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Vague Direct Product in BCK- Algebra

S.Jafari[1], L. Mariapresenti[2] and I. Arockiarani[3]

Department of Mathematics

[1]College of Vestsjaelland south, Herrestraede, Denmark

[2,3]Nirmala College for Women, Coimbatore-18

Email: presentimaria88@gmail.com, jafaripersia@gmail.com

Abstract

In this paper, the notion of direct product of vague set in BCK- algebra is introduced and some related properties are investigated. Also we introduce some of the properties of direct product of finite vague ideal and vague subalgebras of BCK- algebras.

Keywords: vague set, direct product of vague sets, direct product of finite vague ideal, direct product of finite vague subalgebras.

1. Introduction: As crisp set theory does not reflect the real life problems exactly, L. A. Zadeh [16] introduced the concept of fuzzy set to generalize the notion of a member belonging to a set X. The concept of fuzzy set has been applied to various algebraic structures. Imai and Iseki [6,7]introduced two classes of abstract algebras, BCK- algebra and BCI- algebras. Al- Shehri [1], Jun et al[8,9], Saeid et al[13,14] and satyanarayana et al[15], applied the concept of fuzzy set to BCK- algebra. Zhan and Tan [17] introduced the concept of fuzzy H- ideal in BCK- algebras. Gau and Buehrer[3] introduced the concept of vague set. The vague set is developed by means of truth membership function and false membership function. Ranjit Biswas[12] initiated the study of vague algebra by studying vague groups. The objective of this paper is to contribute further to the study of direct product of vague set in BCK algebra and vague ideal in BCK- algebras and discuss some of their results.

2.Preliminaries:

Definition 2.1:[5] A BCI algebra is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms for all $x, y, z \in X$.:

(i) $((x*y)*(x*z))*(z*y)=0$

(ii) $(x*(x*y))*y=0$

(iii) $x*x=0$

(iv) $x*y=0$ and $y*x=0$ implies x=y.

We can define a partial ordering "$\leq$" by $x \leq y$ if and only if $x*y=0$.

If a BCI- algebra X satisfies $0*x=0$, for all $x \in X$, then we say that X is a BCK- algebra. Any BCK- algebra X satisfies the following axioms :

(i) $(x*y)*(x*z) \leq (z*y)$

(ii) $x*(x*y) \leq y$

(iii) $x \leq x$

(iv) $0 \leq x$

(v) $x \leq y$ and $y \leq x$ implies x=y. where $x \leq y$ means $x*y=0$.

Definition 2.2:[4] A non empty subset S of X is called a subalgebra of X if $x*y \in S$ for any $x, y \in S$.

Definition 2.3:[5] A non empty subset I of X is called an ideal of X if it satisfies

(I_1) $0 \in I$ and

(I_2) $x*y \in I$ and $y \in I$ imply $x \in I$.

Definition 2.4:[10] A non empty subset I of X is said to be an H- ideal of X if it satisfies(I_1) and

(I_3) $x*(y*z)\in I$ and $y\in I$ imply $x*z\in I$, for all $x,y,z\in X$.

Definition 2.5: [2] A vague set A in the universe of discourse U is characterized by two membership functions given by:

(i) A true membership function $t_A : U \to [0,1]$ and
(ii) A false membership function $f_A : U \to [0,1]$

where $t_A(x)$ is a lower bound on the grade of membership of x derived from the "evidence for x", $f_A(x)$ is a lower bound on the negation of x derived from the "evidence against x", and $t_A(x)+f_A(x)\le 1$. Thus the grade of membership of U in the vague set A is bounded by a subinterval $[t_A(x),1-f_A(x)]$ of [0,1]. This indicates that if the actual grade of membership of x is μ(x), then, $t_A(x)\le \mu(x)\le 1-f_A(x)$ .The vague set A is written as $A=\{\langle x,[t_A(x),1-f_A(x)]\rangle / u\in U\}$ where, the interval $[t_A(x),1-f_A(x)]$ is called the vague value of x in A, denoted by $V_A(x)$.

