Order in the Collatz: Fractal Symmetry Controlling Syracuse Function Trajectories

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Consider the function $C(n) = \begin{cases} 3n+1, & n \equiv 1 \, mod \, 2. \\ \frac{n}{2}, & n \equiv 0 \, mod \, 2. \end{cases}$ The $3x + 1$ problem or Collatz conjecture asks if all trajectories iterating recursively contain one. This holds empirically but eludes theoretical proof. Remove even entries to obtain the Syracuse function, $T(u)$, mapping odd numbers u, $T(u) = (3u + 1)/2^{r}$, where r maximal for $T(u)$ odd. Here we show by expressing u as $4x \pm 1$, a fractal regularity given by the amplitude of a periodic function passing through the x axis in the curves defined $y = \mathbb{N}\cos\left(\frac{\pi x}{2^{\mathbb{N}}}\right) \equiv \mathbb{N} \bmod \left(\frac{x+2^{\mathbb{N}-1}}{2^{\mathbb{N}}}, 1\right)$, with $\mathbb{N}$ the natural numbers. The expression describes the length of ascending segments R and with translation, r the number of divisions in $T(u)$. Algebraic analysis reveals the Syracuse function's image neatly partitions onto arithmetic progressions $6k \pm 1$, as a function of r. This confirms that all natural trajectories in the $3x + 1$ problem contain one..

Introduction: Here, we use $\mathbb{U}$ to denote the odd natural numbers, and $\mathbb{N}$ without zero by default.

Though current trends see this as a random process [1], we find it useful in order to see the patterns in the trajectories to arrange them with $T^{(0)}$ an odd multiple of three, i.e. $u = 3k + 6$ and proceeding until redundancy with earlier trajectories. This gives the generic form $\boxed{3+6k} \mapsto \boxed{6k \pm 1 \Leftrightarrow 6k \pm 1} \mapsto \boxed{1}$ Figure 0 illustrates the first few such trajectories with color coded points of convergence and an example of self-similarity <Figure 0>.

The $3x + 1$ problem has a well-documented, storied, history, and is expressed in many distinct forms showing it from a variety of perspectives [2]. Despite the remarkable interest and attention, the search for formal structure underlying its' controlling mechanisms has long been abandoned, the canonical pedagogy being that it is a pseudo-random process [3]. Lacking a macroscopic order, despite validation well beyond the point of hand computation, reasonable uncertainty persists in relation to the conjecture that iterative application of the $3x + 1$ protocols ultimately admit exactly one fixed point attractor at 1 with the complete set of natural numbers as the basin of attraction. The infinitude of natural numbers precludes strictly empirical evidence from completely dispelling uncertainty in principle, demanding conceptual analysis of the structure of the problem itself for any hope of resolution.

We complete the work initiated in [1] proving the existence of finite descent characteristics by showing the fractal structure underlying all ascents and descents. In the odd number domain $\mathbb{U}$, we present formula from writing u in terms of $4k - 1$, $8k + 1$, and $8k + 5$ that reveal, r the extent of descent from u to $T(u)$, and R the consecutive mappings $u \longrightarrow T(u)$ with $T(u) > u$. from elementary algorithmic operations. The results manifest precisely as the interleaving of arithmetic progressions, describeable with a family of periodic functions.

$$\boldsymbol{y = N\cos\left(\frac{\pi x}{2^N}\right) \equiv N \bmod \left(\frac{x + 2^{N-1}}{2^N}\right).} \qquad \textbf{(1)}$$

Numbers from $4x - 1$ map to $6x + 5 = (3x + 1)/2$ accounting for half of cases. The $8k + 1$ map to $6x + 1 = (3x + 1)/4$ for a fourth of cases, completing the set of proximal pre-images. The remainnig quarter, $8k + 5$ are the distal preimages, mapping $8x + 5$ to both $6k \pm 1 = (3x + 1)/2r \geq 3$. This allows the visualization of precise trajectory turning points and see the scale free structure, permitting direct confirmation that all natural number trajectories contain one.

