$, $\left|(1-T^-_{Q_1}(u_i))\right|\geq 0$, $\left|(1-T^-_{Q_2}(u_i))\right|\geq 0$,

$\left|(1-T^-_{Q_1}(u_i))-(1-T^-_{Q_2}(u_i))\right|\geq 0$, $\sqrt{1+\left|(1-T^-_{Q_1}(u_i))\right|^2}\geq 0$, $\sqrt{1+\left|(1-T^-_{Q_2}(u_i))\right|^2}\geq 0$

$$\Rightarrow\left[\frac{2\left|T^-_{Q_1}(u_i)-T^-_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T^-_{Q_1}(u_i))-(1-T^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{Q_2}(u_i))\right|^2}}\right]\geq 0 \qquad (7.24)$$

and

$\left|T^+_{Q_1}(u_i)\right|\geq 0$, $\left|T^+_{Q_2}(u_i)\right|\geq 0$, $\left|T^+_{Q_1}(u_i)-T^+_{Q_2}(u_i)\right|\geq 0$, $\sqrt{1+\left|T^+_{Q_1}(u_i)\right|^2}\geq 0$,

$\sqrt{1+\left|T^+_{Q_2}(u_i)\right|^2}\geq 0$,

$\left|(1-T^+_{Q_1}(u_i))\right|\geq 0$, $\left|(1-T^+_{Q_2}(u_i))\right|\geq 0$, $\left|(1-T^+_{Q_1}(u_i))-(1-T^+_{Q_2}(u_i))\right|\geq 0$,

$\sqrt{1+\left|(1-T^+_{Q_1}(u_i))\right|^2}\geq 0$, $\sqrt{1+\left|(1-T^+_{Q_2}(u_i))\right|^2}\geq 0$

$$\Rightarrow\left[\frac{2\left|T^+_{Q_1}(u_i)-T^+_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T^+_{Q_1}(u_i))-(1-T^+_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{Q_2}(u_i))\right|^2}}\right]\geq 0 \qquad (7.25)$$

Similarly, we can show that

$$\left[\frac{2\left|I_{Q_1}^{-}(u_i)-I_{Q_2}^{-}(u_i)\right|}{\sqrt{1+\left|I_{Q_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-I_{Q_1}^{-}(u_i))-(1-I_{Q_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2}^{-}(u_i))\right|^2}}\right]\geq 0 \quad (7.26)$$

$$\left[\frac{2\left|I_{Q_1}^{+}(u_i)-I_{Q_2}^{+}(u_i)\right|}{\sqrt{1+\left|I_{Q_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-I_{Q_1}^{+}(u_i))-(1-I_{Q_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2}^{+}(u_i))\right|^2}}\right]\geq 0 \quad (7.27)$$

$$\left[\frac{2\left|F_{Q_1}^{-}(u_i)-F_{Q_2}^{-}(u_i)\right|}{\sqrt{1+\left|F_{Q_1}^{-}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2}^{-}(u_i)\right|^2}}+\frac{2\left|(1-F_{Q_1}^{-}(u_i))-(1-F_{Q_2}^{-}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1}^{-}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2}^{-}(u_i))\right|^2}}\right]\geq 0 \quad (7.28)$$

and

$$\left[\frac{2\left|F_{Q_1}^{+}(u_i)-F_{Q_2}^{+}(u_i)\right|}{\sqrt{1+\left|F_{Q_1}^{+}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2}^{+}(u_i)\right|^2}}+\frac{2\left|(1-F_{Q_1}^{+}(u_i))-(1-F_{Q_2}^{+}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1}^{+}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2}^{+}(u_i))\right|^2}}\right]\geq 0 \quad (7.29)$$

Adding Equation (7.21) to Equation (7.29), and using $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1$, we have $CE_{NC}^{w}(Q_1, Q_2)\geq 0$.

Hence, this completes the proof.□

ii.

$$\left[\frac{2\left|T_{Q_1}(u_i)-T_{Q_2}(u_i)\right|}{\sqrt{1+\left|T_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{Q_1}(u_i))-(1-T_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow T_{Q_1}(u_i)=T_{Q_2}(u_i) \quad (7.30)$$

$$\left[\frac{2\left|I_{Q_1}(u_i)-I_{Q_2}(u_i)\right|}{\sqrt{1+\left|I_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2}(u)\right|^2}}+\frac{2\left|(1-I_{Q_1}(u_i))-(1-I_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2}(u_i))\right|^2}}\right]=0$$

$$\Leftrightarrow I_{Q_1}(u_i)=I_{Q_2}(u_i) \quad (7.31)$$

$$\left[\frac{2\left|F_{Q_1}(u_i)-F_{Q_2}(u_i)\right|}{\sqrt{1+\left|F_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{Q_1}(u_i))-(1-F_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow F_{Q_1}(u_i)=F_{Q_2}(u_i)$, For all values of $u_i \in U$. (7.32)

Again,

$$\left[\frac{2\left|T^-_{Q_1}(u_i)-T^-_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T^-_{Q_1}(u_i))-(1-T^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{Q_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow T^-_{Q_1}(u_i)=T^-_{Q_2}(u_i)$ (7.33)

$$\left[\frac{2\left|T^+_{Q_1}(u_i)-T^+_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T^+_{Q_1}(u_i))-(1-T^+_{JQ2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{Q_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow T^+_{Q_1}(u_i)=T^+_{Q_2}(u_i)$ (7.34)

$$\left[\frac{2\left|I^-_{Q_1}(u_i)-I^-_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-I^-_{Q_1}(u_i))-(1-I^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^-_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^-_{Q_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow I^-_{Q_1}(u_i)=I^-_{Q_2}(u_i)$ (7.35)

$$\left[\frac{2\left|I^+_{Q_1}(u_i)-I^+_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^+_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-I^+_{Q_1}(u_i))-(1-I^+_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^+_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^+_{Q_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow I^+_{Q_1}(u_i)=I^+_{Q_2}(u_i)$ (7.36)

$$\left[\frac{2\left|F^-_{Q_1}(u_i)-F^-_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F^-_{Q_1}(u_i))-(1-F^-_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^-_{Q_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow F^-_{Q_1}(u_i)=F^-_{Q_2}(u_i)$ (7.37)

$$\left[\frac{2\left|F^+_{Q_1}(u_i)-F^+_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^+_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F^+_{Q_1}(u_i))-(1-F^+_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^+_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^+_{Q_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow F^+_{Q_1}(u_i)=F^+_{Q_2}(u_i)$, for all values of $u_i \in U$. (7.38)

Using Equation (7.30) to Equation (7.38) and $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1, w_i \geq 0$, we have

$CE_{NC}^{w}(Q_1, Q_2) = 0$ iff $T_{Q_1}^{-}(u_i) = T_{Q_2}^{-}(u_i)$, $T_{Q_1}^{+}(u_i) = T_{Q_2}^{+}(u_i)$, $I_{Q_1}^{-}(u_i) = I_{Q_2}^{-}(u_i)$, $I_{Q_1}^{+}(u_i) = I_{Q_2}^{+}(u_i)$, $F_{Q_1}^{-}(u_i) = F_{Q_2}^{-}(u_i)$, $F_{Q_1}^{+}(u_i) = F_{Q_2}^{+}(u_i)$ and $T_{Q_1}(u_i) = T_{Q_2}(u_i)$, $I_{Q_1}(u_i) = I_{Q_2}(u_i)$, $F_{Q_1}(u_i) = F_{Q_2}(u_i)$ for all $u_i \in U$.

Hence, this completes the proof.☐

iii.We have,

$$
\begin{aligned}
&CE_{NC}^{w}(Q_1^c, Q_2^c) \\
&= \frac{1}{8}\left\langle \sum_{i=1}^{n} w_i \left\{ \left[\frac{2\left|T_{Q_1^c}^{-}(u_i) - T_{Q_2^c}^{-}(u_i)\right|}{\sqrt{1+\left|T_{Q_1^c}^{-}(u_i)\right|^2} + \sqrt{1+\left|T_{Q_2^c}^{-}(u_i)\right|^2}} + \frac{2\left|(1-T_{Q_1^c}^{-}(u_i)) - (1-T_{Q_2^c}^{-}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1^c}^{-}(u_i))\right|^2} + \sqrt{1+\left|(1-T_{Q_2^c}^{-}(u_i))\right|^2}} \right] + \right. \right. \\
&\left[\frac{2\left|T_{Q_1^c}^{+}(u_i) - T_{J_2^c}^{+}(u_i)\right|}{\sqrt{1+\left|T_{Q_1^c}^{+}(u_i)\right|^2} + \sqrt{1+\left|T_{Q_2^c}^{+}(u_i)\right|^2}} + \frac{2\left|(1-T_{Q_1^c}^{+}(u_i)) - (1-T_{Q_2^c}^{+}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1^c}^{+}(u_i))\right|^2} + \sqrt{1+\left|(1-T_{Q_2^c}^{+}(u_i))\right|^2}} \right] + \\
&\left[\frac{2\left|I_{Q_1^c}^{-}(u_i) - I_{Q_2^c}^{-}(u_i)\right|}{\sqrt{1+\left|I_{Q_1^c}^{-}(u_i)\right|^2} + \sqrt{1+\left|I_{Q_2^c}^{-}(u_i)\right|^2}} + \frac{2\left|(1-I_{Q_1^c}^{-}(u_i)) - (1-I_{Q_2^c}^{-}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1^c}^{-}(u_i))\right|^2} + \sqrt{1+\left|(1-I_{Q_2^c}^{-}(u_i))\right|^2}} \right] + \\
&\left[\frac{2\left|I_{Q_1^c}^{+}(u_i) - I_{Q_2^c}^{+}(u_i)\right|}{\sqrt{1+\left|I_{Q_1^c}^{+}(u_i)\right|^2} + \sqrt{1+\left|I_{Q_2^c}^{+}(u_i)\right|^2}} + \frac{2\left|(1-I_{Q_1^c}^{+}(u_i)) - (1-I_{Q_2^c}^{+}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1^c}^{+}(u_i))\right|^2} + \sqrt{1+\left|(1-I_{Q_2^c}^{+}(u_i))\right|^2}} \right] +
\end{aligned}
$$

$$\left[\frac{2\left|F^{-}_{Q_1^c}(u_i)-F^{-}_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|F^{-}_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{Q_1^c}(u_i))-(1-F^{-}_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{+}_{Q_1^c}(u_i)-F^{+}_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|F^{+}_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{Q_1^c}(u_i))-(1-F^{+}_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|T_{Q_1^c}(u_i)-T_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|T_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-T_{Q_1^c}(u_i))-(1-T_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{Q_1^c}(u_i)-I_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|I_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2^c}(u)\right|^2}}+\frac{2\left|(1-I_{Q_1^c}(u_i))-(1-I_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left.\left[\frac{2\left|F_{Q_1^c}(u_i)-F_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|F_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-F_{Q_1^c}(u_i))-(1-F_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2^c}(u_i))\right|^2}}\right]\right]\right\}\right\rangle$$

