Gauge and Coordinate Paradox for a Static Charged Sphere

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Abstract

We make gauge and coordinate transformations so that the electromagnetic vector potential has unit time and zero space components. It is shown if the Einstein field equations, with electromagnetic vector potential having this form, have a metric solution then there is a metric related to this metric by a coordinate transformation that is not a solution. Since the Einstein field equations do not determine coordinates we expect this transformed metric to also be a solution.

1 Electromagnetic potential and field

Let $A_\mu(t,r,\theta,\varphi)$ and $g_{\mu\nu}(t,r,\theta,\varphi)$ be the electromagnetic potential and metric tensor respectively. The electromagnetic field is

$$F_{\mu\nu}(t,r,\theta,\varphi)=A_{\nu,\mu}(t,r,\theta,\varphi)-A_{\mu,\nu}(t,r,\theta,\varphi) \tag{1}$$

Let $A^\mu(t,r,\theta,\phi)$ be the electromagnetic vector potential. For a scalar function $\phi(t,r,\theta,\varphi)$ define a gauge transformation of $A^\mu(t,r,\theta,\phi)$ to $\hat{A}^\mu(t,r,\theta,\phi)$ by

$$\hat{A}^\mu(t,r,\theta,\varphi)=A^\mu(t,r,\theta,\varphi)+(g^{\mu\alpha}\phi_{,\alpha})(t,r,\theta,\varphi) \tag{2}$$

We have by (1) and (2) that

$$\begin{aligned}F_{\mu\nu} &= A_{\nu,\mu}-A_{\mu,\nu}=A_{\nu,\mu}-A_{\mu,\nu}+\phi_{,\nu\mu}-\phi_{,\mu\nu}=(A_\nu+\phi_{,\nu})_{,\mu}-(A_\mu+\phi_{,\mu})_{,\nu}\\ &= (g_{\nu\alpha}[A^\alpha+g^{\alpha\beta}\phi_{,\beta}])_{,\mu}-(g_{\mu\alpha}[A^\alpha+g^{\alpha\beta}\phi_{,\beta}])_{,\nu}=(g_{\nu\alpha}\hat{A}^\alpha)_{,\mu}-(g_{\mu\alpha}\hat{A}^\alpha)_{,\nu}\end{aligned} \tag{3}$$

2 Static charged sphere and Einstein field equations

Let there be a static charged sphere with spherically symmetric charge and mass densities. Let

$$A_0(r)>0 \qquad A_1(r)=A_2(r)=A_3(r)=0 \tag{4}$$

and $A_0(r)\to 0$ as $r\to\infty$. Let the metric having form

$$-a(r)dt^2+b(r)dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2) \tag{5}$$

and $a(r)\to 1$ and $b(r)\to 1$ as $r\to\infty$ satisfy the Einstein field equations

$$G_{\mu\nu}=8\pi\left[g^{\sigma\tau}F_{\mu\sigma}F_{\nu\tau}-\frac{1}{4}g_{\mu\nu}g^{\sigma\alpha}g^{\tau\beta}F_{\sigma\tau}F_{\alpha\beta}\right]+8\pi T_{\mu\nu} \tag{6}$$

where $T^{\mu\nu}(r)$ is the energy-momentum tensor of a perfect fluid in thermodynamic equalibrium.

3 Coordinate transformation

Let

$$\phi(t, r, \theta, \varphi) = t \tag{7}$$

hence by (2), (4), and (7)

$$\hat{A}^0(r) = a^{-1}(r)[1 + A_0(r)] \qquad \hat{A}^1(r) = \hat{A}^2(r) = \hat{A}'^3(r) = 0 \tag{8}$$

Consider the coordinate tranformation given by

$$t' = \frac{a(r)t}{1 + A_0(r)} \qquad r' = r \qquad \theta' = \theta \qquad \varphi' = \varphi \tag{9}$$

This transformation transforms $\hat{A}^\mu(r)$ to

$$\hat{A}'^0(t', r') = 1 \qquad \hat{A}'^1(t', r') = \hat{A}'^2(t', r') = \hat{A}'^3(t', r') = 0 \tag{10}$$

Since $a(r) \to 1, b(r) \to 1$, and $A_0(r) \to 0$ as $r \to \infty$ we have $g'_{\mu\nu}$ approaches the Minkowski metric in spherical coordinates as $r \to \infty$.