Definition 2.6:[2] Let A and B be vague sets(VSs) of the form $A=\{\langle x,[t_A(x),1-f_A(x)]\rangle / x\in X\}$ and $B=\{\langle x,[t_B(x),1-f_B(x)]\rangle / x\in X\}$ Then

(i) $A\subseteq B$ if and only if $t_A(x)\le t_B(x)$ and $1-f_A(x)\le 1-f_B(x)$ for all $x\in X$
(ii) A=B if and only if $A\subseteq B$ and $B\subseteq A$
(iii) $A^c=\{\langle x, f_A(x),1-t_A(x)\rangle / x\in X\}$
(iv) $A\cap B=\left\{\left\langle \begin{matrix} x, \min(t_A(x),t_B(x)), \\ \min(1-f_A(x),1-f_B(x))\end{matrix}\right\rangle / x\in X\right\}$
(v) $A\cup B=\left\{\left\langle \begin{matrix} x, \max(t_A(x)\vee t_B(x)), \\ \max(1-f_A(x)\vee 1-f_B(x))\end{matrix}\right\rangle / x\in X\right\}$

For the sake of simplicity, we shall use the notation $A=\langle x,t_A,1-f_A\rangle$ instead of $A=\{\langle x,[t_A(x),1-f_A(x)]\rangle / x\in X\}$.

Definition 2.7:[11] A vague set A on X is called a vague subalgebra of x if, for any $x\in X$, we have $t_A(xy)\ge \min\{t_A(x),t_A(y)\}$ and $1-f_A(xy)\ge \min\{1-f_A(x),1-f_A(y)\}$

Definition 2.8:[11] A vague set A of a BCK- algebra X is called a vague ideal of X if the following conditions are true:

(i) $(V_A(0)\ge V_A(x)),\quad (\forall\ x\in X)$
(ii) $(V_A(x)\ge i\min\{V_A(x*y),V_A(y)\}$ $(\forall\ x,y\in X)$ that is,
$t_A(0)\ge t_A(x),\quad 1-f_A(0)\ge 1-f_A(x),$ and
$(t_A(x)\ge \min\{t_A(x*y),t_A(y)\}$
$(1-f_A(x)\ge \min\{1-f_A(x*y),1-f_A(y)\}$ for all $x,y\in X$.

Definition 2.9:[11] Let A be a vague set of a universe X with the true- membership function t_A and the false- membership function f_A. The (α, β)-cut of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by $A_{(\alpha,\beta)}=\{x\in X / V_A(x)\ge[\alpha,\beta]\}$. Clearly $A_{(0,0)}$=X. The (α,β)- cut of the vague set A are also called vague cuts of A.

Definition 2.10:[11] The α- cut of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$ .Thus A_0=X, and if α ≥ β then, $A_\beta\subseteq A_\alpha$ and A(α, β)= A_α. Equivalently, we define the α-cut as $A_\alpha=\{x : x\in X, t_A(x)\ge\alpha\}$.

3. Direct product of vague ideals in BCK- algebras

Definition 3.1: Let $V_A=[t_A,1-f_A]$ and $V_B=[t_B,1-f_B]$ be two vague sets in BCK- algebras X_1 and X_2 respectively. Then the direct product of vague sets V_A and V_B is denoted by $V_{A\times B}=[t_{A\times B},1-f_{A\times B}]$, and defined as

$V_{A\times B}(x,y) = \min\ \{V_A(x), V_B(y)\}$

(i.e.,) $t_{A\times B}(x,y) = \min\ \{t_A(x), t_B(y)\}$ and

$1 - f_{A\times B}(x,y) = \min\ \{1 - f_A(x), 1 - f_B(y)\}$ for all

$(x,y) \in X_1 \times X_2$.

Definition 3.2: A vague set $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ of $X_1 \times X_2$ is called a vague subalgebra of $X_1 \times X_2$ if $V_{A\times B}((x_1,y_1)*(x_2,y_2)) \geq$

(i.e.,)

$t_{A\times B}((x_1,y_1)*(x_2,y_2)) \geq$
$\min\ \{t_{A\times B}(x_1,y_1), t_{A\times B}(x_2,y_2)\}$

$1 - f_{A\times B}((x_1,y_1)*(x_2,y_2)) \geq$
$\min\ \{1 - f_{A\times B}(x_1,y_1), 1 - f_{A\times B}(x_2,y_2)\}$

for all $(x_1,y_1),(x_2,y_2) \in X_1 \times X_2$.