$$=\frac{1}{8}\left\langle\sum_{i=1}^{n}w_i\left\{\left[\left[\frac{2\left|(1-T^{-}_{Q_1}(u_i))-(1-T^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{Q_2}(u_i))\right|^2}}+\frac{2\left|T^{-}_{Q_1}(u_i)-T^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{Q_2}(u_i)\right|^2}}\right]+\right.\right.\right.$$

$$\left[\frac{2\left|(1-T^{+}_{Q_1}(u_i))-(1-T^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{Q_2}(u_i))\right|^2}}+\frac{2\left|T^{+}_{Q_1}(u_i)-T^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{Q_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{-}_{Q_1^c}(u_i)-F^{-}_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|F^{-}_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{Q_1^c}(u_i))-(1-F^{-}_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{+}_{Q_1^c}(u_i)-F^{+}_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|F^{+}_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{Q_1^c}(u_i))-(1-F^{+}_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|T_{Q_1^c}(u_i)-T_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|T_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-T_{Q_1^c}(u_i))-(1-T_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{Q_1^c}(u_i)-I_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|I_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2^c}(u)\right|^2}}+\frac{2\left|(1-I_{Q_1^c}(u_i))-(1-I_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2^c}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F_{Q_1^c}(u_i)-F_{Q_2^c}(u_i)\right|}{\sqrt{1+\left|F_{Q_1^c}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2^c}(u_i)\right|^2}}+\frac{2\left|(1-F_{Q_1^c}(u_i))-(1-F_{Q_2^c}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1^c}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2^c}(u_i))\right|^2}}\right]\right]\right\}\right\rangle$$

$$=\frac{1}{8}\left\langle\sum_{i=1}^{n}w_i\left\{\left[\left[\frac{2\left|(1-T^{-}_{Q_1}(u_i))-(1-T^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{Q_2}(u_i))\right|^2}}+\frac{2\left|T^{-}_{Q_1}(u_i)-T^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{Q_2}(u_i)\right|^2}}\right]+\right.\right.\right.$$

$$\left[\frac{2\left|(1-T^{+}_{Q_1}(u_i))-(1-T^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{Q_2}(u_i))\right|^2}}+\frac{2\left|T^{+}_{Q_1}(u_i)-T^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{Q_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-I^{-}_{Q_1}(u_i))-(1-I^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{Q_2}(u_i))\right|^2}}+\frac{2\left|I^{-}_{Q_1}(u_i)-I^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{Q_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-I^{+}_{Q_1}(u_i))-(1-I^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{Q_2}(u_i))\right|^2}}+\frac{2\left|I^{+}_{Q_1}(u_i)-I^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{Q_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-F^{-}_{Q_1}(u_i))-(1-F^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{Q_2}(u_i))\right|^2}}+\frac{2\left|F^{-}_{Q_1}(u_i)-F^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{Q_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-F^{+}_{Q_1}(u_i))-(1-F^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{Q_2}(u_i))\right|^2}}+\frac{2\left|F^{+}_{Q_1}(u_i)-F^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{Q_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-T_{Q_1}(u_i))-(1-T_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2}(u_i))\right|^2}}+\frac{2\left|T_{Q_1}(u_i)-T_{Q_2}(u_i)\right|}{\sqrt{1+\left|T_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-I_{Q_1}(u_i))-(1-I_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2}(u_i))\right|^2}}+\frac{2\left|I_{Q_1}(u_i)-I_{Q_2}(u_i)\right|}{\sqrt{1+\left|I_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2}(u)\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|(1-F_{Q_1}(u_i))-(1-F_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2}(u_i))\right|^2}}+\frac{2\left|F_{Q_1}(u_i)-F_{Q_2}(u_i)\right|}{\sqrt{1+\left|F_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2}(u_i)\right|^2}}\right]\right\}\right\rangle$$

$$=\frac{1}{8}\left\langle\sum_{i=1}^{n}w_i\left\{\left[\frac{2\left|T^{-}_{Q_1}(u_i)-T^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{-}_{Q_1}(u_i))-(1-T^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{Q_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T^{+}_{Q_1}(u_i)-T^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{+}_{Q_1}(u_i))-(1-T^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{-}_{Q_1}(u_i)-I^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{Q_1}(u_i))-(1-I^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{+}_{Q_1}(u_i)-I^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{Q_1}(u_i))-(1-I^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{-}_{Q_1}(u_i)-F^{-}_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{Q_1}(u_i))-(1-F^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{+}_{Q_1}(u_i)-F^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{Q_1}(u_i))-(1-F^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|T_{Q_1}(u_i)-T_{Q_2}(u_i)\right|}{\sqrt{1+\left|T_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{Q_1}(u_i))-(1-T_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I_{Q_1}(u_i)-I_{Q_2}(u_i)\right|}{\sqrt{1+\left|I_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2}(u)\right|^2}}+\frac{2\left|(1-I_{Q_1}(u_i))-(1-I_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F_{Q_1}(u_i)-F_{Q_2}(u_i)\right|}{\sqrt{1+\left|F_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{Q_1}(u_i))-(1-F_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2}(u_i))\right|^2}}\right]\right]\right\}\right\rangle$$

$$=CE^{w}_{NC}(Q_1, Q_2), \forall\ u_i \in U.$$

Hence, completes the proof. □

iv.

Since $\forall u_i \in U$, we obtain:

$$\left|T_{Q_1}(u_i)-T_{Q_2}(u_i)\right|=\left|T_{Q_2}(u_i)-T_{Q_1}(u_i)\right|,\ \left|I_{Q_1}(u_i)-I_{Q_2}(u_i)\right|=\left|I_{Q_2}(u_i)-I_{Q_1}(u_i)\right|,$$

$$\left|F_{Q_1}(u_i)-F_{Q_2}(u_i)\right|=\left|F_{Q_2}(u_i)-F_{Q_1}(u_i)\right|,$$

$$\left|(1-T_{Q_1}(u_i))-(1-T_{Q_2}(u_i))\right|=\left|(1-T_{Q_2}(u_i))-(1-T_{Q_1}(u_i))\right|,$$

$$\left|(1-I_{Q_1}(u_i))-(1-I_{Q_2}(u_i))\right|=\left|(1-I_{Q_2}(u_i))-(1-I_{Q_1}(u_i))\right|,$$

$$\left|(1-F_{Q_1}(u_i))-(1-F_{Q_2}(u_i))\right|=\left|(1-F_{Q_2}(u_i))-(1-F_{Q_1}(u_i))\right|.$$

Then, we obtain

$$\sqrt{1+\left|T_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|T_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|T_{Q_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|I_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|I_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|I_{Q_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|F_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|F_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|F_{Q_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_2}(u_i))\right|^2}=\sqrt{1+\left|(-T_{Q_2}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{Q_1}(u_i))\right|^2},$$

$$\sqrt{1+\left|(1-I_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_2}(u_i))\right|^2}=\sqrt{1+\left|(1-I_{Q_2}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{Q_1}(u_i))\right|^2},$$

$$\sqrt{1+\left|(1-F_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_2}(u_i))\right|^2}=\sqrt{1+\left|(1-F_{Q_2}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{Q_1}(u_i))\right|^2},$$

$\forall u_i \in U$.

We have

$$\left|T^-_{Q_1}(u_i)-T^-_{Q_2}(u_i)\right|=\left|T^-_{Q_2}(u_i)-T^-_{Q_1}(u_i)\right|,\ \left|I^-_{Q_1}(u_i)-I^-_{Q_2}(u_i)\right|=\left|I^-_{Q_2}(u_i)-I^-_{Q_1}(u_i)\right|,$$

$$\left|F^-_{Q_1}(u_i)-F^-_{Q_2}(u_i)\right|=\left|F^-_{Q_2}(u_i)-F^-_{Q_1}(u_i)\right|,$$

$$\left|(1-T^-_{Q_1}(u_i))-(1-T^-_{Q_2}(u_i))\right|=\left|(1-T^-_{Q_2}(u_i))-(1-T^-_{Q_1}(u_i))\right|,,$$

$$\left|(1-I^-_{Q_1}(u_i))-(1-I^-_{Q_2}(u_i))\right|=\left|(1-I^-_{Q_2}(u_i))-(1-I^-_{Q_1}(u_i))\right|,$$

$$\left|(1-F^-_{Q_1}(u_i))-(1-F^-_{Q_2}(u_i))\right|=\left|(1-F^-_{Q_2}(u_i))-(1-F^-_{Q_1}(u_i))\right|.$$

Then, we obtain

$$\sqrt{1+\left|T^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|T^-_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|T^-_{Q_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|I^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|I^-_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|I^-_{Q_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|F^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|F^-_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|F^-_{Q_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|(1-T^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|(1-T^-_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|(1-T^-_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|(1-T^-_{Q_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|(1-I^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|(1-I^-_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|(1-I^-_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|(1-I^-_{Q_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|(1-F^-_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|(1-F^-_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|(1-F^-_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|(1-F^-_{Q_1}(u_i)\right|^2},\ \forall u_i \in U.$$

Similarly, $\left|T^+_{Q_1}(u_i)-T^+_{Q_2}(u_i)\right|=\left|T^+_{Q_2}(u_i)-T^+_{Q_1}(u_i)\right|,\ \left|I^+_{Q_1}(u_i)-I^+_{Q_2}(u_i)\right|=\left|I^+_{Q_2}(u_i)-I^+_{Q_1}(u_i)\right|,$

$$\left|F^+_{Q_1}(u_i)-F^+_{Q_2}(u_i)\right|=\left|F^+_{Q_2}(u_i)-F^+_{Q_1}(u_i)\right|,$$

$$\left|(1-T^+_{Q_1}(u_i))-(1-T^+_{Q_2}(u_i))\right|=\left|(1-T^+_{Q_2}(u_i))-(1-T^+_{Q_1}(u_i))\right|,$$

$$\left|(1-I^+_{Q_1}(u_i))-(1-I^+_{Q_2}(u_i))\right|=\left|(1-I^+_{Q_2}(u_i))-(1-I^+_{Q_1}(u_i))\right|,$$

$\left|(1-F^+_{Q_1}(u_i))-(1-F^+_{Q_2}(u_i))\right|=\left|(1-F^+_{Q_2}(u_i))-(1-F^+_{Q_1}(u_i))\right|$, then

$$\sqrt{1+\left|T^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|T^{+}_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{Q_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|I^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|I^{+}_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{Q_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|F^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|F^{+}_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{Q_1}(u_i)\right|^2},$$

$$\sqrt{1+\left|(1-T^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{Q_2}(u_i))\right|^2}=\sqrt{1+\left|(1-T^{+}_{Q_2}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{Q_1}(u_i))\right|^2},$$

$$\sqrt{1+\left|(1-I^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{Q_2}(u_i))\right|^2}=\sqrt{1+\left|(1-I^{+}_{Q_2}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{Q_1}(u_i))\right|^2},$$

$$\sqrt{1+\left|(1-F^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{Q_2}(u_i))\right|^2}=\sqrt{1+\left|(1-F^{+}_{Q_2}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{Q_1}(u_i))\right|^2},\ \forall u_i\in U.$$

In addition, $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1, w_i \geq 0.$

Thus, $CE^{w}_{NC}(Q_1, Q_2) = CE^{w}_{NC}(Q_2, Q_1).$

Hence, this completes the proof.□

7.3 MADM strategy using proposed NC-cross entropy measure in the NCS environment

In this section, we develop an MADM strategy using the proposed NC-cross entropy measure.