4 Coordinate contradiction

We have by (3) transformed to $t', r', \theta', \varphi'$ coordinates and (10) that

$$F'_{\mu\nu} = A'_{\nu,\mu} - A'_{\mu,\nu} = (g'_{\nu\alpha}\hat{A}'^\alpha)_{,\mu} - (g'_{\mu\alpha}\hat{A}'^\alpha)_{,\nu} = g'_{\nu 0,\mu} - g'_{\mu 0,\nu} \tag{11}$$

Transforming (6) to $t', r', \theta', \varphi'$ coordinates and using (11) we have $g'_{\mu\nu}(t', r')$ satisfies

$$\begin{aligned} G'_{\mu\nu} &= 8\pi \left[g'^{\sigma\tau} F'_{\mu\sigma} F'_{\nu\tau} - \frac{1}{4} g'_{\mu\nu} g'^{\sigma\alpha} g'^{\tau\beta} F'_{\sigma\tau} F'_{\alpha\beta} \right] + 8\pi T'_{\mu\nu} \\ &= 8\pi g'^{\sigma\tau}[g'_{\sigma 0,\mu} - g'_{\mu 0,\sigma}][g'_{\tau 0,\nu} - g'_{\nu 0,\tau}] - 2\pi g'_{\mu\nu} g'^{\alpha\sigma} g'^{\beta\tau}[g'_{\tau 0,\sigma} - g'_{\sigma 0,\tau}][g'_{\beta 0,\alpha} - g'_{\alpha 0,\beta}] + 8\pi T'_{\mu\nu} \end{aligned} \tag{12}$$

where $G'_{\mu\nu}$ and $T'_{\mu\nu}$ are constructed using the metric $g'_{\mu\nu}$. The Einstein field equations do not determine coordinates. Consequently if $g'_{\mu\nu}(t', r')$ is a solution of (12) then so is $g''_{\mu\nu}(t''(t', r'), r''(t', r'))$ where $g''_{\mu\nu}$ is related to $g'_{\mu\nu}$ by a coordinate transformation $t', r' \to t'', r''$ [1]. By (12) then

$$G''_{\mu\nu} = 8\pi g''^{\sigma\tau}[g''_{\sigma 0,\mu} - g''_{\mu 0,\sigma}][g''_{\tau 0,\nu} - g''_{\nu 0,\tau}] - 2\pi g''_{\mu\nu} g''^{\alpha\sigma} g''^{\beta\tau}[g''_{\tau 0,\sigma} - g''_{\sigma 0,\tau}][g''_{\beta 0,\alpha} - g''_{\alpha 0,\beta}] + 8\pi T''_{\mu\nu} \tag{13}$$

where $G''_{\mu\nu}$ and $T''_{\mu\nu}$ are constructed using the metric $g''_{\mu\nu}$.

We can choose a coordinate transformation $t', r' \to t'', r''$ so that $g''_{\mu\nu}$ satisfies (13) and the synchronous coordinate conditions

$$g''_{00} = -1 \qquad g''_{10} = g''_{20} = g''_{30} = 0 \tag{14}$$

A static charged sphere has an electromagnetic field hence $G_{\mu\nu}$ is not zero outside the mass. Consequently $G'_{\mu\nu}$ is not zero outside the mass. Since the solution $g''_{\mu\nu}$ is related to the solution $g'_{\mu\nu}$ by a coordinate transformation we then have $G''_{\mu\nu}$ will be related to $G'_{\mu\nu}$ by the same coordinate transformation. Since $G'_{\mu\nu}$ is not zero outside the mass we then have $G''_{\mu\nu}$ is not zero outside the mass. Now $T''_{\mu\nu}$ is zero outside the mass and using (13) and (14) we then have $G''_{\mu\nu}$ is zero outside the mass. Consequently $G''_{\mu\nu}$ both is and is not zero outside the mass. We have a contradiction.

5 Conclusion

We made a gauge and coordinate tranformations so the electromagnetic vector potential has a specific form. If the resulting transformed Einstein field equations for a static charged sphere has a metric solution then there is a metric related to this metric by a coordinate transformation that is not a solution. We however expect since the Einstein field equations don't determine coordinates that the coordinate transformed metric is also a solution.

References

[1] S. Weinberg, *Gravitation and Cosmology*, p.161

[2] K. De Paepe, *Physics Essays*, September 2007

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