Example 3.3: Let $X_1 = \{0,1\}$ and $X_2 = \{0,1,2\}$ are BCK- algebras by the following tables:

*	0	1
0	0	0
1	1	0

*	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Then $X_1 \times X_2 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$ is a BCK algebra. We define vague set $V_A = [t_A, 1 - f_A]$ on X_1 as $t_A : X \to [0,1]$ and $f_A : X \to [0,1]$ by

X	0	1
V_A	[0.5,0.6]	[0.3,0.4]

and $V_B = [t_B, 1 - f_B]$ on X_2 as $t_B : X \to [0,1]$ and $f_B : X \to [0,1]$ by

X	0	1	2
V_B	[0.5,0.7]	[0.5,0.6]	[0.2,0.5]

Then $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ is a vague subalgebra of BCK- algebra.

Definition 3.4: A vague set $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ of $X_1 \times X_2$ is called a vague H-ideal of $X_1 \times X_2$ if,

(i) $V_{A\times B}(0,0) \geq V_{A\times B}(x,y)$ and

(ii) $V_{A\times B}((x_1,y_1)*(x_3,y_3)) \geq$
$\min\{V_{A\times B}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), V_{A\times B}(x_2,y_2)\}$

(i.e.,) $t_{A\times B}(0,0) \geq t_{A\times B}(x,y)$ *and*
$1 - f_{A\times B}(0,0) \geq 1 - f_{A\times B}(x,y)$

$t_{A\times B}((x_1,y_1)*(x_3,y_3)) \geq$
$\min\ \{t_{A\times B}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), t_{A\times B}(x_2,y_2)\}$

$1 - f_{A\times B}((x_1,y_1)*(x_3,y_3)) \geq$
$\min\{1 - f_{A\times B}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))),$
$1 - f_{A\times B}(x_2,y_2)\}$

Definition 3.5: A vague set $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ of $X_1 \times X_2$ is called a vague closed H-ideal of $X_1 \times X_2$ if, $V_{A\times B}((0,0)*(x,y)) \geq V_{A\times B}(x,y)$

(i.e.,) $t_{A\times B}((0,0)*(x,y)) \geq t_{A\times B}(x,y)$ *and*
$1 - f_{A\times B}((0,0)*(x,y)) \geq 1 - f_{A\times B}(x,y)$

Theorem 3.6: Let $V_A = [t_A, 1 - f_A]$ and $V_B = [t_B, 1 - f_B]$ be two vague subalgebras of BCK- algebras X_1 and X_2 respectively. Then $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ is a vague sublagebra of BCK- algebra $X_1 \times X_2$.

Proof: For any $(x_1,y_1),(x_2,y_2) \in X_1 \times X_2$. Then

$V_{A\times B}((x_1,y_1)*(x_2,y_2))$

$= V_{A\times B}(x_1 * x_2, y_1 * y_2)$

$= \min\ \{V_A(x_1 * x_2), V_B(y_1 * y_2)\}$

$\geq \min\ \{\ \min\{V_A(x_1), V_A(x_2)\}, \min\{V_B(y_1) * V_B(y_2)\}\}$

$= \min\ \{\ \min\{V_A(x_1), V_B(y_1)\}, \min\{V_A(x_2) * V_B(y_2)\}\}$

$\geq \min\ \{V_{A\times B}(x_1,y_1), V_{A\times B}(x_2,y_2)\}$

Hence for all $(x_1,y_1),(x_2,y_2) \in X_1 \times X_2$,

$V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ is a vague subalgebra of BCK- algebra $X_1 \times X_2$.

Theorem 3.7: Let $V_A = [t_A, 1 - f_A]$ and $V_B = [t_B, 1 - f_B]$ be two vague H-ideals of BCK-algebras X_1 and X_2 respectively. Then $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ is a vague H- ideals of BCK- algebra $X_1 \times X_2$.