Description of the MADM problem:

Let $A = \{A_1, A_2, A_3, ..., A_m\}$ and $G = \{G_1, G_2, G_3, ..., G_n\}$ be the discrete set of alternatives and attribute, respectively. Let $W = \{w_1, w_2, w_3, ..., w_n\}$ be the weight vector of attributes G_j (j = 1, 2, 3, …, n), where $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1.$

Now, we describe the steps of MADM strategy using NC-cross entropy measure.

Step 1. Formulate the decision matrices

For MADM with neutrosophic cubic information, the rating values of the alternatives $A_i\,(i=1,2,3,...,m)$ on the basis of criterion $G_j\,(j=1,2,3,...,n)$ provided by the decision-maker can be expressed in NCN as $a_{ij} = <[T^{-}_{ij}, T^{+}_{ij}], [I^{-}_{ij}, I^{+}_{ij}], [F^{-}_{ij}, F^{+}_{ij}], (T_{ij}, I_{ij}, F_{ij})>$ (i = 1, 2, 3, …, m; j = 1, 2, 3, …, n). We present these rating values of alternatives provided by the decision-maker in matrix form as follows:

$$M = \begin{array}{c} \\ A_1 \\ A_2 \\ . \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} G_1 & G_2 & \ldots & G_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ . & . & & . \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \end{array} \tag{7.39}$$

Step 2. Formulate priori/ideal decision matrix

In the MADM process, the priori decision matrix is used to select the best alternative from the set of feasible alternatives. In the decision-making situation, we use the following decision matrix as priori decision matrix.

$$P = \begin{array}{c} \\ A_1 \\ A_2 \\ . \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} G_1 & G_2 & \ldots & G_n \end{array} \\ \begin{pmatrix} a_{11}^{*} & a_{12}^{*} & \ldots & a_{1n}^{*} \\ a_{21}^{*} & a_{22}^{*} & & a_{2n}^{*} \\ . & & \ldots & . \\ a_{m1}^{*} & a_{m2}^{*} & \ldots & a_{mn}^{*} \end{pmatrix} \end{array} \tag{7.40}$$

Here, $a_{ij}^{*} = < [1,1],[0,0],[0,0] >$ for benefit attribute and $a_{ij}^{*} = < [0,0],[1,1],[1,1] >$ for cost attribute, (i = 1, 2, 3, …, m; j = 1, 2, 3, …, n).

Step 3. Formulate the weighted NC-cross entropy matrix

Using Equation (7.20), we calculate weighted NC-cross entropy values between decision matrix and priori matrix. The cross entropy value can be presented in matrix form as follows:

$${}^{NC}M_{CE}^{w} = \begin{pmatrix} CE_{NC}^{w}(A_1) \\ CE_{NC}^{w}(A_2) \\ . \quad . \quad . \\ . \quad . \quad . \\ . \quad . \quad . \\ CE_{NC}^{w}(A_m) \end{pmatrix} \tag{7.41}$$

Step 3. Rank the priority

The preference ranking order of all the alternatives is determined according to the increasing order of the cross entropy values $CE_{NC}^{w}(A_i)$ (i = 1, 2, 3, …, m).

7.4 Illustrative Example

In this section, we solve an illustrative example of an MADM problem to reflect the feasibility and efficiency of the proposed strategy in NCS environment.

Now, we use an example (He & Liu, 2013) for cultivation and analysis. A venture capital firm intends to make evaluation and selection to five enterprises with the investment potential:

(1) Automobile company (A_1)

(2) Military manufacturing enterprise (A_2)

(3) TV media company (A_3)

(4) Food enterprises (A_4)

(5) Computer software company (A_5)

On the basis of four attributes namely:

(1) Social and political factor (G_1)

(2) The environmental factor (G_2)

(3) Investment risk factor (G_3)

(4) The enterprise growth factor (G_4).

Weight vector of attributes is $W = \{0.24, 0.25, 0.23, 0.28\}$.

The steps of decision-making strategy to rank alternatives are presented as follows:

Step 1. Formulate the decision matrix

The decision-maker represents the rating values of alternative Ai (i = 1, 2, 3, 4, 5) with respect to the attribute G_j (j = 1, 2, 3, 4) in terms of NCNs and constructs the decision matrix M as follows:

$$
\mathrm{M} = \begin{array}{cc}
 & \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\
\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} &
\begin{pmatrix}
<[.6,.8],[.2,.3],[.3,.4],(.8,.3,.4)> & <[.5,.6],[.2,.4],[.4,.4],(.6,.4,.4)> & <[.6,.8],[.2,.3],[.2,.4],(.8,.4,.4)> & <[.6,.7],[.3,.4],[.3,.4],(.7,.4,.5)> \\
<[.5,.7],[.2,.3],[.3,.4],(.7,.3,.4)> & <[.7,.8],[.2,.3],[.2,.4],(.8,.3,.4)> & <[.6,.8],[.2,.4],[.3,.4],(.8,.4,.4)> & <[.6,.8],[.2,.3],[.2,.3],(.8,.2,.3)> \\
<[.6,.8],[.2,.4],[.3,.4],(.8,.4,.4)> & <[.6,.8],[.2,.3],[.2,.3],(.8,.3,.3)> & <[.8,.9],[.3,.5],[.3,.5],(.9,.5,.5)> & <[.6,.7],[.2,.3],[.2,.4],(.7,.3,.4)> \\
<[.5,.7],[.4,.5],[.3,.5],(.7,.5,.5)> & <[.4,.6],[.1,.3],[.3,.4],(.6,.3,.4)> & <[.5,.6],[.1,.2],[.3,.4],(.6,.2,.4)> & <[.5,.7],[.3,.4],[.4,.5],(.7,.4,.5)> \\
<[.7,.8],[.2,.4],[.2,.3],(.8,.4,.4)> & <[.4,.6],[.2,.4],[.2,.4],(.6,.4,.4)> & <[.5,.7],[.2,.4],[.3,.4],(.7,.4,.4))> & <[.6,.8],[.4,.5],[.4,.5],(.8,.5,.5>
\end{pmatrix}
\end{array}
\quad (7.42)
$$

Step 2. Formulate priori/ideal decision matrix

Priori/ideal decision matrix

$$
\mathrm{M}^1 = \begin{array}{cc}
 & \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\
\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} &
\begin{pmatrix}
<[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> \\
<[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> \\
<[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> \\
<[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> \\
<[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)>
\end{pmatrix}
\end{array}
\quad (7.43)
$$

Step 3. Calculate the weighted NC- cross entropy matrix

Using Equation (7.20), we calculate weighted NC-cross entropy values between ideal matrixes (7.43) and decision matrix (7.42):

$$
{}^{NC}\mathrm{M}_{CE}^{w} = \begin{pmatrix} 0.66 \\ 0.58 \\ 0.60 \\ 0.74 \\ 0.71 \end{pmatrix} \quad (7.44)
$$

Step 4. Rank the priority

The obtained cross entropy values are arranged increasing order as:

$0.58 < 0.60 < 0.66 < 0.71 < 0.74$.

The ranking priority of alternatives is $A_2 > A_3 > A_1 > A_5 > A_4$. Hence, military manufacturing enterprise (A_2) is the best alternative for investment.

Graphical representation of alternatives versus cross entropy is shown in Figure 2. From the Figure 7.1, we see that A_2 is the best preference alternative and A_4 is the least preference alternative.

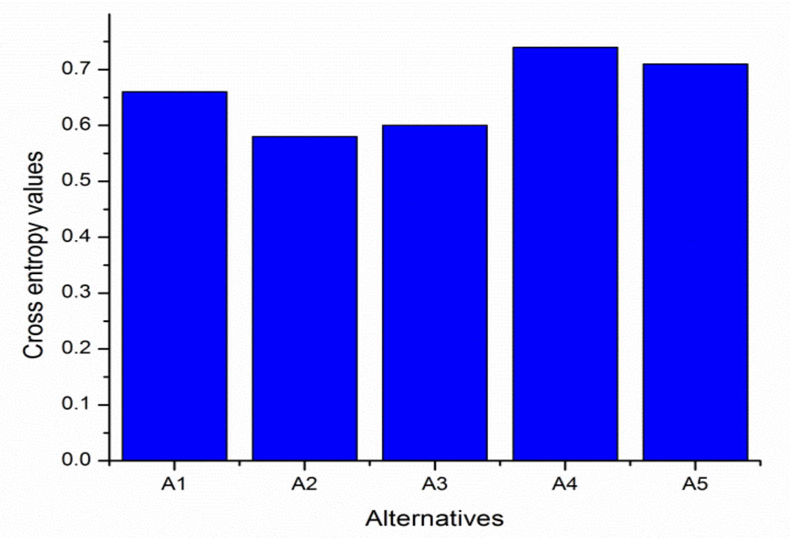

Figure 7.1 Bar diagram of alternatives versus cross entropy values of alternatives.

Figure 7.2 presents relation between cross entropy value and preference ranking of the alternative.

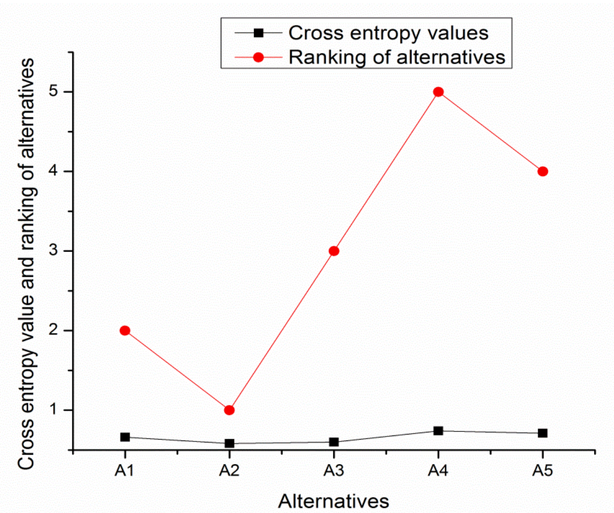

Figure 7.2 Graphical representation of cross entropy values and ranking of alternatives.

7.5 Conclusion

We have introduced NC-cross entropy measure in NCS environment. We have proved the basic properties of the proposed NC-cross entropy measure. We have also introduced weighted NC-cross entropy measure and established its basic properties. Using the weighted NC-cross entropy measure, we have developed a novel MADM strategy. We have also solved an MADM problem to illustrate the proposed MADM strategy. The proposed NC-cross entropy based MADM strategy can be employed to solve a variety of problems such as logistics center selection (Pramanik, Dalapati et al., 2016, 2018), weaver selection (Dey et al., 2015), teacher selection (Pramanik, & Mukhopadhyaya, 2011), brick selection (Mondal, & Pramanik, 2014a), renewable energy selection (San Cristóbal, 2011), etc. The proposed NC-cross entropy based MADM strategy can also be extended to MAGDM strategy using suitable aggregation operators.