Syracuse Function's Fractal[1] Algebraic Image: The image $T(u)$ is the well-ordered interleaving of $6k \pm 1$ progressions. Our equations are proper equations: when u is in the form indicated on the right of an equals sign, $T(u)$ is given by the expression on the left by substituting the same value for x. This partitions $\mathbb{U}$, such that $T(u)$ is calculable without evaluating $\frac{3x+1}{2^r}$ by identifying the unique arithmetic progression to which u belongs. We will see by the form of their controlling expressions that these progressions

[1] The fractal organization permits a direct set theoretic proof of the convergence of all natural Syracuse trajectories to least value cycles.
Proof. Let A be the function mapping $8k + 5 \longrightarrow \frac{3x+1}{2^r}$. So $T \subseteq A$ by 1.1 RHS, and $A \subseteq T$ by 1.1 LHS. Hence $T = A$. But since A is the subset of mappings with $r \geq 3$, no element in A can diverge. But since $A = T$, neither will an element in T. ■.

specify trajectories of u uniquely, and cover the entire domain. The $6k \pm 1$ progressions form the image of T, while mod eight equivalence classes describe the preimage.

$$T(u) = \frac{3x+1}{2^r} = \begin{cases} 6x-1, & u = 4x-1. \\ \frac{3x+1}{2^r}, & u \in 4x+1. \end{cases} = \begin{cases} 6x+5, & u = 4x+3. \quad \equiv r(u) = 1. \\ 6x+1, & u = 8x+1. \quad \equiv r(u) = 2 \\ \frac{3x+1}{2^r}, & u \in 8x+5, \quad \equiv r(u) \geq 3: \ldots \end{cases} \tag{1.1}$$

$$\ldots T(u), \qquad u \in 8x+5: \begin{cases} 6x+5, & u = 4^{(2+n)}x + (13 + \sum_{k=1}^{n} 10(4^k) \\ 6x+1, & u = 2^{(5+2n)}x + (5 + \sum_{k=1}^{n} (2^{3+k}) \end{cases} \tag{1.2}$$

Table 1. Exemplars showing the first four entries in the arithmetic progressions produced by 1.2. This reveals their orderly partitioning of the image onto $6k \pm 1$ arithmetic progressions, from 1.1. By increasing n as required, all odd numbers are captured in exactly one progression corresponding to u with $T(u)$ such that r *is at least three.*

n	$4^{(2+n)}x + (13 + \sum_{k=1}^{n} 10(4^k)$						$2^{(5+2n)}x + (5 + \sum_{k=1}^{n} (2^{3+k})$					
0	x	0	1	2	3	4	x	0	1	2	3	4
	U=16x+13	13	29	45	61	77	u=32x+5	5	37	69	101	133
	$T(u)$ $r = 3$	5	11	17	23	29	$T(u)$ $r = 4$	1	7	13	19	25
1	x	0	1	2	3	4	x	0	1	2	3	4
	u=64x+53	53	117	181	245	309	u=128x+21	21	149	277	405	533
	$T(u)$ $r = 5$	5	11	17	23	29	$T(u)$ $r = 6$	1	7	13	19	25
2	x	0	1	2	3	4	x	0	1	2	3	4
	u=256x+213	213	469	725	981	1237	u=512x+85	85	597	1109	1621	2133
	$T(u)$ $r = 7$	5	11	17	23	29	$T(u)$ $r = 8$	1	7	13	19	25

When a domain and codomain are subsets of the natural numbers, algebraic proof suffices to remove all reasonable uncertainties of a given assertion. For example, if we are reluctant to believe that n^2 - 1 is divisible by 8 for all odd n, we can prove this is true by algebra. Any persisting doubt is unreasonable when the domain and codomain are subsets of the natural numbers, even if a typically trustworthy authority is the voice of skepticism. With respect to the Syracuse function, proofs of (1.1) are all variants of its' algebraic application.

Proof. $3(4\mathrm{x} + 3) + 1 = 12x + 9 + 1 = 12x + 10 = 2(6x + 5)$ ■.

Proof. $3(8\mathrm{x} + 1) + 1 = 24x + 3 + 1 = 24x + 4 = 4(6x + 1)$ ■.

Proof. $3(8\mathrm{x} + 5) + 1 = 24x + 15 + 1 = 24x + 16 = 8(3x + 2)$: Then partitioning odds and evens:

$\underline{x = 2n \rightarrow 8(6n + 2) = 16(3n + 1);}$ $x = 2n + 1 \rightarrow 8(6n + 3 + 2) = 8(6n + 5)$, where the even underlined term will continue to split *ad infinitum* producing the fractal dynamics. ■.