Proof: For any $(x, y) \in X_1 \times X_2$,

$$V_{A\times B}(0,0) = \min\{V_A(0), V_B(0)\}$$
$$\geq \min\{V_A(x), V_B(y)\} = V_{A\times B}(x, y)$$

Now for any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X_1 \times X_2$

$$V_{A\times B}((x_1, y_1) * (x_3, y_3))$$
$$= V_{A\times B}(x_1 * x_3, y_1 * y_3)$$
$$= \min\{V_A(x_1 * x_3), V_B(y_1 * y_3)\}$$
$$\geq \min\{\min\{V_A(x_1 * (x_2 * x_3)), V_A(x_2)\}, \min\{V_B(y_1 * (y_2 * y_3)), V_B(y_2)\}\}$$
$$= \min\{\min\{V_A(x_1 * (x_2 * x_3)), V_B(y_1 * (y_2 * y_3))\}, \min\{V_A(x_2), V_B(y_2)\}\}$$
$$= \min\{\min\{V_{A\times B}((x_1 * (x_2 * x_3)), (y_1 * (y_2 * y_3)))\}, \min\{V_{A\times B}(x_2, y_2)\}\}$$
$$\geq \min\{\min\{V_{A\times B}((x_1, y_1) * (x_2, y_2) * (x_3, y_3)))\}, \min\{V_{A\times B}(x_2, y_2)\}\}.$$

Hence for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X_1 \times X_2$, $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ is a vague H- ideal of BCK- algebra $X_1 \times X_2$.

Theorem 3.8: Let $V_A = [t_A, 1 - f_A]$ and $V_B = [t_B, 1 - f_B]$ be two vague closed H-ideals of BCK- algebras X_1 and X_2, respectively. Then $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ is a vague closed H- ideals of BCK- algebra $X_1 \times X_2$.

Proof: Let $V_A = [t_A, 1 - f_A]$ and $V_B = [t_B, 1 - f_B]$ be two vague closed H-ideals of BCK- algebras X_1 and X_2 respectively. Using theorem 3.7, $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ is a vague H-ideal of $X_1 \times X_2$. Now for any $(x, y) \in X_1 \times X_2$, then

$$V_{A\times B}((0,0) * (x, y)) = V_{A\times B}((0 * x), (0 * y))$$
$$= \min\{V_A(0 * x), V_B(0 * y)\}$$
$$\geq \min\{V_A(x), V_B(y)\} = V_{A\times B}(x, y)$$

Hence $V_A \times V_B = [t_{A\times B}, 1 - f_{A\times B}]$ is a vague closed H- ideal of BCK- algebra $X_1 \times X_2$.

Theorem 3.9: If $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ is a vague H- ideal of BCK- algebra $X_1 \times X_2$. Then we have $(a, b) \subseteq (x, y) \Rightarrow V_{A\times B}(x, y) \subseteq V_{A\times B}(a, b)$ for all $(a, b), (x, y) \in X_1 \times X_2$.

Proof: Let $(a, b), (x, y) \in X_1 \times X_2$, such that $(a, b) \subseteq (x, y) \Rightarrow (a, b) * (x, y) = (0,0)$. Consider

$$V_{A\times B}(x, y) = V_{A\times B}((x, y) * (0,0))$$
$$\geq \min\{V_{A\times B}((x, y) * ((a, b) * (0,0))), V_{A\times B}(a, b)\}$$
$$= \min\{V_{A\times B}((x, y) * ((a, b)), V_{A\times B}(a, b)\}$$
$$= V_{A\times B}(a, b).$$

Hence the proof.

Definition 3.10: Let $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ be a vague set of a BCK- algebra $X_1 \times X_2$ and for any $\alpha, \beta \in [0,1]$. Then the (α, β)- cut of the vague set $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ is a crisp subset $A_{(\alpha, \beta)}$ of the set X given by $A_{(\alpha,\beta)} = \{(x, y) \in X_1 \times X_2 / V_{A\times B}(x, y) \geq [\alpha, \beta]\}$. The (α,β)- cut of the vague set $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ are also called vague cuts of $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$.

Theorem 3.11: Let $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ be a vague subalgebra of BCK- algebra $X_1 \times X_2$. Then $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ is a vague subalgebra of BCK- algebra $X_1 \times X_2$ if and only if for any $\alpha, \beta \in [0,1]$ is a (α,β)- cut of the vague subalgebra of BCK- algebra $X_1 \times X_2$.