Chapter 8

VIKOR based MAGDM strategy in bipolar neutrosophic set environment

8.1 Introduction

Bausys and Zavadskas (2015) extended VIKOR strategy to the interval neutrosophic set environment from crisp VIKOR environment and applied it to solve multi attribute decision making (MADM) problem. Pouresmaeil et al. (2017) proposed a multi attribute group decision making (MAGDM) strategy based on TOPSIS and VIKOR strategies in single valued neutrosophic set environment. Hu et al. (2017) proposed interval neutrosophic projection based VIKOR strategy and applied it for doctor selection. Pramanik, Dalapati, Alam et al. (2018) studied VIKOR strategy for neutrosophic cubic set environment.VIKOR strategy in bipolar neutrosophic set (BNS) is yet to appear in the literature. To fill the research gap, we develop a new MAGDM strategy based on proposed VIKOR strategy in bipolar neutrosophic set environment. To fill the research gaps, we propose VIKOR based strategy, which is capable of dealing with MAGDM problem in bipolar neutrosophic environment. The remainder of this chapter is organized as follows: In the Section 8.2, we introduce a bipolar neutrosophic weighted aggregation operator and prove its basic properties. We develop Normalization procedure of bipolar neutrosophic number. In Section 8.3, we develop a novel MAGDM strategy based on VIKOR strategy to solve the MADGM problems with bipolar neutrosophic information. In Section 8.4, an example is presented to illustrate the proposed strategy. Then in Section 8.5, we present the sensitivity analysis. In section 8.6, conclusion and future direction of research are presented.

The content of this chapter is based on the paper published in **"Neutrosophic Sets and Systems"**, 19, 57-69, 2018.

8.2 Normalization procedure and bipolar neutrosophic number weighted aggregation operator

Definition 8.1 Normalization procedure

Assume that, h_{ij} be a bipolar neutrosophic number (BNN) to express the rating value of i-th alternative with respect to j-th attribute (c_j). If c_j is a cost attribute, then h_{ij} is standardized by employing the complement of h_{ij}. When the attribute c_j is a benefit attribute, h_{ij} is not standardized. We introduce the following formula for normalization:

$$h^{*}_{ij} = <\{1\} - T^{+}_{ij}, \{1\} - I^{+}_{ij}, \{1\} - F^{+}_{ij}, \{-1\} - T^{-}_{ij}, \{-1\} - I^{-}_{ij}, \{-1\} - F^{-}_{ij} >. \quad (8.1)$$

Definition 8.2 Bipolar neutrosophic number weighted aggregation operator

Let $\{h^{1}_{ij}, h^{2}_{ij}, ..., h^{t}_{ij}\}$ be the set of t bipolar neutrosophic numbers and $\{\beta_1, \beta_2, \beta_3, ..., \beta_t\}$ be the set of corresponding weight of t bipolar neutrosophic numbers with conditions $\beta_p \geq 0$ and $\sum_{p=1}^{t} \beta_p = 1$. Then the bipolar neutrosophic number weighted aggregation (BNNWA) operator is defined as follows:

$$h_{ij} = BNNWA_{\beta}(\tilde{h}^{1}_{ij}, \tilde{h}^{2}_{ij}, ..., \tilde{h}^{t}_{ij}) = (\beta_1 \tilde{h}^{1}_{ij} \oplus \beta_2 \tilde{h}^{2}_{ij} \oplus \beta_3 \tilde{h}^{3}_{ij} \oplus ... \oplus \beta_\rho \tilde{h}^{t}_{ij}) =$$

$$< \left(\sum_{p=1}^{t} \beta_p \tilde{T}^{+(p)}_{ij}, \sum_{p=1}^{t} \beta_p \tilde{I}^{+(p)}_{ij}, \sum_{p=1}^{t} \beta_p \tilde{F}^{+(p)}_{ij}, \sum_{p=1}^{t} \beta_p \tilde{T}^{-(p)}_{ij}, \sum_{p=1}^{t} \beta_p \tilde{I}^{-(p)}_{ij}, \sum_{p=1}^{t} \beta_p \tilde{F}^{-(p)}_{ij} \right) > \quad (8.2)$$

The BNNWA operator satisfies the following properties:

8.1 Idempotency

8.2 Monotonicity

8.3Boundedness

Property 8.1 Idempotency

If all h^{1}_{ij}, $h^{2}_{ij}, ..., h^{t}_{ij} = h$ are equal, then $h_{ij} = BNNWA_{\beta}(h^{1}_{ij}, h^{2}_{ij}, ..., h^{t}_{ij}) = h$

Proof:

Since $h_{ij}^{1} = h_{ij}^{2} = ... = h_{ij}^{t} = h$, based on the Equation (3) and with conditions, $\beta_p \geq 0$ and $\sum_{p=1}^{t} \beta_p = 1$, we obtain $h_{ij} = BNNWA_{\beta}(h_{ij}^{1}, h_{ij}^{2},..., h_{ij}^{t}) = (\beta_1 h_{ij}^{1} \oplus \beta_2 h_{ij}^{2} \oplus \beta_3 h_{ij}^{3} \oplus ... \oplus \beta_t h_{ij}^{t}) =$

$$(\beta_1 h \oplus \beta_2 h \oplus \beta_3 h \oplus ... \oplus \beta_t h) = <\left([T^{+} \sum_{p=1}^{t} \beta_p, I^{+} \sum_{p=1}^{t} \beta_p, F^{+} \sum_{p=1}^{t} \beta_p, T^{-} \sum_{p=1}^{t} \beta_p, I^{-} \sum_{p=1}^{t} \beta_p, F^{-} \sum_{p=1}^{t} \beta_p] \right) > =$$

$$<\left(T^{+}, I^{+}, F^{+}, T^{-}, I^{-}, F^{-} \right) > = h.$$

Property 8.2 Monotonicity

Assume that $\{h_{ij}^{1}, h_{ij}^{2},..., h_{ij}^{t}\}$ and $\{h_{ij}^{*1}, h_{ij}^{*2},..., h_{ij}^{*t}\}$ be any two set of collections of t bipolar neutrosophic numbers with the condition $t_{ij}^{p} \leq t_{ij}^{*p}$ (p = 1, 2,...,t), then

$$BNNWA_{\beta}(h_{ij}^{1}, h_{ij}^{2},..., h_{ij}^{t}) \leq BNNWA_{\beta}(h_{ij}^{*1}, h_{ij}^{*2},..., h_{ij}^{*t}).$$

Proof:

From the given condition $T_{ij}^{+(p)} \leq T_{ij}^{+*(p)}$, we have $\beta_p T_{ij}^{+(p)} \leq \beta_p T_{ij}^{+*(p)}$

$$\Rightarrow \sum_{p=1}^{t} \beta_p T_{ij}^{+(p)} \leq \sum_{p=1}^{t} \beta_p T_{ij}^{+*(p)}.$$

From the given condition $I_{ij}^{+(p)} \geq I_{ij}^{+*(p)}$, we have $\beta_p I_{ij}^{+(p)} \geq \beta_p I_{ij}^{+*(p)}$

$$\Rightarrow \sum_{p=1}^{t} \beta_p I_{ij}^{+(p)} \geq \sum_{p=1}^{t} \beta_p I_{ij}^{+*(p)}.$$

From the given condition $F_{ij}^{+(p)} \geq F_{ij}^{+*(p)}$, we have $\beta_p F_{ij}^{+(p)} \geq \beta_p F_{ij}^{+*(p)}$

$$\Rightarrow \sum_{p=1}^{t} \beta_p F_{ij}^{+(p)} \geq \sum_{p=1}^{t} \beta_p F_{ij}^{+*(p)}.$$

From the given condition $T_{ij}^{-(p)} \geq T_{ij}^{-*(p)}$, we have $\beta_p T_{ij}^{-(p)} \geq \beta_p T_{ij}^{-*(p)}$

$$\Rightarrow \sum_{p=1}^{t} \beta_p T_{ij}^{-(p)} \geq \sum_{p=1}^{t} \beta_p T_{ij}^{-*(p)}.$$

From the given condition $I_{ij}^{-(p)} \leq I_{ij}^{-*(p)}$, we have $\beta_p I_{ij}^{-(p)} \leq \beta_p I_{ij}^{-*(p)}$

$$\Rightarrow \sum_{p=1}^{t} \beta_p I_{ij}^{-(p)} \leq \sum_{p=1}^{t} \beta_p I_{ij}^{-*(p)}.$$

From the given condition $F_{ij}^{-(p)} \leq F_{ij}^{-*(p)}$, we have $\beta_p F_{ij}^{-(p)} \leq \beta_p F_{ij}^{-*(p)}$

$$\Rightarrow \sum_{p=1}^{t} \beta_p F_{ij}^{-(p)} \leq \sum_{p=1}^{t} \beta_p F_{ij}^{-*(p)}.$$

From the above relations, we obtain, $BNNWA_{\beta}(h_{ij}^{1}, h_{ij}^{2}, ..., h_{ij}^{t}) \leq BNNWA_{\beta}(h_{ij}^{*1}, h_{ij}^{*2}, ..., h_{ij}^{*t})$.

Property: 8.3 Boundedness

Let $\{h_{ij}^{1}, h_{ij}^{2}, ..., h_{ij}^{t}\}$ be any collection of t bipolar neutrosophic numbers.

If $h^{+} = < \max_p \{T_{ij}^{+(p)}\}, \min_p \{I_{ij}^{+(p)}\}, \min_p \{F_{ij}^{+(p)}\}, \min_p \{T_{ij}^{-(p)}\}, \max_p \{I_{ij}^{-(p)}\}, \max_p \{F_{ij}^{-(p)}\} >$

$h^{-} = < \min_p \{T_{ij}^{+(p)}\}, \max_p \{I_{ij}^{+(p)}\}, \max_p \{F_{ij}^{+(p)}\}, \max_p \{T_{ij}^{-(p)}\}, \min_p \{I_{ij}^{-(p)}\}, \min_p \{F_{ij}^{-(p)}\} >$

(p= 1, 2, 3, ...,t).

Then, $h^{-} \leq BNNWA_{\beta}(h_{ij}^{1}, h_{ij}^{2}, ..., h_{ij}^{t}) \leq h^{+}$.

Proof:

From Property 1 and Property 2, we obtain

$BNNWA_{\beta}(h_{ij}^{1}, h_{ij}^{2}, ..., h_{ij}^{t}) \geq BNNWA_{\beta}(h^{-}, h^{-}, ..., h^{-}) = h^{-}$

and $BNNWA_{\beta}(h_{ij}^{1}, h_{ij}^{2}, ..., h_{ij}^{t}) \leq BNNWA_{\beta}(h^{+}, h^{+}, ..., h^{+}) = h^{+}$.