Table 2 *Examples of $R(u)$* trajectory sub-sets $6k \pm 1 \leftrightarrow 6k \pm 1$

$R(u) = \log_2 \frac{3u+1}{d(3u+1)}$.
(1) $u = 7. \therefore R(u) = \log_2 \frac{8}{1} = \log_2 8 = 3.$
$\{7 \rightarrow 11 \rightarrow 17\}$.
(2) $u = 15, \therefore R(u) = \log_2 \frac{16}{1} = \log_2 16 = 4.$
$\{15 \rightarrow 23 \rightarrow 35 \rightarrow 53\}$.
(3) $u = 27, \therefore R(u) = \log_2 \frac{28}{4} = \log_2 4 = 2.$
$\{27 \rightarrow 41\}$.
(4) $u = 47, \therefore R(u) = \log_2 \frac{48}{3} = \log_2 16 = 4.$
$\{47 \rightarrow 71 \rightarrow 107 \rightarrow 161\}$
(5) $u = 167, \therefore R(u) = \log_2 \frac{168}{21} = \log_2 8 = 3.$
$\{167 \rightarrow 251 \rightarrow 377\}$

The only uncertainty remaining about $T(u)$ after expressing u modulo eight is the specification of how much more than four $3u + 1$ will be divided by to obtain $T(u)$. That is, what is $r(u)$? Uncertainty also exists in how high the rising maps can rise. Let $R(u)$ denote the length of the trajectory segment beginning with u such that $T(u)$ is greater than u and counting all $T(u)$ for which this remains the case. To discuss both r and R together we shall write the double-struck, "$\mathbb{R}$."

Discovering order in $\mathbb{R}$ can resolve the remaining reasonable uncertainties in the Syracuse function. If the mean and mean average (the indicated statistic describing full data) deviation of R positively accelerates, this could allow rising trajectories to grow arbitrarily with increasing arguments; paired with r negatively accelerating, and there is justification not to rule out exotic arguments diverging. On the other hand, if R negatively accelerates ior r positively accelerates, exotic fixed point attractors, i.e. cycles other than $1 \rightarrow 1$ could reasonably appear. But, if $\mathbb{R}$ is stable asymptotically, we need not examine exotic regions of the domain to be certain that the present observations will persist and all trajectories include one.

Table 3. R(u) and d(u+1). Here we see that R(u) is predictable as a function of u. More importantly, it is scale free. The last two columns show that removing the highest frequency elements and subtracting 1 from the list of R(u) returns the original list.

u	$u+1$	$d(u+1)$	$R(u)$: u $4x\pm1$		$R(u)$: u $4x+3$	R-1
1	2	1	1		2	1
3	4	1	2		3	2
5	6	3	1		2	1
7	8	1	3		4	3
9	10	5	1		2	1
11	12	3	2		3	2
13	14	7	1		2	1
15	16	1	4		5	4
17	18	9	1		2	1
19	20	5	2		3	2
21	22	11	1			
23	24	3	3			
25	26	13	1			
27	28	7	2			
29	30	15	1			
31	32	1	5			
33	34	17	1			
35	36	9	3			
37	38	19	1			
39	40	5	3			

Computing $\mathbb{R}$ from u requires identifying the greatest odd number that divides a quantity without remainder; let $d(n)$ denote this function. The quotient after the division will always be a power of two when n is even since prime factors are either odd or two.

We have,

$$d(n) = \max\left(\mathbb{U}\left\lfloor 1 - mod\left(\frac{n}{\mathbb{U}}, 1\right)\right\rfloor\right), \tag{2.1}$$

where the modulo (mod 1) function returns zero if and only if the odd number u in $\mathbb{U}$ causes no remainder in dividing n; otherwise, the floor function (parenthetical brackets) prevents the maximization function from considering that instance of u.

From this we establish,

$$R(u) = log_2 \frac{u+1}{d(u+1)}, \qquad r(u) = log_2 \frac{3u+1}{d(3u+1)}. \qquad (2.2)$$

Since $r(u)$ is familiar from the Syracuse function, we provide explicit examples demonstrating $R(u)$ only in table 2. <Tables 3 and 4 > contain data for the first 20 u.

Beside 2.2 $\mathbb{R}(u)$ are also obtainable from the expressions,

$$y = \mathbb{N}\cos\left(\frac{\pi x}{2^{\mathbb{N}}}\right), \qquad \textit{and equivalently}, \qquad y = \mathbb{N}\, mod\left(\frac{x + 2^{\mathbb{N}-1}}{2^{\mathbb{N}}}, 1\right). \qquad (3.0)$$