Proof: Let $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ be a vague subalgebra of BCK- algebra $X_1 \times X_2$. Now for any $\alpha, \beta \in [0,1]$ and $(x_1, y_1), (x_2, y_2) \in A_{(\alpha,\beta)}$ then $t_{A\times B}(x_1, y_1) \geq \alpha$ and $t_{A\times B}(x_2, y_2) \geq \alpha$. Since

$$t_{A\times B}((x_1, y_1) * (x_2, y_2))$$
$$\geq \min\{t_{A\times B}(x_1, y_1), t_{A\times B}(x_2, y_2)\} \geq \min\{\alpha, \alpha\} = \alpha$$
$$t_{A\times B}((x_1, y_1) * (x_2, y_2)) \geq \alpha.$$
$$\Rightarrow (x_1, y_1) * (x_2, y_2) \in A_{(\alpha,\beta)}$$

and for any $(x_1, y_1), (x_2, y_2) \in A_{(\alpha,\beta)}$ then $1 - f_{A\times B}(x_1, y_1) \geq \beta$ and $1 - f_{A\times B}(x_2, y_2) \geq \beta$. Since

$$1 - f_{A\times B}((x_1, y_1) * (x_2, y_2))$$
$$\geq \min\{1 - f_{A\times B}(x_1, y_1) 1 - f_{A\times B}(x_2, y_2)\}$$
$$\geq \min\{\beta, \beta\} = \beta$$
$$1 - f_{A\times B}((x_1, y_1) * (x_2, y_2)) \geq \beta$$
$$\Rightarrow (x_1, y_1) * (x_2, y_2) \in A_{(\alpha,\beta)}.$$

Therefore $A_{(\alpha,\beta)}$ is a vague subalgebra of BCK- algebra $X_1 \times X_2$.
Conversely, suppose that $A_{(\alpha,\beta)}$ is a vague subalgebra of BCK- algebra $X_1 \times X_2$. Suppose that $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ is not a vague subalgebra of a BCK- algebra $X_1 \times X_2$. Then there exist $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$, such that

$$t_{A\times B}((x_1, y_1) * (x_2, y_2)) < \min\{t_{A\times B}(x_1, y_1), t_{A\times B}(x_2, y_2)\}.$$

Now let

$$t_0 = \frac{1}{2}\{t_{A\times B}((x_1, y_1) * (x_2, y_2)) + \min\{t_{A\times B}(x_1, y_1), t_{A\times B}(x_2, y_2)\}\}.$$

This implies

$$t_{A\times B}((x_1, y_1) * (x_2, y_2)) < t_0 < \min\{t_{A\times B}(x_1, y_1), t_{A\times B}(x_2, y_2)\}.$$

So $(x_1, y_1), (x_2, y_2) \notin A_{(\alpha,\beta)}$ but $(x_1, y_1) \in A_{(\alpha,\beta)}$ and $(x_2, y_2) \in A_{(\alpha,\beta)}$, which is contradiction. Hence $V_{A\times B} = [t_{A\times B}, 1 - f_{A\times B}]$ is a vague subalgebra of BCK- algebra $X_1 \times X_2$.

Definition 3.12: Let $V_{A_i} = [t_{A_i}, 1 - f_{A_i}]$ be n-vague set of BCK- algebras X_i, respectively i=1,2,…..n. Then $\prod_{i=1}^{n} V_{A_i}$ is called direct product of finite vague sets of $\prod_{i=1}^{n} X_i$ if,

$$\prod_{i=1}^{n} V_{A_i}((x_1 * y_i, \ldots, x_n * y_n) = \min\{V_{A_1}((x_1 * y_1), \ldots, V_{A_n}(x_n * y_n))\}$$

for all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \prod_{i=1}^{n} X_i$.

Definition 3.13: Let $V_{A_i} = [t_{A_i}, 1 - f_{A_i}]$ be n-vague set of BCK- algebras X_i, respectively i=1,2,…..n. Then $\prod_{i=1}^{n} V_{A_i}$ is called direct product of finite vague subalgebra of $\prod_{i=1}^{n} X_i$ if,

$$\prod_{i=1}^{n} V_{A_i}((x_1, \ldots, x_n) * (y_1, \ldots, y_n) \geq \min\{\prod_{i=1}^{n} V_{A_i}((x_1, \ldots, x_n), \prod_{i=1}^{n} V_{A_i}(y_1, \ldots, y_n)\}$$