So, we have $h^{-} \leq BNNWA_{\beta}(h_{ij}^{1}, h_{ij}^{2}, ..., h_{ij}^{t}) \leq h^{+}$.

8.3 VIKOR strategy for solving MAGDM problem in bipolar neutrosophic set environment

Assume that, $A=\{A_1, A_2, A_3, ..., A_r\}$ be a set of r alternatives and $C=\{c_1, c_2, c_3, ..., c_s\}$ be a set of s attributes. Assume that, $\alpha=\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_s\}$ be the weight vector of the attributes, where $\alpha_k \geq 0$ and $\sum_{k=1}^{s} \alpha_k = 1$. Let $DM=\{DM_1, DM_2, DM_3, ..., DM_t\}$ be the set of t decision makers and $\beta=\{\beta_1, \beta_2, \beta_3, ..., \beta_t\}$ be the set of weight vector of decision makers, where $\beta_p \geq 0$ and $\sum_{p=1}^{t} \beta_p = 1$.

The VIKOR strategyconsisting of the following steps:

Step 1. Construction of the decision matrix

Let $M^p = (h_{ij}^p)_{r\times s}$ (p = 1, 2, 3,…, t) be the p-th decision matrix, where information about the alternative A_i is provided by the decision maker DM_p with respect to attribute c_j (j = 1, 2, 3, …, s). The p-th decision matrix denoted by M^p (See Equation (8.3)) is constructed as follows:

$$M^p = \begin{array}{c} \\ A_1 \\ A_2 \\ . \\ A_r \end{array} \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & ... & c_s \end{array} \\ \begin{pmatrix} h_{11}^p & h_{12}^p & ... & h_{1s}^p \\ h_{21}^p & h_{22}^p & ... & h_{2s}^p \\ . & . & & . \\ h_{r1}^p & h_{r2}^p & ... & h_{rs}^p \end{pmatrix} \end{array} \tag{8.3}$$

Here, p = 1, 2, 3,…, t; i = 1, 2, 3,…, r; j = 1, 2, 3,…, s.

Step 2. Normalization of the decision matrix

To normalize we use the following equation:

$$h_{ij}^* = <\{1\} - T_{ij}^+, \{1\} - I_{ij}^+, \{1\} - F_{ij}^+, \{-1\} - T_{ij}^-, \{-1\} - I_{ij}^-, \{-1\} - F_{ij}^- >.$$

Using the normalization technique, we obtain the following normalized decision matrix (See eq. (8.4)):

$$M^p = \begin{pmatrix} & c_1 & c_2 & \ldots c_s \\ A_1 & \tilde{h}_{11}^p & \tilde{h}_{12}^p & \ldots \tilde{h}_{1s}^p \\ A_2 & \tilde{h}_{21}^p & \tilde{h}_{22}^p & \ldots \tilde{h}_{2s}^p \\ . & . & . & . \\ A_r & \tilde{h}_{r1}^p & \tilde{h}_{r2}^p & \ldots \tilde{h}_{rs}^p \end{pmatrix} \tag{8.4}$$

Here, $\tilde{h}_{ij}^p = \begin{cases} h_{ij}^p & \text{if } c_j \text{ is benefit attribute.} \\ \left(h_{ij}^*\right)^p & \text{if } c_j \text{ is cost attribute.} \end{cases}$

Step 3. Aggregation of the decision matrices

Using BNNWA operator in Equation (8.2), we obtain the aggregated decision matrix as follows:

$$M = \begin{pmatrix} & c_1 & c_2 \ldots & .c_s \\ A_1 & h_{11} & h_{12} & \ldots h_{1s} \\ A_2 & h_{21} & h_{22} & \ldots h_{2s} \\ . & . & . & . \\ A_r & h_{r1} & h_{r2} & \ldots h_{rs} \end{pmatrix} \tag{8.5}$$

where , (i = 1, 2, 3, …, r; j = 1, 2, 3, …, s; p=1, 2, ….t).

Step 4. Define the positive ideal solution and negative ideal solution

$$h_{ij}^{+} = < \max_i T_{ij}^{+}, \min_i I_{ij}^{+}, \min_i F_{ij}^{+}, \min_i T_{ij}^{-}, \max_i I_{ij}^{-}, \max_i F_{ij}^{-} > \tag{8.6}$$

$$h_{ij}^{-} = < \min_i T_{ij}^{+}, \max_i I_{ij}^{+}, \max_i F_{ij}^{+}, \max_i T_{ij}^{-}, \min_i I_{ij}^{-}, \min_i F_{ij}^{-} > \tag{8.7}$$

Step 5. Define and compute the value of Γ_i and Z_i (i = 1, 2, 3, ..., r)

Γ_i and Z_i represent the average and worst group scores for the alternative A_i respectively, with the relations

$$\Gamma_i = \sum_{j=1}^{s} \frac{\alpha_j \times D(h_{ij}^{+}, \tilde{h}_{ij})}{D(h_{ij}^{+}, h_{ij}^{-})} \tag{8.8}$$

$$Z_i = \max_j \left\{ \frac{\alpha_j \times D(h_{ij}^{+}, \tilde{h}_{ij})}{D(h_{ij}^{+}, h_{ij}^{-})} \right\} \tag{8.9}$$

where, α_j is the weight of c_j.

The smaller values of Γ_i and Z_i correspond to the better average and worse group scores for alternative A_i, respectively.

Step 6. Calculate the values of index VIKOR φ_i (i = 1, 2, 3, …, r) by the relation

$$\varphi_i = \gamma \frac{(\Gamma_i - \Gamma^{-})}{(\Gamma^{+} - \Gamma^{-})} + (1-\gamma) \frac{(Z_i - Z^{-})}{(Z^{+} - Z^{-})} \tag{8.10}$$

Here, $\Gamma_i^{-} = \min_i \Gamma_i$, $\Gamma_i^{+} = \max_i \Gamma_i$, $Z_i^{-} = \min_i Z_i$, $Z_i^{+} = \max_i Z_i$ (8.11)

and γ depicts the decision making mechanism coefficient. If $\gamma > 0.5$, it is for "the maximum group utility"; if $\gamma < 0.5$, it is " the minimum regret"; it has been inferred that the VIKOR index value is mostly taken as $v = 0.5$.

Step 7. Rank the priority of alternatives

We rank the alternatives using the traditional VIKOR strategy.

Step 8. Determine the compromise solution

The procedure of determining the compromise solution has been discussed in Section 6.5 of Chapter 6.

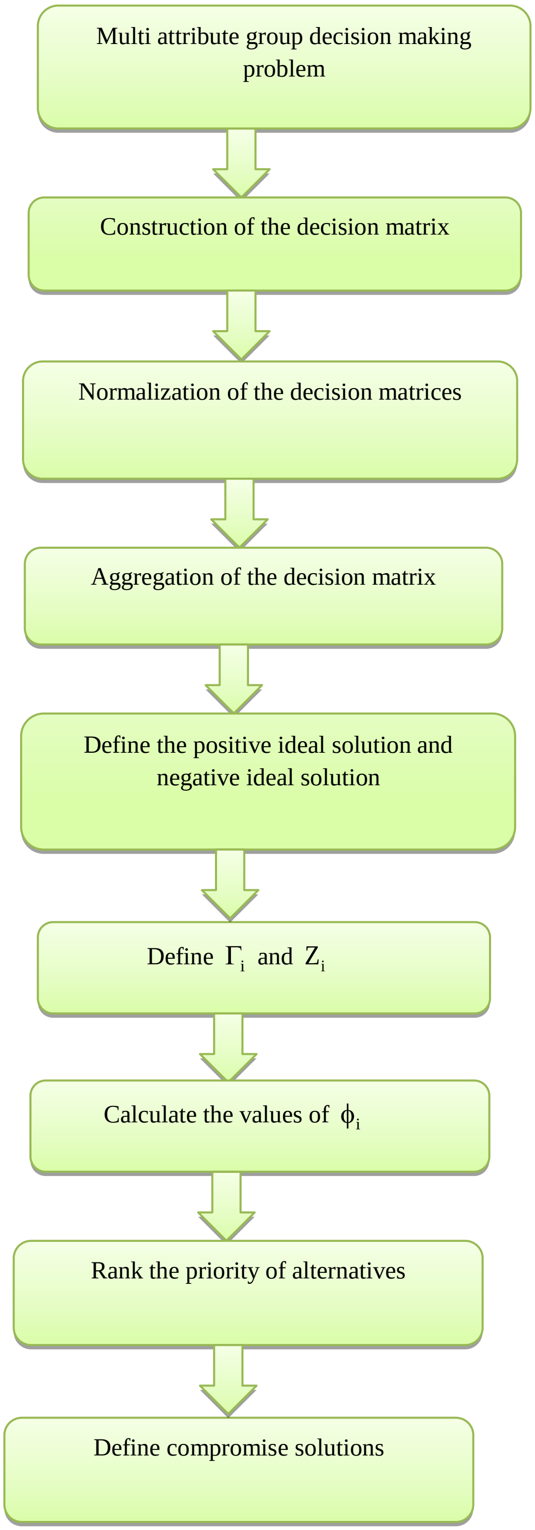

Figure 8.1 Decision making procedure of proposed MAGDM strategy.

8.4 Illustrative example

To demonstrate the applicability and feasibility of the proposed strategy, we solve an MAGDM problem adapted from (He & liu, 2013). We assume that an investment company wants to invest a sum of money in the best option. The investment company forms a decision making board involving of three members (DM_1, DM_2, DM_3) who evaluate the four alternatives to invest money. The alternatives are Car company (A_1), Food company (A_2), Computer company (A_3) and Arms company (A_4). Decision makers make decision to evaluate alternatives based on the criteria namely, risk factor (c_1), growth factor (c_2), environment impact (c_3). We consider three criteria as benefit type based on Zhang et al. (2016). Assume that the weight vector of attributes is $\alpha = (0.37, 0.33, 0.3)^T$ and weight vector of decision makers is $\beta = (0.38, 0.32, 0.3)^T$. Now, we apply the proposed MAGDM strategy which has following steps.