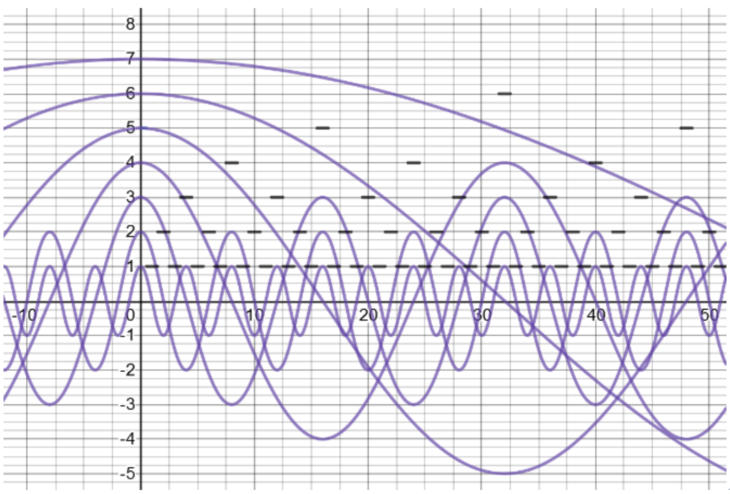

Figure 1. Continuous graph of $R(u)$ and (3.1) with $\mathbb{N}$ up to six.

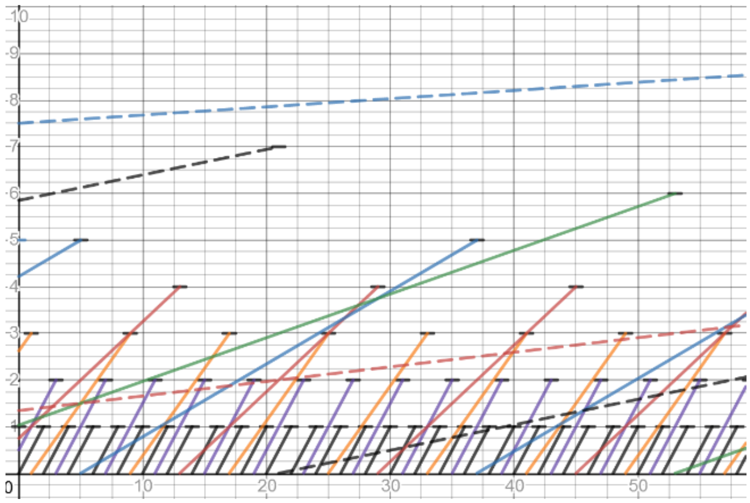

Figure 2 Discrete graph of r(u) and (3.23) with $\mathbb{N}$ up to nine.

Alongside the cosine governed expression, we present the governing expression using the modulo function to induce periodicity. This is in the interest of computation without pi. Interestingly, this suggests a tie between offset powers of two and pi. Now where $\mathbb{N}$ is the set of natural numbers, the values of $\mathbb{R}(u)$ are periodic governed by $\mathbb{N}\cos\left(\frac{\pi x}{2^{\mathbb{N}}}\right)$ continuously, and $\mathbb{N}\,mod\left(\frac{x+2^{\mathbb{N}-1}}{2^{\mathbb{N}}},1\right)$ discretely. Let Υ denote the set of curves determined by the relevant expression. Let $Amp(\ \Upsilon\ x\ )$ return the amplitude of the curve with y value zero at x. The x value is obtained from the desired u expressed as $4x \pm 1$. To equate $Amp(\ \Upsilon, x\ )$ and $\mathbb{R}(u)$ we subtract one from $\mathbb{R}$. Subtracting one from R precisely matches the pattern given by (3.0) as (3.1). For r we must remove the $4x - 1$ elements, now subtracting one from r becomes equivalent to (3.2) by including the translation b (3.3). Now we concisely describe the discussed variability observed in trajectories obtained by iterating the $3x + 1$ over $\mathbb{U}$, ,

$$R(u) - 1 = Amp\left(\mathbb{N}\cos\left(\frac{\pi x}{2^{\mathbb{N}}}\right)\right), \qquad R(u) - 1 = Amp\left(\mathbb{N} mod\left(\frac{x + 2^{\mathbb{N}-1}}{2^{\mathbb{N}}}, 1\right)\right). \tag{3.1}$$

$$r(u) - 1 = Amp\left(\mathbb{N}\cos\left(\frac{\pi x - b}{2^{\mathbb{N}}}\right)\right), \qquad r(u) - 1 = Amp\left(\mathbb{N} mod\left(\frac{x + 2^{\mathbb{N}-1} - b}{2^{\mathbb{N}}}, 1\right)\right), \tag{3.2}$$

$$where, \qquad b = \left(mod(\mathbb{N},2)\sum_{i=0}^{\frac{\mathbb{N}-1}{2}} 4^i + (1 - mod(\mathbb{N},2))\sum_{i=0}^{\frac{\mathbb{N}-1}{2}} 4^i\right). \tag{3.3}$$

Table 4. Since dividing 3u+1 by d(u) produces T(u) we report this data as it is. Notice the orderly periodic organization.