(i.e.,) $\prod_{i=1}^{n} t_{A_i}((x_1,\ldots\ldots,x_n)*(y_1,\ldots\ldots,y_n) \geq$

$\min\{\prod_{i=1}^{n} t_{A_i}((x_1,\ldots\ldots,x_n), \prod_{i=1}^{n} t_{A_i}(y_1,\ldots\ldots,y_n)\}$

$\prod_{i=1}^{n} 1-f_{A_i}((x_1,\ldots\ldots,x_n)*(y_1,\ldots\ldots,y_n) \geq$

$\min\{\prod_{i=1}^{n} 1-f_{A_i}((x_1,\ldots\ldots,x_n), \prod_{i=1}^{n} 1-f_{A_i}(y_1,\ldots\ldots,y_n)\}$

for all $(x_1,x_2,\ldots\ldots,x_n),(y_1,y_2,\ldots\ldots,y_n) \in \prod_{i=1}^{n} X_i$.

Definition 3.14: Let $V_{A_i} = [t_{A_i}, 1-f_{A_i}]$ be n-vague set of BCK- algebras X_i, respectively i=1,2,…..n. Then $\prod_{i=1}^{n} V_{A_i}$ is called direct product of finite vague ideal of $\prod_{i=1}^{n} X_i$ if, (i)

$\prod_{i=1}^{n} V_{A_i}(0,\ldots\ldots,0) \geq \prod_{i=1}^{n} V_{A_i}(x_1,\ldots\ldots,x_n)$ and

(ii) $\prod_{i=1}^{n} V_{A_i}(x_1,\ldots\ldots,x_n) \geq$

$\min\{\prod_{i=1}^{n} V_{A_i}((x_1,\ldots\ldots,x_n)*(y_1,\ldots\ldots,y_n)), \prod_{i=1}^{n} V_{A_i}(y_1,\ldots\ldots,y_n)\}$

(i.e.,) (i) $\prod_{i=1}^{n} t_{A_i}(0,\ldots\ldots,0) \geq \prod_{i=1}^{n} t_{A_i}(x_1,\ldots\ldots,x_n)$ *and*

$\prod_{i=1}^{n} 1-f_{A_i}(0,\ldots\ldots,0) \geq \prod_{i=1}^{n} 1-f_{A_i}(x_1,\ldots\ldots,x_n)$

(ii) $\prod_{i=1}^{n} t_{A_i}(x_1,\ldots\ldots,x_n) \geq$

$\min\{\prod_{i=1}^{n} t_{A_i}((x_1,\ldots\ldots,x_n)*(y_1,\ldots\ldots,y_n)), \prod_{i=1}^{n} t_{A_i}(y_1,\ldots\ldots,y_n)\},$

$\prod_{i=1}^{n} 1-f_{A_i}(x_1,\ldots\ldots,x_n) \geq$

$\min\{\prod_{i=1}^{n} 1-f_{A_i}((x_1,\ldots\ldots,x_n)*(y_1,\ldots\ldots,y_n)), \prod_{i=1}^{n} 1-f_{A_i}(y_1,\ldots\ldots,y_n)\},$

for all $(x_1,\ldots\ldots,x_n),(y_1,\ldots\ldots,y_n) \in \prod_{i=1}^{n} X_i$.

Definition 3.15: Let $V_{A_i} = [t_{A_i}, 1-f_{A_i}]$ be n-vague set of BCK- algebras X_i, respectively i=1,2,…..n. Then $\prod_{i=1}^{n} V_{A_i}$ is called direct product of finite vague H- ideal of $\prod_{i=1}^{n} X_i$ if,

(i) $\prod_{i=1}^{n} V_{A_i}(0,\ldots\ldots,0) \geq \prod_{i=1}^{n} V_{A_i}(x_1,\ldots\ldots,x_n)$ and

(ii) $\prod_{i=1}^{n} V_{A_i}((x_1,\ldots,x_n)*(z_1,\ldots,z_n)$

$\geq \min \prod_{i=1}^{n} V_{A_i}((x_1,\ldots\ldots,x_n)*((y_1,\ldots\ldots,y_n)*(z_1,\ldots\ldots,z_n)),$