Step 1. Construction of the decision matrix

We constructed the decision matrix using rating values provided by the decision makers in terms of BNNs with respect to the criteria as follows:

Decision matrix for DM_1 in BNN form

$$M^1 = \begin{pmatrix} & c_1 & c_2 & c_3 \\ A_1 & (.5, .6, .7, -.3, -.6, -.3) & (.8, .5, .6, -.4, -.6, -.3) & (.9, .4, .6, -.1, -.6, -.5) \\ A_2 & (.6, .2, .2, -.4, -.5, -.3) & (.6, .3, .7, -.4, -.3, -.5) & (.7, .5, .3, -.4, -.3, -.3) \\ A_3 & (.8, .3, .5, -.6, -.4, -.5) & (.5, .2, .4, -.1, -.5, -.3) & (.4, .2, .8, -.5, -.3, -.2) \\ A_4 & (.7, .5, .3, -.6, -.3, -.3) & (.8, .7, .2, -.8, -.6, -.1) & (.6, .3, .4, -.3, -.4, -.7) \end{pmatrix}$$

Decision matrix for DM_2 in BNN form

$$M^2 = \begin{array}{cccc} & c_1 & c_2 & c_3 \\ A_1 & (.6, .3, .4, -.5, -.3, -.7) & (.5, .3, .4, -.3, -.3, -.4) & (.1, .5, .7, -.5, -.2, -.6) \\ A_2 & (.7, .4, .5, -.3, -.2, -.1) & (.8, .4, .5, -.7, -.3, -.2) & (.6, .2, .7, -.5, -.2, -.9) \\ A_3 & (.8, .3, .2, -.5, -.2, -.6) & (.3, .2, .1, -.6, -.3, -.4) & (.7, .5, .4, -.4, -.3, -.2) \\ A_4 & (.3, .5, .2, -.5, -.5, -.2) & (.5, .6, .4, -.3, -.6, -.7) & (.4, .3, .8, -.5, -.6, -.5) \end{array}$$

Decision matrix for DM_3 in BNN form

$$M^3 = \begin{array}{cccc} & c_1 & c_2 & c_3 \\ A_1 & (.9, .6, .4, -.7, -.3, -.2) & (.7, .5, .3, -.6, -.2, -.5) & (.4, .2, .3, -.2, -.5, -.7) \\ A_2 & (.5, .3, .2, -.6, -.4, -.1) & (.5, .2, .7, -.3, -.2, -.5) & (.6, .3, .2, -.7, -.6, -.3) \\ A_3 & (.2, .5, .6, -.4, -.5, -.7) & (.3, .2, .7, -.2, -.3, -.5) & (.8, .2, .4, -.2, -.3, -.6) \\ A_4 & (.8, .5, .5, -.4, -.6, -.3) & (.9, .3, .4, -.5, -.6, -.7) & (.7, .4, .3, -.2, -.5, -.7) \end{array}$$

Step 2. Normalization of the decision matrix

Since all the criteria are considered as benefit type, we do not need to normalize the decision matrices (M^1, M^2, M^3).

Step 3. Aggregated decision matrix

Using Equation (8.2), the aggregated decision matrix is presented below:

M =

$$\begin{array}{cccc} & c_1 & C_2 & C_3 \\ A_1 & (.22, .17, .17, -.16, -.14, -.13) & (.22, .14, .15, -.14, -.13, -.13) & (.16, .12, .18, -.10, -.10, -.20) \\ A_2 & (.20, .10, .10, -.14, -.12, -.10) & (.21, .10, .21, -.15, -.10, -.13) & (.21, .11, .13, -.17, -.12, -.16) \\ A_3 & (.21, .12, .16, -.17, -.12, -.20) & (.13, .10, .13, -.10, -.12, -.13) & (.21, .10, .18, -.13, -.10, -.11) \\ A_4 & (.20, .17, .11, -.17, -.15, -.10) & (.24, .18, .11, -.19, -.20, -.16) & (.19, .11, .17, -.11, -.16, -.21) \end{array}$$

Step 4. Determine the positive ideal solution and negative ideal solution

The positive ideal solution

$$h_{ij}^{+} = \begin{array}{ccc} c_1 & c_2 & c_3 \\ (.22, .10, .10, -.14, -.12, -.10) & (.24, .10, .11, -.19, -.10, -.13) & (.21, .10, .13, -.17, -.10, -.11) \end{array}$$

and the negative ideal solution

$$h_{ij}^{-} = \begin{array}{ccc} c_1 & c_2 & c_3 \\ (.20, .17, .17, -.14, -.15, -.20) & (.13, .18, .21, -.10, -.20, -.16) & (.16, .12, .18, -.10, -.16, -.11) \end{array}$$

Step 5. Compute Γ_i and Z_i

We have computed the values of Γ_i and Z_i as:

$\Gamma_1 = 0.75$, $\Gamma_2 = 0.38$, $\Gamma_3 = 0.60$, $\Gamma_4 = 0.75$ and $Z_1 = 0.34$, $Z_2 = 0.16$, $Z_3 = 0.33$, $Z_4 = 0.34$.

Step 6. Calculate the values of φ_i

For $\gamma = 0.5$, we obtain, $\varphi_1 = 1$, $\varphi_2 = \mathbf{0}$, $\varphi_3 = 0.77$, $\varphi_4 = 1$.

Step 7. Rank the priority of alternatives

The preference ranking order of the alternatives is presented in Table 8.1

Table 8.1 Preference ranking order and compromise solution based on Γ, Z and φ

	A_1	A_2	A_3	A_4	Reference ranking order	Compromise solution
Γ	0.75	0.38	0.60	0.75	$A_2 \succ A_3 \succ A_4 = A_1$	A_2
Z	0.34	0.16	0.33	0.34	$A_2 \succ A_3 \succ A_4 = A_1$	A_2
φ $(\gamma = 0.5)$	1	0	0.77	1	$A_2 \succ A_3 \succ A_4 = A_1$	A_2

Step 8. Determine the compromise solution

From Table 8.1, we have $\varphi(A_2) = 0$, and $\varphi(A_3) = 0.77$.

Therefore, $\varphi(A_3) - \varphi(A_2) = 0.77 > 0.333$ (since, r = 4; 1/(r-1) = 0.333), which satisfies the condition $1(\varphi(A^2) - \varphi(A^1) \ge \frac{1}{(r-1)})$.

Also we observe that the alternative A_2 is the best ranked by Γ, Z, which satisfies the condition 2.

So A_2 is the compromise solution. Since A_2 satisfies the both conditions, no need to calculate the compromise solution.

8.5 The influence of parameter γ

Table 8.2: shows that the ranking order of alternatives (A_i) with the value of γ changing from 0.1 to 0.9.

Table 8.2 Values of φ_i (i = 1, 2, 3, 4) and ranking of alternatives for different values of γ

Values of γ	Values of ϕ_i	Preference order of alternatives
$\gamma = 0.1$	$\varphi_1 = 1$, $\varphi_2 = \mathbf{0}$, $\varphi_3 = 0.915$, $\varphi_4 = 1$	$A_2 \succ A_3 \succ A_4 = A_1$.
$\gamma = 0.2$	$\varphi_1 = 1$, $\varphi_2 = \mathbf{0}$, $\varphi_3 = 0.880$, $\varphi_4 = 1$	$A_2 \succ A_3 \succ A_4 = A_1$.
$\gamma = 0.3$	$\varphi_1 = 1$, $\varphi_2 = \mathbf{0}$, $\varphi_3 = 0.845$, $\varphi_4 = 1$	$A_2 \succ A_3 \succ A_4 = A_1$.
$\gamma = 0.4$	$\varphi_1 = 1$, $\varphi_2 = \mathbf{0}$, $\varphi_3 = 0.810$, $\varphi_4 = 1$	$A_2 \succ A_3 \succ A_4 = A_1$.
$\gamma = 0.5$	$\varphi_1 = 1$, $\varphi_2 = \mathbf{0}$, $\varphi_3 = 0.770$, $\varphi_4 = 1$	$A_2 \succ A_3 \succ A_4 = A_1$.
$\gamma = 0.6$	$\varphi_1 = 1$, $\varphi_2 = \mathbf{0}$, $\varphi_3 = 0.740$, $\varphi_4 = 1$	$A_2 \succ A_3 \succ A_4 = A_1$.
$\gamma = 0.7$	$\varphi_1 = 1$, $\varphi_2 = \mathbf{0}$, $\varphi_3 = 0.700$, $\varphi_4 = 1$	$A_2 \succ A_3 \succ A_4 = A_1$.
$\gamma = 0.8$	$\varphi_1 = 1$, $\varphi_2 = \mathbf{0}$, $\varphi_3 = 0.670$, $\varphi_4 = 1$	$A_2 \succ A_3 \succ A_4 = A_1$.
$\gamma = 0.9$	$\varphi_1 = 1$, $\varphi_2 = \mathbf{0}$, $\varphi_3 = 0.640$, $\varphi_4 = 1$	$A_2 \succ A_3 \succ A_4 = A_1$.

Figure 8.2 represents the graphical representation of alternatives (A_i) versus φ_i (i = 1, 2, 3, 4) for different values of γ .

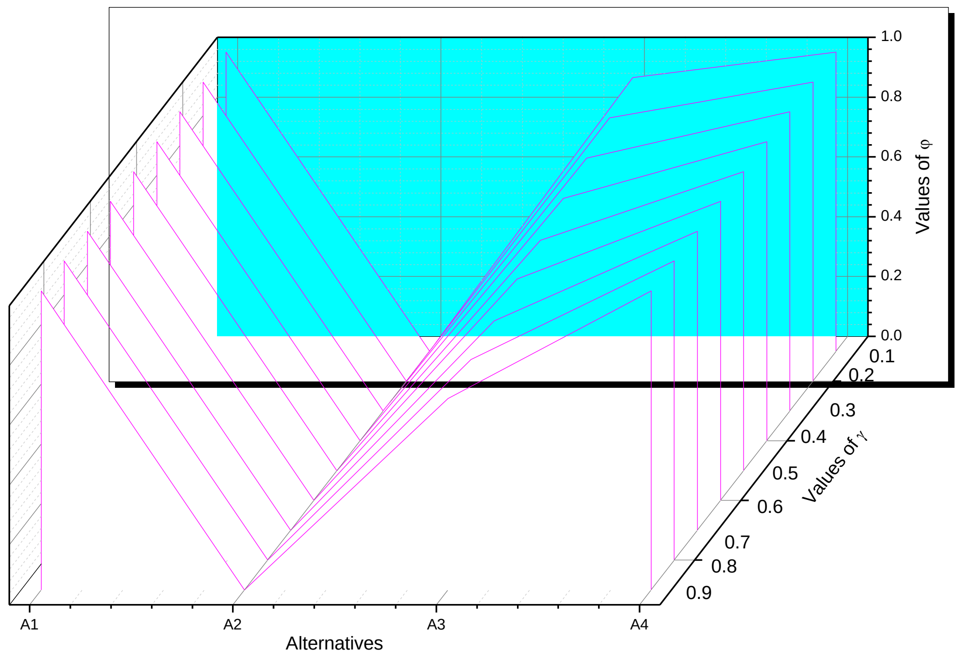

Figure 8.2 Graphical representation of ranking order of alternatives for different values of γ .

8.6 Conclusion

In this Chapter, we developed a VIKOR strategy to bipolar neutrosophic set environment. We have introduced bipolar neutrosophic number weighted aggregation operator and applied it to aggregate the individual opinion to one group opinion. Finally, we solve an MAGDM problem to show the feasibility and efficiency of the propose VIKOR strategy. We present a sensitivity analysis to reflect the impact of different values of the decision making mechanism coefficient on ranking order of the alternatives. The proposed VIKOR based MAGDM strategy can be employed to solve a variety of problems such as logistics center selection, renewable energy selection, fault diagnosis, weaver selection, etc.