$r(u)$	$T(u)$	$3u + 1$	u
2	1	4	1
1	5	10	3
4	1	16	5
1	11	22	7
2	7	28	9
1	17	34	11
3	5	40	13
1	23	46	15
2	13	52	17
1	29	58	19
6	1	64	21
1	35	70	23
2	19	76	25
1	41	82	27
3	11	88	29
1	47	94	31
2	25	100	33
1	53	106	35
4	7	112	37
1	59	118	39

The form of (3.3) ultimately interleaves the $6k \pm 1$ arithmetic progressions (Table 1) of $T(u)$ as a function of the parity of r as encoded by the modular term prefixing the summation. We can see by the form of (3) that every x is accompanied by exactly one Υ. If no curve passes through x, extending $\mathbb{N}$ will resolve this. The result is that $\mathbb{R}$ is a multilevel fractal interleaving of $\mathbb{N}$, where one occurs twice as often as two, occurring twice as often as three, etc., with the caveat being that in r, but not R, this pattern is increasingly offset from the origin. In other words, larger numbers appear prior to smaller numbers of divisions, but not consecutive increasing iterates, demanding that all trajectories descend. Moreover, with the scale-free nature of the rising segments, their diversity is predictable. It is clear without needing calculations that the dynamics of $\mathbb{R}$ are constant from the governing equations (3). The self-symmetry of R, and nested appearances of T when unpacked algebraically imply that observing the descending trajectories suffices to understand the full range of possible outcomes. .

Conclusion We opine that all the disorderly conduct in Syracuse function trajectories is due to the linear mapping constraints imposed by the nonlinear expansion of the domain as the elements increase in size. The relative impact, or disturbance relative to an untranslated but scaled by three image, of the unit translation is constantly decreasing. It begins at a maximum of $1 + 1/3$ at u equals 1, and goes asymptotically to 1 with large arguments. But the relative impact of the translation to the division step is constant, the change is always 0.5. This discrepancy may prevent a transparent emergence of self-similar trajectories even though the protocol is perfectly static and well defined. All of the disorderly conduct occurs from the squeezing of the domain by the mapping to fit into the codomain, itself arranged in an orderly fashion. Despite the overall descent, some elements are not allowed a direct path as per the interaction between the properties of natural numbers and the constraints of the problem. This is similar to planes required to remain in holding patterns above their destination airport when there is a lot of air traffic. We have demonstrated that the uncertainties caused by these chaotic holding patterns retain no merit against the conjecture that all trajectories among natural numbers contain 1 in light of the constant dynamics observed in $\mathbb{R}(u)$.

References

[1] T. Brox, "Collatz cycles with few descents.," *Acta Arithmetica*, pp. 181-188, 2000.

[2] J. Lagarias, "The 3x + 1 Problem: An annotated Bilbiography II (2000, 2009)," *arXiv*, vol. arXiv.math/0608208v6[Math.NT], p. 42, 2012.

[3] T. Tao, "Almost all orbits of the Collatz map attain almost bounded values," *arXiv*, vol. arXiv:1909.03562v2, p. 49, 2019.

3	**5**	1			
9	**7**	**11**	**17**	**13**	**5**
15	**23**	**35**	**53**	**5**	

21	1																	
27	**41**	**31**	**47**	**71**	**107**	**161**	121	**91**	**137**	103	155	233	175	263	395	593	445	**167**
33	25	**19**	**29**	**11**														251
39	**59**	89	67	**101**	**19**													377
45	**17**																	**283**
51	77	**29**																**425**
57	**43**	65	49	**37**	**7**													**319**
63	95	143	215	323	485	**91**												**479**
69	**13**																	**719**
75	113	85	1															**1079**
81	**61**																	**1619**
87	131	197	**37**															2429
93	**35**																	**911**
99	149	**7**																**1367**
105	79	119	179	269	**101**													2051
111	**167**																	3077
117	**11**																	577
123	185	139	209	157	**59**													433
129	97	73	55	83	125	**47**												325
135	203	305	229	**43**														**61**
141	**53**																	**23**

Figure 0. Efficient ordering of 3x+1 problem trajectories.