$\prod_{i=1}^{n} V_{A_i}(y_1,\ldots\ldots,y_n)\}$

(i.e.,) (i) $\prod_{i=1}^{n} t_{A_i}(0,\ldots\ldots,0) \geq \prod_{i=1}^{n} t_{A_i}(x_1,\ldots\ldots,x_n)$ *and*

$\prod_{i=1}^{n} 1-f_{A_i}(0,\ldots\ldots,0) \geq \prod_{i=1}^{n} 1-f_{A_i}(x_1,\ldots\ldots,x_n)$

(ii) $\prod_{i=1}^{n} t_{A_i}((x_1,\ldots,x_n)*(z_1,\ldots,z_n)$

$\geq \min \prod_{i=1}^{n} t_{A_i}((x_1,\ldots\ldots,x_n)*((y_1,\ldots\ldots,y_n)*(z_1,\ldots\ldots,z_n)),$

$\prod_{i=1}^{n} t_{A_i}(y_1,\ldots\ldots,y_n)\}$

(ii) $\prod_{i=1}^{n} 1-f_{A_i}((x_1,\ldots,x_n)*(z_1,\ldots,z_n) \geq$

$\min\{\prod_{i=1}^{n} 1-f_{A_i}((x_1,\ldots\ldots,x_n)*((y_1,\ldots\ldots,y_n)*(z_1,\ldots\ldots,z_n)),$

$\prod_{i=1}^{n} 1-f_{A_i}(y_1,\ldots\ldots,y_n)\}$

for all $(x_1,\ldots\ldots,x_n),(y_1,\ldots\ldots,y_n) \in \prod_{i=1}^{n} X_i$.

Theorem 3.16: Let $V_{A_i} = [t_{A_i}, 1-f_{A_i}]$ be n-vague subalgebras of BCK- algebras X_i, respectively i=1,2,…..n. Then $\prod_{i=1}^{n} V_{A_i}$ is vague subalgebras of $\prod_{i=1}^{n} X_i$.

Proof: Let $V_{A_i} = [t_{A_i}, 1-f_{A_i}]$ be n-vague subalgebras of BCK- algebras X_i, respectively. Let $(x_1,x_2,\ldots\ldots,x_n),(y_1,y_2,\ldots\ldots,y_n) \in \prod_{i=1}^{n} X_i$. Then

$$\prod_{i=1}^{n} V_{A_i}((x_1,......,x_n)*(y_1,......,y_n))$$
$$= \prod_{i=1}^{n} V_{A_i}((x_1 * y_i,......,x_n * y_n)$$
$$= \min\{V_{A_1}((x_1 * y_1),......,V_{A_n}(x_n * y_n))\}$$
$$\geq \min\{\min\{V_{A_1}(x_1),V_{A_1}(y_1)\},......,\min\{V_{A_n}(x_n),V_{A_n}(y_n))\}$$
$$= \min\{\min\{V_{A_1}(x_1),........,V_{A_n}(x_n)\},......,$$
$$\min\{V_{A_1}(y_1),........,V_{A_n}(y_n))\}$$

$$\prod_{i=1}^{n} V_{A_i}((x_1,......,x_n)*(y_1,......,y_n)) \geq$$
$$\min\{\prod_{i=1}^{n} V_{A_i}(x_1,......,x_n), \prod_{i=1}^{n} V_{A_i}(y_1,........,y_n)\}$$

This completes the proof.

Theorem 3.17: Let $V_{A_i} = [t_{A_i}, 1 - f_{A_i}]$ be n-vague ideals of BCK- algebras $\prod_{i=1}^{n} X_i$, respectively. If $V_{A_i}(x_i * y_i) \geq V_{A_i}(x_i)$ where i=1,2,…..n, then $\prod_{i=1}^{n} V_{A_i}$ is vague H- ideal of $\prod_{i=1}^{n} X_i$