Chapter 9

Conclusion

9.1 Conclusion

In this study, different MADM and MAGDM strategies have been addressed in NS, INS, and bipolar neutrosophic set and neutrosophic cubic set environment. This thesis covers neutrosophic decision strategies based on NS- cross entropy, IN-cross entropy, NC-TODIM, neutrosophic cubic similarity measure, NC-cross entropy, VIKOR, and NC-VIKOR in neutrosophic set environment, interval neutrosophic set environment, neutrosophic cubic set and bipolar neutrosophic set environment.

9.2 Scope of future research

The field of decision making in neutrosophic and neutrosophic hybrid environment have grown rapidly due to the significant works done by many researchers mainly in NS, INS, NCS, BNS, rough neutrosophic set, neutrosophic hesitant fuzzy set, etc. However, MADM in neutrosophic hybrid environment is a new field of research. After the inception of the international journal "Neutrosophic Sets and System", researchers have shown great interest to further develop the theory and its applications in a wide variety of areas. In a word, a wave of new research and new trends in neutrosophic theory and its applications have been observed (Smarandache & Pramanik, 2016, 2018). As a promising tool, different neutrosophic hybrid systems such as rough neutrosophic set (Broumi et al., 2014; Pramanik & Mondal 2015b, 2015c), rough bipolar neutrosophic set (Pramanik & Mondal, 2015e), tri-complex rough neutrosophic set (Mondal and Pramanik, 2015e), Hyper-complex rough neutrosophic set (Mondal et al., 2017), bipolar neutrosophic set (Deli et al., 2015), neutrosophic hesitant fuzzy set (Ye, 2015c; Biswas et al. 2016b), for MADM and MAGDM are open for new research.

Future research areas may be summarized as follows:

- The proposed NS-cross entropy-based MAGDM strategy can be applied in real decision making problem such as pattern recognition, personnel selection problem, etc.
- The proposed IN-cross entropy-based MAGDM strategy can be also extended to the neutrosophic hybrid set environment.
- The proposed NC-TODIM strategy can be extended to the neutrosophic hybrid environment.
- The proposed neutrosophic cubic MAGDM strategy based on similarity measure can be extended to the neutrosophic hybrid environment.
- The proposed NC-cross entropy strategy offers a novel and effective strategy for decision makers under the NCS environment, and will open up a new avenue of research into the neutrosophic hybrid environment.
- The proposed NC-VIKOR based MAGDM strategy can be employed to solve a variety of problems such as logistics center selection, teacher selection, renewable energy selection, fault diagnosis, etc. in neutrosophic cubic set environment.
- The proposed VIKOR based MAGDM strategy can be employed to solve a variety of problems such as logistics center selection, teacher selection, renewable energy selection, fault diagnosis, weaver selection in bipolar neutrosophic set environment.

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NS-Cross Entropy-Based MAGDM under Single-Valued Neutrosophic Set Environment

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Received: 29 December 2017; Accepted: 6 February 2018; Published: 9 February 2018

Abstract: A single-valued neutrosophic set has king power to express uncertainty characterized by indeterminacy, inconsistency and incompleteness. Most of the existing single-valued neutrosophic cross entropy bears an asymmetrical behavior and produces an undefined phenomenon in some situations. In order to deal with these disadvantages, we propose a new cross entropy measure under a single-valued neutrosophic set (SVNS) environment, namely NS-cross entropy, and prove its basic properties. Also we define weighted NS-cross entropy measure and investigate its basic properties. We develop a novel multi-attribute group decision-making (MAGDM) strategy that is free from the drawback of asymmetrical behavior and undefined phenomena. It is capable of dealing with an unknown weight of attributes and an unknown weight of decision-makers. Finally, a numerical example of multi-attribute group decision-making problem of investment potential is solved to show the feasibility, validity and efficiency of the proposed decision-making strategy.

Keywords: neutrosophic set; single-valued neutrosophic set; NS-cross entropy measure; multi-attribute group decision-making

1. Introduction

To tackle the uncertainty and modeling of real and scientific problems, Zadeh [1] first introduced the fuzzy set by defining membership measure in 1965. Bellman and Zadeh [2] contributed important research on fuzzy decision-making using max and min operators. Atanassov [3] established the intuitionistic fuzzy set (IFS) in 1986 by adding non-membership measure as an independent component to the fuzzy set. Theoretical and practical applications of IFSs in multi-criteria decision-making (MCDM) have been reported in the literature [4–12]. Zadeh [13] introduced entropy measure in the fuzzy environment. Burillo and Bustince [14] proposed distance measure between IFSs and offered an axiomatic definition of entropy measure. In the IFS environment, Szmidt and Kacprzyk [15] proposed a new entropy measure based on geometric interpretation of IFS. Wei et al. [16] developed an entropy measure for interval-valued intuitionistic fuzzy set (IVIFS) and presented its applications in pattern recognition and MCDM. Li [17] presented a new multi-attribute decision-making (MADM) strategy combining entropy and Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) in an IVIFS environment. Shang and Jiang [18] introduced the cross entropy in the fuzzy environment. Vlachos and Sergiadis [19] presented intuitionistic fuzzy cross entropy by extending fuzzy cross entropy [18]. Ye [20] defined a new cross entropy under an IVIFS environment and presented an

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IN-cross Entropy Based MAGDM Strategy under Interval Neutrosophic Set Environment

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Abstract. Cross entropy measure is one of the best way to calculate the divergence of any variable from the priori one variable. We define a new cross entropy measure under interval neutrosophic set (INS) environment, which we call IN-cross entropy measure and prove its basic properties. We also develop weighted IN-cross entropy measure and investigats its basic properties. Based on the weighted IN-cross entropy measure, we develop a novel strategy for multi attribute group decision making (MAGDM) strategy under interval neutrosophic environment. The proposed multi attribute group decision making strategy is compared with the existing cross entropy measure based strategy in the literature under interval neutrosophic set environment. Finally, an illustratative example of multi attribute group decision making problem is solved to show the feasibility, validity and efficiency of the proposed MAGDM strategy.

Keywords: Interval neutrosophic set, IN-cross entropy measure, MAGDM strategy.

1. Introduction

In our daily life we frequently meet with the quantitative measure to take appropriate decision for solving many problems. Entropy measure provides us a quantitative measure of two variables. In 1968, Zadeh [1] introduced fuzzy entropy measure. According to Liu [2], under fuzzy environment, entropy should meet at least three basic following requirements: the entropy of a crisp number is zero; the entropy of an equipossible fuzzy variable is maximum and the entropy is applicable not only to finite and infinite cases but also to discrete and continuous cases. Shang and Jiang [3] proposed a cross entropy measure and symmetric discrimination measure between fuzzy sets. Atanassov [4] introduced intuitionistic fuzzy set (IFS) in 1989, which is the extension of fuzzy set. Some recent applications of IFS are found in [5-11] in the literature. Vlachos and Sergiadis [12] defined cross entropy measure in IFS environment and showed a mathematical connection between the notions of entropy for fuzzy sets and IFSs in terms of fuzziness and intuitionism. In 1998, Smarandache [13] introduced the concept of neutrosophic set (NS) by introducing truth membership, falsity membership and indeterminacy membership functions as independent components and their sum lies ($^-0$, 3^+). Thereafter, Wang et al. [14] introduced single valued neutrosophic set (SVNS) as a subclass of NS. Thereafter, many researchers paid attention to apply NS and SVNS in many field of research such as conflict resolution [15], clustering analysis [16, 17], decision making [18-47], educational problem [48, 49], image processing [50, 52], medical diagnosis [53], optimization [54-59], social problem [60, 61]. Ye [62] introduced cross entropy measure in SVNS and applied it to multi criteria decision- making (MCDM) problems. Ye [63] defined an improved cross entropy measure for SVNS to overcome drawbacks in [62]. In 2005, Wang et al. [64] introduced interval neutrosophic set (INS) considering truth membership, indeterminate membership and falsity membership as interval number in [0, 1]. Broumi and Smarandache [65] defined correlation coefficient of INS and proved its basic properties. Zhang et al. [66] defined correlation coefficient for

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NC-TODIM-Based MAGDM under a Neutrosophic Cubic Set Environment

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Received: 19 October 2017; Accepted: 14 November 2017; Published: 18 November 2017

Abstract: A neutrosophic cubic set is the hybridization of the concept of a neutrosophic set and an interval neutrosophic set. A neutrosophic cubic set has the capacity to express the hybrid information of both the interval neutrosophic set and the single valued neutrosophic set simultaneously. As newly defined, little research on the operations and applications of neutrosophic cubic sets has been reported in the current literature. In the present paper, we propose the score and accuracy functions for neutrosophic cubic sets and prove their basic properties. We also develop a strategy for ranking of neutrosophic cubic numbers based on the score and accuracy functions. We firstly develop a TODIM (Tomada de decisao interativa e multicritévio) in the neutrosophic cubic set (NC) environment, which we call the NC-TODIM. We establish a new NC-TODIM strategy for solving multi attribute group decision making (MAGDM) in neutrosophic cubic set environment. We illustrate the proposed NC-TODIM strategy for solving a multi attribute group decision making problem to show the applicability and effectiveness of the developed strategy. We also conduct sensitivity analysis to show the impact of ranking order of the alternatives for different values of the attenuation factor of losses for multi-attribute group decision making strategies.

Keywords: neutrosophic cubic set; single valued neutrosophic set; interval neutrosophic set; multi attribute group decision making; TODIM strategy; NC-TODIM

1. Introduction

While modelling multi attribute decision making (MADM) and multi attribute group decision making (MAGDM), it is often observed that the parameters of the problem are not precisely known. The parameters often involve uncertainty. To deal with uncertainty, Zadeh [1] left an important mark to represent and compute with imperfect information by introducing the fuzzy set. The fuzzy set fostered a broad research community, and its impact has also been clearly felt at the application level in MADM [2–4] and MAGDM [5–9].

Atanassov [10] incorporated the non-membership function as an independent component and defined the intuitionistic fuzzy set (IFS) at first to express uncertainty in a more meaningful way. IFSs have been applied in many MADM problems [11–13]. Smarandache [14] proposed the notion of the neutrosophic set (NS) by introducing indeterminacy as an independent component. Wang et al. [15] grounded the concept of the single valued neutrosophic set (SVNS), an instance of the neutrosophic set, to deal with incomplete, inconsistent, and indeterminate information in a realistic way. Wang et al. [16] proposed the interval neutrosophic set (INS) as a subclass of neutrosophic sets in which the values of truth, indeterminacy, and falsity membership degrees are interval numbers. Theoretical development

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Neutrosophic Cubic MCGDM Method Based on Similarity Measure

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Abstract. The notion of neutrosophic cubic set is originated from the hybridization of the concept of neutrosophic set and interval valued neutrosophic set. We define similarity measure for neutrosophic cubic sets and prove some of its basic properties. We present a new multi criteria group decision making method with linguistic variables in neutrosophic cubic set environment. Finally, we present a numerical example to demonstrate the usefulness and applicability of the proposed method.

Keywords: Cubic set, Neutrosophic cubic set, similarity measure, multi criteria group decision making.

1. Introduction

In practical life we frequently face decision making problems with uncertainty that cannot be dealt with the classical methods. Therefore sophisticated techniques are required for modification of classical methods to deal decision making problems with uncertainty. L. A. Zadeh [1] first proposed the concept of fuzzy set to deal non-statistical uncertainty called fuzziness. K. T. Atanassov [2, 3] introduced the concept of intuitionistic fuzzy set (IFS) to deal with uncertainty by introducing the non-membership function as an independent component. F. Smarandache [4, 5, 6, 7, 8] introduced the notion of neutrosophic set by introducing indeterminacy as independent component. The theory of neutrosophic sets is a powerful tool to deal with incomplete, indeterminate and inconsistent information involed in real world decision making problem. Wang et al. [9] defined single valued neutrosophic set (SVNS) which is an instance of neutrosophic set. SVNS can independently express a truth-membership degree, an indeterminacy-membership degree and non-membership (falsity-membership) degree. SVNS is capable of representing human thinking due to the imperfection of knowledge received from real world problems. SVNS is obviously suitable for representing incomplete, inconsistent and indeterminate information.

Neutrosophic sets and SVNSs have become hot research topics in different areas of research such as conflict resolution [10], clustering analysis [11, 12], decision making [13-41], educational problem [42, 43], image processing [44, 45, 46], medical diagnosis [47], optimization [48-53], social problem [54, 55].

By combining neutrosophic sets and SVNS with other sets, several neutrosophic hybrid sets have been proposed in the literature such as neutrosophic soft sets [56, 57, 58, 59, 60, 61], neutrosophic soft expert set [62, 63], single valued neutrosophic hesitant fuzzy sets [64, 65, 66, 67, 68], interval neutrosophic hesitant sets [69], interval neutrosophic linguistic sets [70], single valued neutrosophic linguistic sets [71], rough neutrosophic set [72, 73, 74, 75, 76, 77, 78, 79], interval rough neutrosophic set [80, 81, 82], bipolar neutrosophic set [83, 84], bipolar rough neutrosophic set [85] Tri-complex rough neutrosophic set[86], hyper complex rough neutrosophic set [87]. Neutrosophic refined set [88, 89, 90, 91, 92, 93], Bipolar neutrosophic refined sets [94], rough complex set neutrosophic cubic set [95].

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NC-VIKOR Based MAGDM Strategy under Neutrosophic Cubic Set Environment

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Abstract. Neutrosophic cubic set consists of interval neutrosophic set and single valued neutrosophic set simultaneously. Due to its unique structure, neutrosophic cubic set can express hybrid information consisting of single valued neutrosophic information and interval neutrosophic information simultaneously. VIKOR (VIsekriterijumska optimizacija i KOmpromisno Resenje) strategy is an important decision making strategy which selects the optimal alternative by utilizing maximum group utility and minimum of an individual regret. In this paper, we propose VIKOR strategy in neutrosophic cubic set environment, namely NC-VIKOR. We first define NC-VIKOR strategy in neutrosophic cubic set environment to handle multi-attribute group decision making (MAGDM) problems, which means we combine the VIKOR with neutrosophic cubic number to deal with multi-attribute group decision making problems. We have proposed a new strategy for solving MAGDM problems. Finally, we solve MAGDM problem using our newly proposed NC-VIKOR strategy to show the feasibility, applicability and effectiveness of the proposed strategy. Further, we present sensitivity analysis to show the impact of different values of the decision making mechanism coefficient on ranking order of the alternatives.

Keywords: MAGDM, NCS, NC-VIKOR strategy.

1. Introduction

Smarandache [1] introduced neutrosophic set (NS) by defining the truth membership function, indeterminacy function and falsity membership function as independent components by extending fuzzy set [2] and intuitionistic fuzzy set [3]. Each of three independent component of NS belons to $[^-0, 1^+]$. Wang et al. [4] introduced single valued neutrosophic set (SVNS) where each of truth, indeterminacy and falsity membership degree belongs to [0, 1]. Many researchers developed and applied the NS and SVNS in various areas of research such as conflict resolution [5], clustering analysis [6-9], decision making [10-39], educational problem [40, 41], image processing [42-45], medical diagnosis [46, 47], social problem [48, 49]. Wang et al. [50] proposed interval neutrosophic set (INS). Ye [51] defined similarity measure of two interval neutrosophic sets and applied it to solve multi criteria decision making (MCDM) problem. By combining SVNS and INS Jun et al. [52], and Ali et al. [53] proposed neutrosophic cubic set (NCS). Thereafter, Zhan et al. [54] presented two weighted average operators on NCSs and applied the operators for MADM problem. Banerjee et al. [55] introduced the grey relational analysis based MADM strategy in NCS environment. Lu and Ye [56] proposed three cosine measures between NCSs and presented MADM strategy in NCS environment. Pramanik et al. [57] defined similarity measure for NCSs and proved its basic properties and presented a new multi criteria group decision making strategy with linguistic variables in NCS environment. Pramanik et al. [58] proposed the score and accuracy functions for NCSs and prove their basic properties. In the same study, Pramanik et al. [59] developed a strategy for ranking of neutrosophic cubic numbers (NCNs) based on the score and accuracy functions. In the same study, Pramanik et al. [58] first developed a TODIM (Tomada de decisao interativa e multicritévio), called the NC-TODIM and presented new NC-TODIM [58] strategy for solving (MAGDM) in NCS environment. Shi and Ye [59] introduced Dombi aggregation operators of NCSs and applied them for MADM problem. Pramanik et al. [60] proposed an ex-

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NC-Cross Entropy Based MADM Strategy in Neutrosophic Cubic Set Environment

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Received: 29 March 2018; Accepted: 26 April 2018; Published: 29 April 2018

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Abstract: The objective of the paper is to introduce a new cross entropy measure in a neutrosophic cubic set (NCS) environment, which we call NC-cross entropy measure. We prove its basic properties. We also propose weighted NC-cross entropy and investigate its basic properties. We develop a novel multi attribute decision-making (MADM) strategy based on a weighted NC-cross entropy measure. To show the feasibility and applicability of the proposed multi attribute decision-making strategy, we solve an illustrative example of the multi attribute decision-making problem.

Keywords: single valued neutrosophic set (SVNS); interval neutrosophic set (INS); neutrosophic cubic set (NCS); multi attribute decision-making (MADM); NC-cross entropy measure

1. Introduction

In 1998, Smarandache [1] introduced the neutrosophic set by considering membership (truth), indeterminacy, non-membership (falsity) functions as independent components to uncertain, inconsistent and incomplete information. In 2010, Wang et al. [2] defined the single valued neutrosophic set (SVNS), a subclass of neutrosophic sets to deal with real and scientific and engineering applications. In the medical domain, Ansari et al. [3] employed the neutrosophic set and neutrosophic inference to knowledge based systems. Several researchers applied neutrosophic sets effectively for image segmentation problems [4–9]. Neutrosophic sets are also applied for integrating geographic information system data [10] and for binary classification problems [11].

Pramanik and Chackrabarti [12] studied the problems faced by construction workers in West Bengal in order to find its solutions using neutrosophic cognitive maps [13]. Based on the experts' opinion and the notion of indeterminacy, the authors formulated a neutrosophic cognitive map and studied the effect of two instantaneous state vectors separately on a connection matrix and neutrosophic adjacency matrix. Mondal and Pramanik [14] identified some of the problems of Hijras (third gender), namely, absence of social security, education problems, bad habits, health problems, stigma and discrimination, access to information and service problems, violence, issues of the Hijra community, and sexual behavior problems. Based on the experts' opinion and the notion of indeterminacy, the authors formulated a neutrosophic cognitive map and presented the effect of two instantaneous state vectors separately on a connection matrix and neutrosophic adjacency matrix.

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VIKOR Based MAGDM strategy under Bipolar Neutrosophic Set Environment

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Abstract. In this paper, we extend the VIKOR (VIsekriterijumska optimizacija i KOmpromisno Resenje) strategy to multiple attribute group decision-making (MAGDM) with bipolar neutrosophic set environment. In this paper, we first define VIKOR strategy in bipolar neutrosophic set environment to handle MAGDM problems, which means we combine the VIKOR with bipolar neutrosophic number to deal with MAGDM. We propose a new strategy for solving MAGDM. Finally, we solve MAGDM problem using our newly proposed VIKOR strategy under bipolar neutrosophic set environment. Further, we present sensitivity analysis to show the impact of different values of the decision making mechanism coefficient on ranking order of the alternatives.

Keywords: Bipolar neutrosophic sets, VIKOR strategy, Multi attribute group decision making.

1 Introduction

In 1965, Zadeh [1] first introduced the fuzzy set to deal with the vague, imprecise data in real life specifying the membership degree of an element. Thereafter, in 1986 Atanassov [2] introduced intuitionistic fuzzy set to tackle the uncertainity in data in real life expressing membership degree and non-membership degree of an element as independent component. As a generalization of classical set, fuzzy set and intuitionistic fuzzy set, Smarandache [3] introduced the neutrosophic set by expressing the membership degree (truth membership degree), indeterminacy degree and non-membership degree (falsity membership degree) of an element independently. For real applications of neutrosophic set, Wang et al. [4] introduced the single valued neutrosophic set which is a sub class of neutrosophic set.

Decision making process involves seleting the best alternative from the set of feasible alternatives. There exist many decision making strategies in crisp set environment[5-7], fuzzy [8-12], intuitionistic fuzzy set environment [13-19]. vauge set environment [20, 21]. Theoretical as well as practical applications multi attribute decision making (MADM) of SVNS environment [22-42] and interval neutrosophic set (INS) environment [43-56] have been reported in the literaure. Recently, decision making in hybrid neutrosophic set environment have drawn much attention of the researches such as rough neutrosophic environment [57-73], neutrosophic soft set environment [74-80], neutrosophic soft expert set environment [81-82], neutrosophic hesitant fuzzy set environment [83-87], neutrosophic refined set environment [88-93], neutrosophic cubic set environment [94-104], etc. In 2015, Deli *et al.* [105] proposed bipolar neutrosophic set (BNS) using the concept of bipolar fuzzy sets [106, 107] and neutrosophic sets [3]. A BNS consists of two fully independent parts, which are positive membership degrees $T^+ \to [0, 1]$, $I^+ \to [0, 1]$, $F^+ \to [0, 1]$, and negative membership degrees $T^- \to [-1, 0]$, $I^- \to [-1, 0]$, $F^- \to [-1, 0]$ where the positive membership degrees T^+, I^+, F^+ represent truth membership degree, indeterminacy membership degree and false membership degree respectively of an element and the negative membership degrees T^-, I^-, F^- represent truth membership degree, indeterminacy membership degree and false membership degree respectively of an element to some implicit counter property corresponding to a BNS. Deli *et al.* [105] defined some operations namely, score function, accuracy function, and certainty function to compare BNSs and provided some operators in order to aggregate BNSs. Deli and Subas [108] defined correlation coefficient similarity measure for dealing with MADM problems under bipolar set