Proof: Let $V_{A_i} = [t_{A_i}, 1 - f_{A_i}]$ be n-vague ideals of BCK- algebras $\prod_{i=1}^{n} X_i$, respectively and $V_{A_i}(x_i * y_i) \geq V_{A_i}(x_i)$ for any $(x_1, x_2,......., x_n), (y_1, y_2,......., y_n) \in \prod_{i=1}^{n} X_i$. Then we have,

$$\prod_{i=1}^{n} V_{A_i}((x_1,.....,x_n)*(y_1,.....,y_n))$$
$$= \prod_{i=1}^{n} V_{A_i}(x_1 * y_1,.....,x_n * y_n)$$
$$= \min\{V_{A_1}(x_1 * y_1),.....,V_{A_n}(x_n * y_n)\}$$
$$\geq \min\{V_{A_1}(x_1),.....,V_{A_n}(x_n)\}$$
$$= \prod_{i=1}^{n} V_{A_i}((x_1,.....,x_n)$$

Hence $\prod_{i=1}^{n} V_{A_i}$ is vague H- ideal of $\prod_{i=1}^{n} X_i$

Theorem 3.18: Let $V_{A_i} = [t_{A_i}, 1 - f_{A_i}]$ be n-vague ideals of BCK- algebras $\prod_{i=1}^{n} X_i$, respectively. If $x_i * y_i \leq z_i$ where i=1,2,…..n ,holds in $\prod_{i=1}^{n} X_i$. Then

$$\prod_{i=1}^{n} V_{A_i}(x_1,.....,x_n) \geq$$
$$\min\{\prod_{i=1}^{n} V_{A_i}(y_1,.....,y_n), \prod_{i=1}^{n} V_{A_i}(z_1,.....,z_n)\}$$

Proof: Let $(x_1, x_2,......., x_n), (y_1, y_2,......., y_n), (z_1, z_2,......., z_n) \in \prod_{i=1}^{n} X_i$ be such that, $x_i * y_i \leq z_i$. Then $(x_i * y_i) * z_i = 0$, and thus

$$\prod_{i=1}^{n} V_{A_i}(x_1,.......,x_n)$$
$$\geq \min\{\prod_{i=1}^{n} V_{A_i}((x_1,.......,x_n)*(y_1,.......,y_n)),$$
$$\prod_{i=1}^{n} V_{A_i}(y_1,.......,y_n)\}$$
$$\geq \min\{\min\{\prod_{i=1}^{n} V_{A_i}(((x_1,.......,x_n)*(y_1,.......,y_n))$$
$$*(z_1,.......,z_n)), \prod_{i=1}^{n} V_{A_i}(z_1,.......,z_n)\},$$
$$\prod_{i=1}^{n} V_{A_i}(y_1,.......,y_n$$
$$= \min\{\min\{\prod_{i=1}^{n} V_{A_i}(0,...,0), \prod_{i=1}^{n} V_{A_i}(z_1,.......,z_n)\},$$
$$\prod_{i=1}^{n} V_{A_i}(y_1,.......,y_n)\}$$
$$= \min\{\prod_{i=1}^{n} V_{A_i}(y_1,.......,y_n), \prod_{i=1}^{n} V_{A_i}(z_1,.......,z_n)\}$$

Theorem 3.19: Let $V_{A_i} = [t_{A_i}, 1 - f_{A_i}]$ be n-vague ideal of BCK- algebras X_i, respectively, If $x_i \leq y_i$, whenever $x_i * y_i = 0$ where i=1,2,…..n. Then $\prod_{i=1}^{n} V_{A_i}(x_1,.....,x_n) \geq \prod_{i=1}^{n} V_{A_i}(y_1,.....y_n)$.

Proof: Let $(x_1, x_2,......., x_n), (y_1, y_2,......., y_n) \in \prod_{i=1}^{n} X_i$ be such that $x_i \leq y_i$, whenever $x_i * y_i = 0$ and then we have ,

$$\prod_{i=1}^{n} V_{A_i}(x_1, x_2, \ldots\ldots, x_n)$$
$$\geq \min\{\prod_{i=1}^{n} V_{A_i}((x_1, x_2, \ldots\ldots, x_n) * (y_1, y_2, \ldots\ldots, y_n)), \prod_{i=1}^{n} V_{A_i}(y_1, y_2, \ldots\ldots, y_n)$$
$$= \min\{\prod_{i=1}^{n} V_{A_i}((0,0,\ldots,0), \prod_{i=1}^{n} V_{A_i}(y_1, y_2, \ldots\ldots, y_n)\}$$
$$= \prod_{i=1}^{n} V_{A_i}(y_1, y_2, \ldots\ldots, y_n)$$

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